Appendix
Money and Capital in a Persistent Liquidity Trap

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This appendix investigates empirically the increase in cash holdings in the US, and provides an assessment of the net position of investors, also in the US. Then it analyzes welfare, and finally develops extensions to the model. These include: 1) bubbles; 2) preference and growth shocks; 3) partial capital depreciation; 4) financial intermediaries; 5) inefficient saving technology; 6) idiosyncratic uncertainty about investment opportunities; 7) nominal bonds; 8) nominal rigidities.

1 Empirical analysis

1.1 Decomposing cash holdings in the US

This section investigates the rise in cash holdings occurring at the ZLB by focusing on the US. First, we use the balance-sheet tables of the Households and Nonprofit Organizations (B.101),

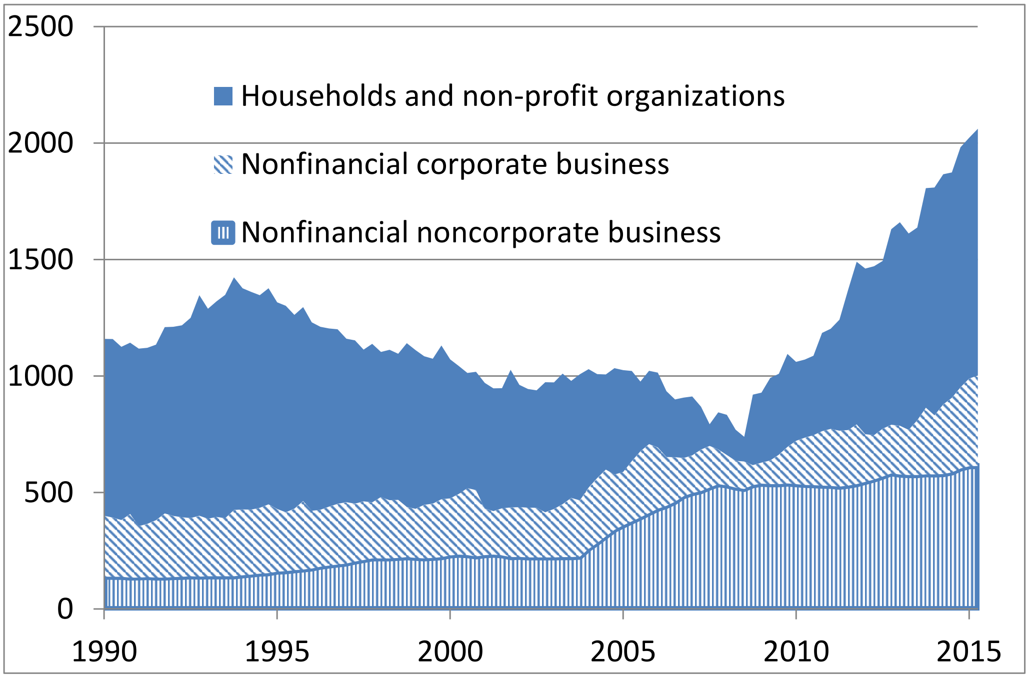


Figure 1: Checkable Deposits and Currency, billions of 2010 USD

Source: Financial accounts of the US, Federal Reserve, International Financial Statistics, authors' calculations.

of the Nonfinancial Corporate Business (B.103) and of the Nonfinancial Noncorporate Business (B.104) from the Financial Accounts of the US. The series used are the Checkable Deposits and Currency (FL153020005, FL103020005 and FL113020005). These are deflated by the CPI, obtained from the International Financial Statistics. The results are shown in Figure 1. The figure shows that both households and the nonfinancial businesses increase their cash holdings at the end of 2008, when the Fed funds rate started approaching zero. Among the nonfinancial businesses, the corporate sector accounts for most of the increase. This is consistent with our model, as the demand for money for saving purposes arises for the *less* constrained agents in the economy.

Second, we use the Survey of Consumer Finances (SCF) to decompose cash holdings by households into different household categories. We consider checking account holdings only, as currency is not available in the survey. We aggregate this variable within some categories of households, trying to reflect the split between households who participate in financial markets (our investors) and households who do not participate in financial markets (our hand-to-

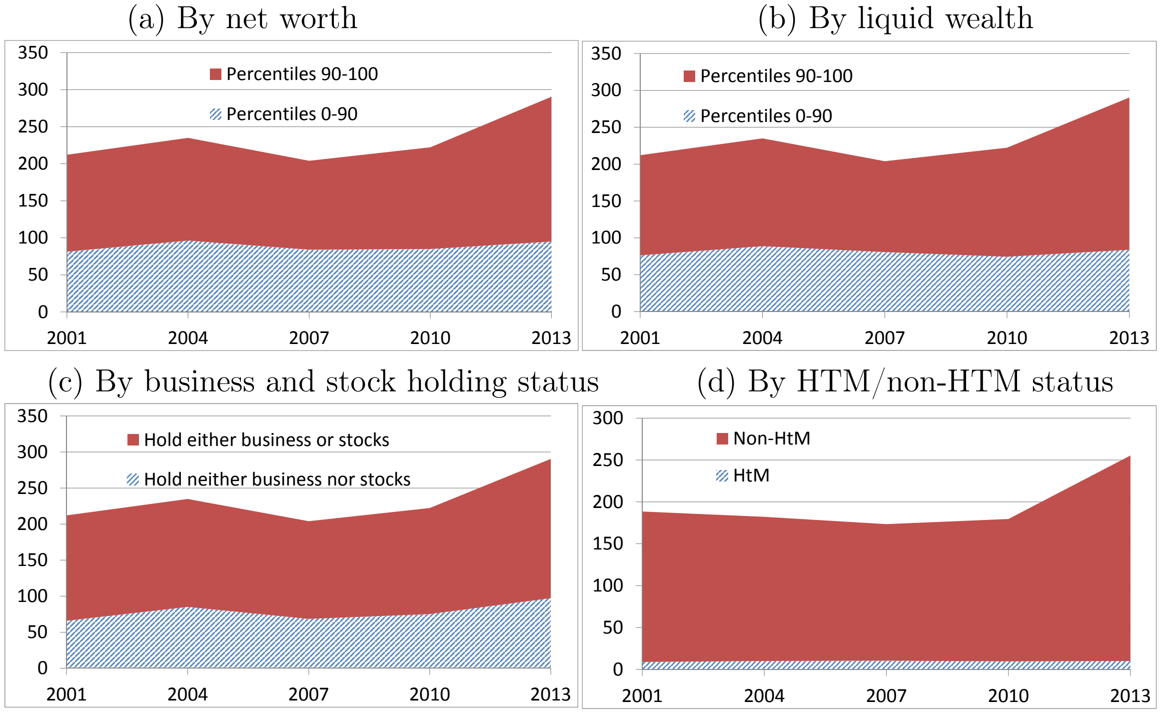


Figure 2: Households' checking account holdings, billions of 2010 USD

Source: Survey of Consumer Finances, Federal Reserve System, authors' calculations.

mouth workers). We use several proxies: households above the 90^{th} percentile of net worth versus those below; households above the 90^{th} percentile of liquid wealth versus those below; households owning a business or some stocks versus those owning neither; non hand-to-mouth households versus hand-to-mouth households (HTM). Aggregated holdings are deflated by the CPI, obtained from the International Financial Statistics.

The definition of liquid wealth follows Kaplan et al. (2014): it consists of checking, saving, money market, and call accounts as well as directly held mutual funds, stocks, corporate bonds, and government bonds, minus the sum of all credit card balances that accrue interest, after the most recent payment. The definition of HTM households also follows Kaplan et al. (2014), but is even more inclusive. HTM households are those whose average liquid wealth balances are positive (to capture the fact they are not borrowing), but are equal to or less than half their earnings per pay period. as a pay period, we use a month instead of two weeks, to obtain a higher share of HTM households. The proposrtion of non-HTM is therefore more conservative. We obtain a proportion of HTM that is around 0.36 on average between 2001 and 2013.

The results are represented in Figure 2. The figure shows that the bulk of the increase in cash holdings between the 2007 and 2013 surveys, among households, comes mainly from households with a high net worth, liquid wealth, households owning a business or stocks and especially, non-HTM households.

1.2 Estimation of investors' net position

We evaluate the investors' net position as the net position of the nonfinancial corporate business and households who participate to financial markets. We always find a negative position, suggesting that $\bar{l} < 0$.

To calculate the corporate net position in interest-bearing assets, we use the balance-sheet tables of the Nonfinancial Corporate Business (B.103) from the Financial Accounts of the US. We calculate the net positions as follows: Time and savings deposits (FL103030003) + Money market fund shares (FL103034003) + Security repurchase agreements (FL102051003) + Credit market instruments (FL104004005) + Trade receivables (FL103070005) - Credit market instruments (FL104104005) - Trade payables (FL103170005)- Taxes payable (FL103178000). We find a large negative net position of -5000 Billion USD.

To calculate the net position of "participating" households, we use the same proxies as in Section 1.1: households above the 90^{th} percentile of net worth; households above the 90^{th} percentile of liquid wealth; households owning a business or some stocks; non hand-to-mouth households following the definition of Kaplan et al. (2014). Using the 2013 Survey of Consumer Finances, we compute for each group the average net position in interest-bearing assets as Certificates of deposit + Savings in bonds + Directly held bonds - Debt. We exclude pensions and life insurance as these assets are usually not liquid and hence cannot be used for investment. The weighted average of these net positions is about -160 thousand USD for households above the 90^{th} percentile of net worth; -135 thousand USD for households above the 90^{th} percentile of liquid wealth; -145 thousand USD for households owning a business or some stocks; -100 thousand USD for non hand-to-mouth households.

2 Welfare

2.1 Intertemporal utility of investors

In a steady state, the utility of investors in their investing phase is given by

$$U = \frac{\log(c^I) + \beta \log(c^S)}{1-\beta^2} = \frac{\log(\beta r) + (1+\beta)\log\big((1-\beta)(1-\phi)\alpha y\big)}{1-\beta^2}.$$

The first term on the numerator reflects consumption smoothing and depends positively on the real interest rate. The second term reflects the level of the whole consumption path and depend on output. In the case of autarkic investors, the difference of utility in the liquidity trap and the cashless steady state is then given by:

$$\begin{aligned}(1-\beta^2)(U^{\mathrm{LT}} - U^{\mathrm{cashless}}) &= -\log(\theta r^S) + (1+\beta)\frac{\alpha}{1-\alpha}\log((1-\phi)\beta/\theta + \phi\theta/\beta) \\ &= -\log(1-\theta\Delta) + (1+\beta)\frac{\alpha}{1-\alpha}\log\Big(1 - \frac{(\theta-\beta)\Delta}{1+\beta/\theta - \beta\Delta}\Big)\end{aligned}$$

where Δ is the interest rate gap. Both logarithms are strictly negative when $\Delta > 0$. The first term is therefore positive, reflecting better consumption smoothing in the liquidity trap. The second term is negative, due to lower capital and output. However, the second term is first order in $\theta - \beta$, and is likely to be small for any realistic calibration of these parameters, since they are both close to 1. The first term can be rewritten $-\log(1-\beta\Delta) - \log\big(1 - \frac{(\theta-\beta)\Delta}{1-\beta\Delta}\big)$ and is 0-order in $\theta - \beta$. Therefore, unless α is very close to 1, the first term should be strictly larger than the second term.

A similar reasoning applies to the utility of investors in their saving phase.

2.2 Efficient allocations

The following proposition characterizes efficient allocations.

Proposition 1 (Efficient allocations) *An allocation $\{k_{t+1}, c_t^I, c_t^S, c_t^w\}$ satisfying the resource constraint $y_t = k_{t+1} + c_t^I + c_t^S + c_t^w$ is Pareto efficient if and only if $k_{t+1} = \beta\alpha y_t$ and $c_{t+1}^w/c_t^w = c_{t+1}^I/c_t^S = c_{t+1}^S/c_t^I = \beta\rho_{t+1} = y_{t+1}/y_t$.*

Proof. Denote by the superscript 1 (2) the group of investors in their saving (investing) phase in even (odd) periods. A Pareto-efficient allocation maximizes $\sum_{t=0}^{\infty} \beta^t(\lambda^1 \log c_t^1 + \lambda^2 \log c_t^2 + \lambda^w \log c_t^w)$ under the resource constraint $y_t = k_{t+1} + c_t^1 + c_t^2 + c_t^w$, where $\lambda^i, i = 1, 2, w$ are Pareto weights associated to both groups of investors and workers which sum to 1. The maximization can be carried out in two steps. First, maximize $(c_t^1)^{\lambda^1}(c_t^2)^{\lambda^2}(c_t^w)^{\lambda^w}$ s.t. $C_t = c_t^1 + c_t^2 + c_t^3$ in any period, which gives constant shares of aggregate consumption $c_t^i = \lambda^i C_t$. Then, maximize $\sum_{t=0}^{\infty} \beta^t \log C_t$ s.t. $y_t = k_{t+1} + C_t$. This well-known maximization problem has the first-order condition $C_{t+1}/C_t = \beta\rho_{t+1} = \beta\alpha k_{t+1}^{\alpha-1}$ and is solved by $k_{t+1} = \beta\alpha y_t$. Having individual consumptions equal to constant shares of aggregate consumption is equivalent to having all individual consumption grow at the same rate, which is also the rate of output growth. ■

We can check that for $\bar{l} \geq \bar{l}_{max}$, the steady state is Pareto-efficient and satisfies the characteristics described in Proposition 1.

2.3 Pareto-improving Policy with Additional Policy Instruments

Consider the following additional policy instruments: a capital subsidy γ_t (such that one unit of capital is paid $1-\gamma_t$ by investors), a consumption tax η_t (such that one unit of consumption costs $1+\eta_t$ to consumers), and a corporate tax τ_t^k paid by S-investors on their profits. With these additional policy instruments, the (binding) borrowing constraint (3) becomes $b_{t+1} = \phi_t(1-\tau_{t+1}^k)\rho_{t+1}k_{t+1} = \phi_t(1-\tau_{t+1}^k)\alpha y_{t+1}$, and Equations (14) and (15) respectively become:

$$\beta\alpha(1-\tau_t^k)(1-\phi_{t-1})y_t = \frac{1}{r_{t+1}}\left[\left((1-\tau_{t+1}^k)\phi_t\alpha + \bar{l}_t\right)y_{t+1} + m_{t+1}^S\right], \tag{32}$$

$$(1-\gamma_t)k_{t+1} + \pi_{t+1}m_{t+1}^S + \frac{1}{r_{t+1}}\bar{l}_t y_{t+1} = \beta\left[\left((1-\tau_t^k)\alpha + \bar{l}_{t-1}\right)y_t + m_t^S\right]. \tag{33}$$

Consumption of all three agents follows:

$$c_t^S = \frac{1}{1+\eta_t}(1-\beta)[\alpha(1-\tau_t^k)y_t - b_t], \tag{34}$$

$$c_t^I = \frac{1}{1+\eta_t}(1-\beta)\left[b_t + \bar{l}_{t-1}y_t + m_t^S\right], \tag{35}$$

$$c_t^w = \frac{1}{1+\eta_t}\left[\frac{T_t^w}{P_t} + \frac{M^w}{P_t} + \frac{l_{t+1}^w}{r_{t+1}} - l_t^w\right]. \tag{36}$$

Finally, the government budget constraint is now:

$$\frac{M_{t+1}}{P_t} + \frac{l^g_{t+1}}{r_{t+1}} + \tau^k_t \alpha y_t + \eta_t(c^w_t + c^I_t + c^S_t) = \frac{M_t}{P_t} + \frac{T^w}{P_t} + l^g_t + \gamma_t k_{t+1}. \tag{37}$$

Consider an economy in a liquidity trap steady state at $t = -1$. Suppose that the government has already implemented an open-market operation to increase debt to the limit of the cashless equilibrium, such that $\bar{l}_{-1} = \bar{l}_T(\phi)$ (using the definitions of Proposition 3) and $m^S_0 = 0$ in period $t = -1$. The following Proposition shows that an appropriate debt policy, together with the three policy instruments mentioned above, can lead to a Pareto-improving and Pareto-efficient equilibrium from $t = 0$ on.

Intuitively, this policy consists in increasing debt sufficiently to be in the cashless equilibrium in all periods. Getting the right level of investment during the transition is done with the capital subsidy. Engineering transfers from investors to workers is done through a consumption tax (together with the lump-sum transfers to workers already assumed in the baseline model). Finally, smoothing investors' consumption during the transition is done through the tax on corporate profits.

Proposition 2 (Pareto-efficient policy) *Consider constant leverage $\{\phi, \bar{l}^w\}$. Suppose the economy is initially in a liquidity trap steady state at $t = -1$ with $\bar{l}^g_{-1} + \bar{l}^w = \bar{l}_T(\phi)$, $m^S_0 = 0$ and zero taxes and subsidies: $\gamma_{-1} = \eta_{-1} = \tau^k_{-1} = 0$. Define a policy by a sequence $\{\bar{l}^g_t, \gamma_t, \eta_t, \tau^k_t\}$ for $t \geq 0$ and suppose that $M_{t+1} = \theta M_t$ and that transfers T^w_t adjust the government budget constraint* (37). *There is a policy such that the associated equilibrium:*

- *is not a liquidity trap ($i_{t+1} > 1$ for all $t \geq 0$),*
- *is Pareto-efficient as described in Proposition 1,*
- *Pareto-improves on the initial steady state.*

Proof. We provide a proof by construction. Consider arbitrary $\lambda^1, \lambda^2, \lambda^w$ in $(0, 1)$ with $\lambda^1 + \lambda^2 + \lambda^w = 1$.

Consider now the candidate policy defined in the following way.

$$
\begin{aligned}
1+\eta_t &= \frac{\alpha(1-\beta)}{1-\alpha\beta}\frac{\bar{l}_{-1}+\alpha\phi}{\alpha\lambda^2} \qquad t \geq 0,\\
1-\tau_0^k &= \phi + \frac{\lambda^1}{\lambda^2}\Big(\frac{\bar{l}_{-1}}{\alpha}+\phi\Big),\\
1-\tau_1^k &= (1-\tau_0^k)\frac{\phi}{1-\phi} + \frac{1}{1-\phi}\frac{\bar{l}_{-1}}{\alpha},\\
\tau_t^k &= \tau_{t-2}^k \qquad t \geq 2,\\
\gamma_t &= \tau_{t+1}^k \qquad t \geq 0,\\
\bar{l}_0^g &= \alpha(1-\tau_0^k)\frac{1-2\phi}{1-\phi} - \frac{\phi}{1-\phi}\bar{l}_{-1} - \bar{l}^w,\\
\bar{l}_t^g &= \bar{l}_{t-2}^g \qquad t \geq 1.
\end{aligned}
$$

We now show all three statements of the Proposition in turn. First, the equilibrium is cashless. Indeed, start looking for a cashless equilibrium with $m_t^s = 0$. Plugging the candidate policy into (32) we get $r_{t+1} = y_{t+1}/(\beta y_t)$. From the money market equilibrium (16), we have $P_{t+1}/P_t = \theta y_t/y_{t+1}$. As a result, we get a gross nominal rate $i_{t+1} = P_{t+1}r_{t+1}/P_t = \theta/\beta > 1$ from Assumption 1, which confirms that we are indeed in a cashless equilibrium.

Second, plugging the candidate policy together with the equilibrium real interest rate into (33), we get $k_{t+1} = \beta\alpha y_t$. Plugging the policy in (34) and (35), we get $c_{t+1}^I/c_t^S = c_{t+1}^S/c_t^I = y_{t+1}/y_t$. Consumption growth of workers $c_{t+1}^w/c_t^w = y_{t+1}/y_t$ follows from the aggregate resource constraint $y_t = k_{t+1} + c_t^I + c_t^S + c_t^w$. From Proposition 1, this implies that the equilibrium is Pareto efficient.

Finally, plugging the policy into (34) and (35) at $t = 0$ with $b_0 = \phi\alpha y_0$, we get $c_0^S = \lambda^1(y_0 - k_1) = \lambda^1(c_0^S + c_0^I + c_0^w)$ and $c_0^I = \lambda^2(c_0^S + c_0^I + c_0^w)$. This implies that $\lambda^1, \lambda^2, \lambda^w$ are the Pareto weights associated to both groups of investors and workers, where the superscript 1 (2) denotes the group of investors in their saving (investing) phase in even (odd) periods, as in the proof of Proposition 1. As the choice of these weights was not constrained, it is always possible to choose them in such a way that all agents get at least the utility they had in the initial steady state. Therefore, the equilibrium Pareto-improves on the initial steady state. ■

3 Extensions

3.1 Bubbles

Consider an infinitely-lived asset in fixed unitary supply with no intrinsic value—a bubble. Denote z_t its relative price in terms of consumption goods. The real return of the bubble as of time t is z_{t+1}/z_t. For the bubble to be traded, this rate of return must be equal to the real interest rate: $z_{t+1}/z_t = r_{t+1}$. With r_{t+1} different from 1, the bubble would either asymptotically disappear or diverge to an infinite value. Then, a bubbly steady state necessarily has a zero *real* interest rate: $r = 1$. With positive long run inflation, $1 > 1/\theta$, the bubble strictly dominates money as a saving instrument. Therefore, S-investors would hold the bubble and would not hold money.[1] In the case of autarkic investors, such a bubbly steady state is described by:

$$z = \alpha[(1-\phi)\beta - \phi]y \tag{38}$$

$$k = \beta\alpha y - (1-\beta)z \tag{39}$$

where (38) is the Euler equation of savers and (39) the aggregate budget constraint of investors. As can be seen from equations (19) and (21), the bubbly steady state is formally equivalent to a liquidity trap steady state with $m^S = z$ and $\theta = 1$. The bubble plays the same role as investor-held money in the liquidity trap, but offers a higher real return. This allows us to derive the following Proposition.

Proposition 3 (Bubbly steady state with autarkic investors) *Suppose* $0 < \phi < \phi_{max}$ *and* $\theta > 1$. *Define* $\phi_B = \beta/(1+\beta)$ *and* $\phi_K = \beta^2/(\theta + \beta^2)$. *We have* $\phi_B > \phi_T > \phi_K$.

(i) *If* $\phi \geq \phi_B$, *there is a unique cashless steady state as described by Proposition 1.*

(ii) *If* $\phi_T \leq \phi < \phi_B$, *there is a cashless steady state with* $r = \phi/[\beta(1-\phi)] < 1$ *and a bubbly steady state with* $r = 1$.

(iii) *If* $\phi < \phi_T$, *there is a liquidity-trap steady state with* $r = 1/\theta < 1$ *as described in Proposition 1 and a bubbly steady state with* $r = 1$.

[1] With negative long-run inflation, bubbles would be dominated by money and could never arise in equilibrium.

(iv) *In the bubbly steady state, the real (nominal) interest rate is given by* $r = 1$ $(i = \theta)$*,* z/y *is decreasing in* ϕ *and* k *is increasing in* ϕ*.*

(v) *Capital and output are strictly lower in the bubbly steady state than in the cashless steady state. They are lower in the bubbly steady state than in the liquidity-trap steady state when* $\phi_K \leq \phi < \phi_T$ *and larger in the bubbly steady state than in the liquidity-trap steady state when* $\phi < \phi_K$*.*

Proof. Points (i) to (iv) directly follow from Proposition 1 using the formal equivalence between bubbly steady states and liquidity-trap steady states mentioned in the text. From (38) and (39), we get $k^{1-\alpha} = \alpha\big(\beta - (1-\beta)[(1-\phi)\beta - \phi]\big)$ in the bubbly steady state. Comparing this with the cashless and liquidity trap steady states, we get point (v). ■

As the Proposition shows, a bubble can help the economy exit the liquidity trap if $\theta > 1$. The bubble raises the nominal interest rate from $i = 1$ to $i = \theta$. S-investors then substitute the bubble for money in their portfolio. For a given money supply, this also reflates the economy as the price level increases to accomodate the lower money demand.

However, the bubbly steady state is qualitatively similar to a liquidity trap. As with money, holding the bubble takes out resources from investment and output is lower in the bubbly equilibrium than in the cashless steady state. In the intermediate case where $\phi_T \leq \phi < \phi_B$, a bubble prevents the downward interest rate adjustment that would restore the cashless level of capital and output. In the case of low leverage $\phi < \phi_T$, bubbles increase the real interest rate, which may or may not increase capital and output compared to the liquidity trap. A higher real interest rate decreases the price of liquidity but increases the net liquidity of investors, with an ambiguous total effect on investment depending on the level of net liquidity. This is similar to the ambiguous effect of inflation described in Section 4.

3.2 Preference and Growth Shocks

To study the effect of β on output, we make the simplifying assumption of autarkic investors: $\bar{l} = 0$. This is without loss of generality as the investors' net debt matters only in the cashless economy. We derive the following Proposition:

Proposition 4 (Effect of β on the steady state with autarkic investors) *Define $\beta_T = \theta\phi/(1-\phi)$ and $\phi_{max} = 1/2$.*

If $0 < \phi < \phi_{max}$, then there exists a locally constrained steady state with $r < 1/\beta$.

(i) *If, additionally, $\beta \leq \beta_T$, then the steady state is cashless.*

(ii) *If $\beta > \beta_T$, then the steady state is a liquidity trap.*

(iii) *In the cashless steady state, the real interest rate r and the nominal interest rate i are decreasing in β, $m^S = 0$ and k is increasing in β.*

(iv) *In the liquidity-trap steady state, the real interest rate r is invariant in β, m^S/y is increasing in β and k is increasing in β.*

Proof. The proof derives from Lemma 1. Note simply that $\beta > \beta_T$ is equivalent to $\phi < \phi_T$, which defines the liquidity trap steady state. We then derive r, k and m^S with respect to β in the cashless and liquidity trap steady states. ■

An increase in β makes the long-run interest rate fall, and eventually hit the zero-lower bound. In both the cashless and liquidity-trap steady states, an increase in β increases the investors' propensity to save, which increases the capital stock in the long run. As a result, whereas an increase in β can explain the emergence of a liquidity trap, it cannot explain the slowdown in investment. In the presence of trend growth, the same conclusions would hold in case of a growth slowdown. In particular, with lower trend growth, less investment is required to keep the capital stock on its trend. Therefore a given amount of saving leads to an upward shift in the capital intensity of production, and hence in the investment rate.

3.3 Partial Capital Depreciation

We assume here that $\delta < 1$, so that capital depreciates only partially from period to period. For consistency, we focus on the case where investors are net debtors $\bar{l} \leq 0$. All our results generalize provided some mild condition on $\bar{l}$, as shown in the following Proposition:

Proposition 5 (Steady state when entrepreneurs are net debtors) *Define $\phi_{max}(\bar{l}) = (1 - [1-\beta(1-\delta)]\bar{l}/\alpha)/2$ and $\phi_T(\bar{l}) = (\beta - [\theta - \beta^2(1-\delta)]\bar{l}/\alpha)/[\theta + \beta - (\theta^2 - \beta^2)(1-\delta)\bar{l}/\alpha]$. If $\bar{l} \leq 0$ and $0 < \phi < \phi_{max}$, then there exists a locally constrained steady state with $0 < r < 1/\beta$.*

(i) *If, additionally, $\phi \geq \phi_T$, then the steady state is cashless.*

(ii) *If, $\phi < \phi_T$, then the steady state is a liquidity trap.*

(iii) *In the cashless steady state, the real interest rate r and the nominal interest rate i are increasing in ϕ if $\bar{l} > -1/\beta(1-\delta)$, and increasing in $\bar{l}$ if $\bar{l} > -1/[1+\beta(1-\delta)]$, $m^S = 0$, k is decreasing in ϕ and decreasing in $\bar{l}$ in the neighborhood of $\bar{l} = 0$.*

(iv) *In the liquidity-trap steady state, the real interest rate r and the nominal interest rate i are invariant in ϕ and $\bar{l}$, $m^S/\rho k$ is decreasing in ϕ and $\bar{l}$ and k is increasing in ϕ and independent of $\bar{l}$.*

Proof. With partial depreciation, using $f(k) = [\rho(k) - (1-\delta)]k/\alpha$, we can show that the dynamic system at the cashless steady state satisfies

$$r\beta(1-\phi)\rho(k) = \phi\rho(k) + \frac{\bar{l}}{\alpha}[\rho(k) - (1-\delta)] \tag{40}$$

$$r = \beta r \left[\rho(k) + \frac{\bar{l}}{\alpha}\left(1 - \frac{1}{\beta r}\right)[\rho(k) - (1-\delta)]\right] \tag{41}$$

We derive r^* as follows. We use (41) to express ρ as a function of r and replace in (40). This gives $P(r) = 0$ where P is a second-order polynomial defined by

$$P(r) = \beta(1-\phi)\left[1 + \beta(1-\delta)\frac{\bar{l}}{\alpha}\right]r^2 - \left(\phi + \frac{\bar{l}}{\alpha}\right)r + \phi(1-\delta)\frac{\bar{l}}{\alpha}$$

This polynomial admits two roots. We have $P(0) = \phi(1-\delta)\frac{\bar{l}}{\alpha} \leq 0$ as $\bar{l} \geq 0$ and $\phi > 0$, so it admits only one positive root, which we then take as our solution for r.

This solution is lower than $1/\beta$ as long as $P(1/\beta) > 0$. This is equivalent to $\phi < \phi_{max}$. Finally, $i = r\theta > 0$ is guaranteed by $P(1/\theta) < 0$, which implies $\phi > \phi_T$. In that case, the economy is cashless and follows 40 and 41. This proves result (i). Otherwise, the economy is in a liquidity trap and follows

$$\frac{\beta(1-\phi)}{\theta}\rho = \phi\rho + \frac{\bar{l}}{\alpha}[\rho - (1-\delta)] + \frac{m^S}{k} \tag{42}$$

$$1 = \beta\rho - (\theta - \beta)\left(\frac{\bar{l}}{\alpha}[\rho - (1-\delta)] + \frac{m^S}{k}\right) \tag{43}$$

This proves result(ii).

To establish result (iii), we totally differentiate P with respect to ϕ . Using the fact that r is the upper root of P so that $P'(r) > 0$, we can show that r is increasing in ϕ if and only if

$$\beta\left[1+\beta(1-\delta)\frac{\bar{l}}{\alpha}\right]r^2+r-(1-\delta)\frac{\bar{l}}{\alpha}>0$$

This is the case for $-1/\beta(1-\delta) \leq \bar{l} \leq 0$ as r is positive.

Similarly, we totally differentiate P with respect to $\bar{l}$. Using the fact that r is the upper root of P so that $P'(r) > 0$, we can show that r is increasing in $\bar{l}$ if and only if

$$r-(1-\phi)(1-\delta)\beta^2r^2-\phi(1-\delta)>0$$

As $\beta^2r^2 < 1$, a sufficient condition is $r > 1-\delta$. This is the case for $\bar{l} > -1/[1+\beta(1-\delta)]$, as this guarantees $P(1-\delta) < 0$.

We then express r as a function of ρ using (40) and replace in (41). We find that $Q(\rho) = 0$ where Q is a second-order polynomial defined by

$$Q(\rho)=\beta\left[\left(\phi+\frac{\bar{l}}{\alpha}\right)\left(1+\frac{\bar{l}}{\alpha}\right)-(1-\phi)\frac{\bar{l}}{\alpha}\right]\rho^2-\left(\phi+\frac{\bar{l}}{\alpha}\right)\left[1+2\beta(1-\delta)\frac{\bar{l}}{\alpha}\right]\rho+(1-\delta)\frac{\bar{l}}{\alpha}\left[1+\beta(1-\delta)\frac{\bar{l}}{\alpha}\right]$$

We select the upper root of this polynomial for similar reasons. We thus have

$$\rho=\frac{\left(\phi+\frac{\bar{l}}{\alpha}\right)\left[1+2\beta(1-\delta)\frac{\bar{l}}{\alpha}\right]+\sqrt{\left(\phi+\frac{\bar{l}}{\alpha}\right)^2-\phi(1-\phi)\beta(1-\delta)\left[1+\beta(1-\delta)\frac{\bar{l}}{\alpha}\right]}}{2\beta\left[\left(\phi+\frac{\bar{l}}{\alpha}\right)\left(1+\frac{\bar{l}}{\alpha}\right)-(1-\phi)\frac{\bar{l}}{\alpha}\right]}$$

We compute $Q(1/\beta)$ and show

$$Q(1/\beta)=-[1-\beta(1-\delta)]\frac{\bar{l}}{\alpha}\left[1-2\phi-[1-\beta(1-\delta)]\frac{\bar{l}}{\alpha}\right]$$

This is positive if $\bar{l} < 0$, which implies that $\rho < 1/\beta$.

To study the effect of ϕ on k, we totally differentiate Q with respect to ϕ. Using the fact that ρ is the upper root of Q so that $Q'(\rho) > 0$, we can show that ρ is increasing in ϕ if and only if

$$(\beta\rho - 1) + 2\frac{\bar{l}}{\alpha}\beta[\rho - (1-\delta)] < 0$$

For $\bar{l} < 0$, $\beta\rho < 1$. Besides, as the non-negativity on k imposes $\rho \geq 1-\delta$, then the second term is also negative in that case. As a result, ρ is increasing in ϕ, which implies that k is decreasing in ϕ.

Similarly, to study the effect of $\bar{l}$ on k, we totally differentiate Q with respect to $\bar{l}$. Using the fact that ρ is the upper root of Q so that $Q'(\rho) > 0$, we can show that ρ is increasing in $\bar{l}$ if and only if

$$\left[1 + 2\beta(1-\delta)\left(\phi + 2\frac{\bar{l}}{\alpha}\right)\right]\rho - 2\beta\left(\phi + \frac{\bar{l}}{\alpha}\right)\rho^2 - (1-\delta)\left[1 + 2\beta(1-\delta)\frac{\bar{l}}{\alpha}\right] > 0$$

This is the case both for $\bar{l} = 0$, for which $\rho = 1/\beta$. Therefore, ρ is increasing in $\bar{l}$ in the neighborhood of $\bar{l} = 0$. Since k is inversely related to ρ, k is decreasing in $\bar{l}$ in the neighborhood of $\bar{l} = 0$.

To derive result (iv), consider Equations (42) and (43), which describe the liquidity-trap steady state. They yield

$$\begin{aligned} \rho &= \tfrac{\theta}{\beta^2 + (\theta^2 - \beta^2)\phi} \\ \tfrac{m^S}{k} &= \left[\tfrac{\beta}{\theta}(1-\phi) - \phi - \tfrac{\bar{l}}{\alpha}\right]\rho + (1-\delta)\tfrac{\bar{l}}{\alpha} \end{aligned}$$

As $\theta > \beta$, ρ is decreasing in ϕ, which implies that k is increasing in ϕ. We can also see that ρ and hence k are independent of $\bar{l}$. Similarly, as $i = 1$ and $r = 1/\theta$ in a liquidity trap, i and r are independent of ϕ and $\bar{l}$. Regarding m^S/k, since ρ is decreasing in ϕ, then m^S/k is decreasing in ϕ. Finally, since ρ is independent of $\bar{l}$ and $\rho > 1-\delta$, then m^S/k is decreasing in $\bar{l}$. ■

3.4 Financial Intermediation

In the benchmark model, money is modeled as *outside money* directly supplied by the government. However, in practice, cash holdings usually take the form of deposits, which are a liability of banks, and could in principle be intermediated to capital investment. This extension shows that this is not the case. At the ZLB, banks are unable to channel deposits to credit constrained I-investors for the same reason that savers are unable to do it in the benchmark model. Instead, banks increase their excess reserves at the central bank.

Consider a simple model of *endogenous money.* The monetary authority now only controls base money M^0_{t+1}, which is assumed to be made entirely of banks' reserves. Total money M_{t+1} is made of deposits endogenously supplied by banks. In Equations (8) and (9) of the benchmark model, money supply M_{t+1} has then to be replaced by base money M^0_{t+1}.

There is a unit measure of banks owned by the representative worker. Banks receive a charter from the government which allows them to issue deposits M_{t+1}, a zero nominal interest liability that can be used for transactions in the cash-in-advance constraint of workers. On their asset side, banks buy central bank reserves M^0_{t+1} and bonds for a nominal amount $M_{t+1} - M^0_{t+1}$. Banks maximize next-period profits, which they rebate (period by period) to households. In order to limit money creation, the bank charter subjects them to a reserve requirement: their buying of bonds cannot exceed a fraction μ of the net present value of deposits:[2]

$$M_{t+1} - M^0_{t+1} \leq \mu \frac{M_{t+1}}{i_{t+1}}.$$

The market clearing condition for bonds, given by Equation (10) in the benchmark model, has to be modified to account for bond demand by banks:

$$b_{t+1} + l^w_{t+1} + l^g_{t+1} = a_{t+1} + r_{t+1} \frac{M_{t+1} - M^0_{t+1}}{P_t}. \tag{44}$$

It is useful to define $\tilde{M}^0_{t+1}$, an indicator of *excess* reserves of banks, by:

$$\tilde{M}^0_{t+1} = M^0_{t+1} - \big(1 - \frac{\mu}{i_{t+1}}\big) P_t (1-\alpha) Y_t.$$

We obviously have $\tilde{M}^0_{t+1} = 0$ in the cashless equilibrium.[3] In the general case, the bond market equilibrium can be rewritten

$$b_{t+1} + l^w_{t+1} + l^g_{t+1} = a_{t+1} + \mu \frac{P_t}{P_{t+1}} (1-\alpha) Y_t + \frac{M^S_{t+1} - \tilde{M}^0_{t+1}}{P_{t+1}}.$$

[2]While the precise form of the reserve requirement does not matter, this expression yields tractable results.

[3]When $i_{t+1} > 1$, banks issue as much money and buy as little reserves as they can and the reserve requirement is binding. Banks' reserves are then equal to $M^0_{t+1} = (1 - \mu/i_{t+1}) M_{t+1} = (1 - \mu/i_{t+1})(1-\alpha) P_t y_t$.

Note that a fraction μ of workers's money holdings for transaction purposes is channeled by banks to the bond market. At the zero-lower bound, banks are indifferent between buying bonds or reserves, the reserve requirement does not bind, and excess reserves $\tilde{M}^0 \geq 0$.

We can now rewrite the main equations of the benchmark model, the Euler equation (14) and the aggregate budget constraint (15) as

$$\beta\alpha(1-\phi_{t-1})y_t = \frac{1}{r_{t+1}}\left[(\phi_t\alpha + \bar{l}_t)y_{t+1} - \mu\frac{P_t}{P_{t+1}}(1-\alpha)y_t + \frac{\tilde{M}^0_{t+1}}{P_{t+1}}\right], \tag{45}$$

$$k_{t+1} + \frac{\tilde{M}^0_{t+1}}{P_t} + \bar{l}_t\frac{y_{t+1}}{r_{t+1}} - \mu\frac{P_t}{P_{t+1}}\frac{y_t}{r_{t+1}} = \beta\left[(\alpha + \bar{l}_{t-1})y_t - \mu\frac{P_{t-1}}{P_t}(1-\alpha)y_{t-1} + \frac{\tilde{M}^0_t}{P_t}\right]. \tag{46}$$

There are only two changes compared to the benchmark model. First, the net supply of bonds from the rest of the economy is decreased by the share μ of workers' deposits lent by banks to investors: $\bar{l}_t y_{t+1}$ has to be replaced by $\bar{l}_t y_{t+1} - \mu P_t/P_{t+1}(1-\alpha)y_t$. Second, money holdings by investors M^S is replaced by excess reserves $\tilde{M}^0$ at the Central Bank. In this extended model, the increase in cash holdings by investors at the zero lower bound shows up as an increase in excess reserves at the Central Bank. Results on the steady state of the benchmark model extend to the case of endogenous money with the simple change of parameter $\bar{l} \to \bar{l} - \mu(1-\alpha)/\theta$.

3.5 Inefficient saving technology

Suppose there is an inefficient storage technology available to savers with return $\sigma \in (\theta^{-1}, \beta^{-1})$. This technology provides an alternative saving instrument to bonds and money holdings. There is an installation cost: investing a fraction x of saving in this technology only yields a fraction $\Psi(x)$ that is actually stored, with Ψ twice differentiable, $\Psi(0) = 0$, $\Psi'(0) = 1$, $\Psi'(1) > 0$, $\Psi'(x) > 0$, and $\Psi''(x) < 0$. For simplicity, we focus on the case of autarkic investors.

Investors in their saving phase choose x to maximize the total return on their saving $\rho^S_{t+1} = (1-x_t)r_{t+1} + \sigma\Psi(x_t)$. When ϕ is large enough so that $r_{t+1} \geq \sigma$, the storage technology is too inefficient to be used. For lower values of ϕ, the storage technology starts being used and the first-order condition with respect to x is $r_{t+1} = \sigma\Psi'(x_t)$. The real interest rate decreases with the use of the inefficient technology. The cashless steady state is described by the following equations:

$$\beta\sigma(1-\phi)(1-x)\Psi'(x) = \phi, \tag{47}$$

$$k = \beta\alpha y - \beta\alpha y(1-\phi)[x - \beta\sigma\Psi(x)], \tag{48}$$

where (47) replaces the Euler equation and (48) is the aggregate budget constraint of investors. From (47), it is clear that a lower leverage ϕ is associated with a higher use x of the inefficient storage technology, and therefore with a lower interest rate. From (48), this crowds out investment k in the efficient production technology. It is easy to check that the average productivity of capital invested in both technologies is decreasing in ϕ. This negative reallocative effect of low interest rates on aggregate productivity is similar to the one studied by Buera and Nicolini (2016).

In a liquidity trap equilibrium, the use of the inefficient technology is pinned down by inflation: $\theta\sigma\Psi'(x) = 1$. Then, deleveraging shocks are adjusted by higher real money holdings which crowd out good capital as in the benchmark case, while leaving investment in inefficient storage unaffected. The liquidity trap equilibrium is indeed described by:

$$m^S = \alpha\left[(1-x)(1-\phi)\frac{\beta}{\theta} - \phi\right]y, \tag{49}$$

$$k = \beta\alpha y - \beta\alpha y(1-\phi)[x - \beta\sigma\Psi(x)] - (\theta-\beta)m^S. \tag{50}$$

The key result of the benchmark model remains valid: m^S/y (k/y) decreases (increases) with ϕ.

Note that the storage technology puts a strictly positive lower bound to the shadow rate, contrary to the benchmark model where the shadow rate went to 0 in the limit $\phi \to 0$. Indeed, setting ϕ to 0 in (47), we get $x = 1$, with a corresponding shadow rate $r^S = \sigma\Psi'(1) > 0$.

These results are summarized by the following Proposition.

Proposition 6 (Inefficient storage technology) *Suppose* $\theta < 1/[\sigma\Psi'(1)]$. *Define* $\phi_E = \beta\sigma/(1+\beta\sigma)$ *and* $\phi_{TE} = \dfrac{\beta\big(1-\Psi'^{-1}(1/\sigma\theta)\big)}{\theta+\beta\big(1-\Psi'^{-1}(1/\sigma\theta)\big)}$. *We have* $\phi_{max} > \phi_E > \phi_{TE} > 0$.

(i) *If* $\phi_E \le \phi < \phi_{max}$, *there is a unique cashless steady state with* $x = 0$ *similar to the one described by Proposition 1.*

(ii) *If $\phi_{TE} \leq \phi < \phi_E$, there is a unique cashless steady state with $x > 0$, where r and k are increasing in ϕ, and x is decreasing in ϕ.*

(iii) *If $0 \leq \phi < \phi_{TE}$, there is a unique liquidity-trap steady state with $r = 1/\theta < 1$ and $x > 0$, where x is invariant in ϕ, m^S/y is is decreasing in ϕ, and k is increasing in ϕ.*

(iv) *The shadow rate r^S is increasing in ϕ. When ϕ goes to 0, the shadow rate goes to a lower bound $\sigma\Psi'(1)$ corresponding to $x = 1$.*

Proof. We start by deriving Equations (47) to (50). The optimization problem of investors is the same as in the benchmark model, with the total return ρ^S replacing the interest rate r. With log utility, investors still choose to save a fraction β of their wealth. The demand for bonds and money by saving investors is a fraction $(1 - x)$ of their saving $\beta(1 - \phi)\alpha y$. In the cashless steady state, it has to be equal to the supply of bonds by investors $\phi\alpha y/r$. Using the first-order condition with respect to x, $r = \sigma\Psi'(x)$, we get (47). In the liquidity trap steady state, the demand for bond and money has to be equal to the supply of bonds $\theta\alpha y$ plus real money holdings θm^S, which gives (49).

To get the aggregate budget constraint of investors, note that their aggregate wealth is equal to $\alpha y + \sigma\Psi(x)\beta(1 - \phi)\alpha y + m^S$. The first term is profits from the efficient sector, the second term is the return of the inefficient storage technology, and the last term is money holdings. The save a fraction β of this wealth to buy capital k, invest $x\beta(1 - \phi)\alpha y$ in the storage technology, and acquire money θm^S. This gives Equations (48) and (50).

The storage technology is not used as long as the first-order condition with respect to x is a corner solution: $-r + \sigma\Psi'(0) \leq 0$. Then, we are in the cashless steady state of the benchmark model with $r = \phi/[\beta(1 - \phi)]$. With $\Psi'(0) = 1$, the first-order condition becomes $\phi \geq \phi_E$. This proves Point (i). The comparative statics of Point (ii) directly derive from Equations (47) and (48), together with the first-order condition $r = \sigma\Psi'(x)$. Note in particular that $x - \beta\sigma\Psi(x)$ on the right-hand side of (48) is strictly increasing in x. Indeed, its derivative is given by $1 - \beta\sigma\Psi'(x) > 1 - \beta\sigma > 0$ since $\Psi'(x) < \Psi'(0) = 1$.

When the inefficient technology is in use, the shadow rate is the one that solves

$$\beta r^S(1 - \phi)\big(1 - \Psi'^{-1}(r^S/\sigma)\big) = \phi$$

where we have substituted the first-order condition with respect to x in (47). It is decreasing in ϕ. For $\phi = 0$, we have $x = 1$ from (47) and the shadow rate is then $r^S = \sigma\Psi'(1)$, which proves Point (iv). The steady state is cashless as long as $r^S\theta > 1$. This obtains for $\phi > \phi_{TE}$, which ends proving Point (ii).

The comparative statics of Point (iii) are straightforward given Equations (49) and (50) when $x = \Psi'^{-1}(1/\theta\sigma)$. ■

3.6 Idiosyncratic Uncertainty

In this Appendix we examine a stochastic transition between saving and investing phases. We assume the following 2-state Markov process for individual investors:

- an investor with no investment opportunity at time $t-1$ receives an investment opportunity at time t with probability $\omega \in (0, 1]$,
- an investor with an investment opportunity at time $t-1$ receives no investment opportunity at time t.

While investors face some risk at the individual level, they do not face risk at the aggregate level, as the fraction of investors with investment opportunity is always ω.

A modified aggregate Euler equation of savers Consider an investor i, and denote Ω_t^i her wealth at the beginning of period t. With log utility, her consumption c_t^i is a fraction $1-\beta$ of wealth Ω_t^i, and the Euler equation of an (unconstrained) saver is $1/c_t^i = \beta r_{t+1} \mathrm{E}_t[1/c_{t+1}^i]$, which implies $1/\Omega_t^i = \beta r_{t+1} \mathrm{E}_t[1/\Omega_{t+1}^i]$. For an investor in her saving phase in period t, wealth in period $t+1$ is given by $\Omega_{t+1}^i = a_{t+1}^i + M_{t+1}^i/P_{t+1}$. As there is no aggregate risk, P_{t+1} is known in t, so Ω_{t+1}^i is known in t and we have $\beta\Omega_t^i = \Omega_{t+1}^i/r_{t+1}$. Aggregating over saving investors, we get

$$\beta \int S_t(i)\Omega_t^i di = \frac{1}{r_{t+1}} \int S_t(i)[a_{t+1}^i + M_{t+1}^i/P_t]di = \frac{1}{r_{t+1}} \left(a_{t+1} + \frac{M_{t+1}^S}{P_{t+1}} \right) \tag{51}$$

where $S_t(i)$ is an indicator equal to 1 if investor i has no investment opportunity at time 1 and 0 if she has, and a and M^S denote aggregate bond and money holdings by savers, as in the benchmark model. To compute the left-hand side of (51), note that investors in their saving phase at time t are made of a fraction $1-\omega$ of investors in their saving phase at time $t-1$ and all investors in their investment phase at time $t-1$. The latter enter period t with wealth $\Omega_t^i = \rho_t k_t^i - b_t^i$. This implies:

$$\int S_t(i)\Omega_t^i di = (1-\omega)\int S_{t-1}(i)\Omega_t^i di + \int [1 - S_{t-1}(i)]\Omega_t^i di$$
$$= (1-\omega)\left(a_t + \frac{M_t^S}{P_t}\right) + \rho_t k_t - b_t,$$

where k and b are aggregate capital and aggregate debt of borrowers. As long as $\rho_t > r_t$, which will be the case in equilibrium, investors with an investment opportunity will leverage up as much as possible until they hit their borrowing constraint. Thus, we have $b_t^i = \phi_{t-1}\rho_t k_t^i$, which aggregates to $b_t = \phi_{t-1}\rho_t k_t = \phi_{t-1}\alpha y_t$. Substituting these expressions back into Equation (51), and using the market-clearing condition (10), we find:

$$\beta(1-\omega)\left[(\phi_{t-1}\alpha + \bar{l}_{t-1})y_t + \frac{M_t^S}{P_t}\right] + \beta\alpha(1-\phi_{t-1})y_t = \frac{1}{r_{t+1}}\left[(\phi_t\alpha + \bar{l}_t)y_{t+1} + \frac{M_{t+1}^S}{P_{t+1}}\right]. \quad (52)$$

This equation extends Equation (14) from the benchmark model to the case of idiosyncratic uncertainty. It only differs by the first term on the left hand side. This term represents demand for saving instruments at time t from savers that were already savers at time $t-1$. The lower ω, the larger the share of savers, the higher this additional demand for saving instruments compared to the benchmark model. The term vanishes when $\omega = 1$.

This is the only difference between the extended model and the benchmark. Indeed, we can aggregate the budget constraints of all investors, regardless of whether they save or borrow, to get the same aggregate budget constraint (15) as in the benchmark model.

Steady state with autarkic equilibrium This extended model behaves quite similarly to the benchmark model. Consider for example the case of autarkic investors ($\bar{l} = 0$) treated in Proposition 1 for the benchmark model. In the extended model, the steady state is determined by:

$$\beta(1-\omega)(\phi\alpha y + m^S) + \beta\alpha(1-\phi)y = \frac{1}{r}(\phi\alpha y + m^S),$$
$$k + (\theta - \beta)m^S = \beta\alpha y.$$

When $\beta/(\theta+\omega\beta) \leq \phi < 1/(1+\omega)$, the steady state is cashless with $m^S = 0$, a constant capital stock $k = (\beta\alpha)^{1/(1-\alpha)}$ as in the benchmark model, and

$$r = \frac{\phi}{\beta(1-\omega\phi)}.$$

A lower ω is associated with a lower interest rate: because there are more savers, channeling saving to investment is more difficult and requires a lower interest rate compared to the benchmark model. The interest rate is still strictly increasing in ϕ, but $dr/d\phi$ is increasing in ω: with a larger share of savers (i.e. a lower ω), the interest rate is lower but less responsive to ϕ. Note also that the upper bound on ϕ in the cashless equilibrium is larger than $\phi_{\max} = 1/2$: it is easier to have binding borrowing constraints when there are more savers. Likewise, the lower bound is larger than ϕ_T: it is easier to be in the liquidity trap equilibrium when there are more savers.

When $0 < \phi < \beta/(\theta+\omega\beta)$, the steady state is a liquidity trap with $r = 1/\theta$, and

$$k^{1-\alpha} = \alpha\,\frac{\omega\beta^2 + \phi\big(\theta^2 - \beta[\omega\beta + (1-\omega)\theta]\big)}{\theta - (1-\omega)\beta},$$
$$m^S = \alpha\left[\frac{(1-\omega\phi)\beta - \phi\theta}{\theta - (1-\omega)\beta}\right] y.$$

A lower ω, that is, a higher share of savers, leads to a stronger demand for money m^S/y and a lower stock of capital k. In the liquidity trap, we get the unusual result that more saving actually leads to less investment. As in Proposition 1, k is strictly increasing in ϕ, and m^S/y is strictly decreasing in ϕ. In addition, $dk/d\phi$ is decreasing in ω and $d(m^S/y)/d\phi$ is increasing in ω. A larger share of savers (i.e. a lower ω) implies steeper slopes of k and m^S/y with respect to ϕ.

Overall, the results we have in the benchmark model become stronger when investment opportunity arrive randomly to saving investors instead of in deterministic way.

3.7 Nominal Government Bonds

We have assumed so far that government bonds were issued in real terms. In reality though, a large share of government bonds are nominal. In our deterministic setting, assuming that bonds are nominal instead of real is innocuous, but the capacity of producing real saving instruments by issuing nominal bonds is somehow hampered in the liquidity trap as nominal bonds generate inflation.

In the cashless case, prices are determined by the stock of money through a classical quantity equation. Indeed, the money equilibrium (16) becomes $P = \theta M/(1-\alpha)y$ when $M^S = 0$. Therefore, for a given level of money M, the amount L^g of outstanding nominal bonds has no effect on the price level and directly determines the amount of real debt $l^g = L^g/P$. A 1% increase in nominal debt then translates into a 1% increase in real debt.

This is no longer true in the liquidity trap, as prices are now determined by the total stock of government nominal liabilities. From Equation (27), we now have $P = \theta(M+L^g)/[(1-\alpha)y+\theta(s-\bar{l}^w)]$. Because the market for money merges with the market for bonds, L^g is inflationary, just like money. However, the increase in prices following an increase in nominal debt, for M constant, is less than proportional, so the increase in nominal debt is not fully offset by the increase in prices. A 1% increase in nominal debt then does translate into an increase of real debt, though by less than 1%. The assumption of real bonds is therefore without loss of generality.

3.8 Sticky Wages, Employment, and Output

3.8.1 Labor market with Calvo wage-setting

This section extends the model to a New Keynesian framework with sticky wages and endogenous labor supply. Workers supply labor h_t to a unit measure of employment agencies which produce differentiated labor and engage in monopolistic competition. Agency i transforms $h_{i,t}$ units of homogenous labor into $H_{i,t}$ units of variety i with nominal wage $W_{i,t}$. Employment agencies are owned by workers and transfer their profits to them period by period. A competitive sector then aggregates those differentiated varieties of labor into composite labor H_t with production function $H_t = \left[\int H_{i,t}^{\frac{\epsilon-1}{\epsilon}} di\right]^{\frac{\epsilon}{\epsilon-1}}$. The corresponding aggregate nominal wage is $W_t = \left[\int W_{i,t}^{1-\epsilon} di\right]^{\frac{1}{1-\epsilon}}$. Firms then hire this composite labor to produce $y_t = F(k_t, H_t)$.

Workers

The representative worker has a utility function $U_t^w = \mathrm{E}_t \sum_{s=0}^{\infty} \beta^s u(c_{t+s}^w, h_{t+s})$ and is subject to the budget constraint

$$c_t^w + \frac{M_{t+1}^w}{P_t} + l_t^w = w_t h_t + D_t + \frac{M_t^w}{P_t} + \frac{T_t^w}{P_t} + \frac{l_{t+1}^w}{r_{t+1}}, \tag{53}$$

where w_t is the real wage paid to workers by employment agencies. Compared to Equation (4), this budget constraint adds dividends D_t paid by employment agencies (which can be negative) as a source of income. Workers are also subject to the borrowing constraint (6) and the cash-in-advance constraint.

Denote respectively λ_t, μ_t, and γ_t the Lagrange multipliers on the budget constraint, the cash-in-advance constraint, and the borrowing constraint. Maximization implies the following first order conditions:

$$\begin{aligned}
u_c'(c_t^w, h_t^w) &= \lambda_t + \mu_t, \\
-u_h'(c_t^w, h_t) &= w_t \lambda_t, \\
\lambda_t &= \beta \frac{P_t}{P_{t+1}} (\lambda_{t+1} + \mu_{t+1}), \\
\gamma_t &= \lambda_t + \mu_t - \beta r_{t+1} (\lambda_{t+1} + \mu_{t+1}).
\end{aligned}$$

We can show that $\mu_t = \gamma_t + (i_{t+1} - 1)\lambda_t$. This implies that the cash-in-advance constraint is always binding, even at the zero lower bound, as long as the borrowing constraint is binding, which we assume throughout. The optimal decision by the worker is then given by:

$$c_t^w = \frac{M_t^w + T_t^w}{P_t} + l_{t+1}^w - r_t l_t^w \tag{54}$$

$$-u_h'(c_t^w, h_t) = w_t \lambda_t \tag{55}$$

$$l_{t+1}^w = r_{t+1} l_{t+1}^w, \tag{56}$$

with

$$\lambda_t = \beta \frac{P_t}{P_{t+1}} u_c'(c_{t+1}^w, h_{t+1}).$$

Employment agencies

Employment agencies are subject to a nominal friction, namely Calvo wage-setting. In each period, agencies can only reoptimize their wage $W_{i,t}$ with probability $1-\eta$. With probability η, agencies simply adjust their nominal wage of the preceding period by the gross rate of steady state inflation θ: $W_{i,t} = \theta W_{i,t-1}$. We make the usual assumption that agencies receive a subsidy τ per unit of output, financed out of their profits by a lump sum tax. Later, we will set the subsidy to $\tau = (\epsilon - 1)^{-1}$ to offset the distortion from monopolistic competition in the steady state. This assumption, together with wage indexation, makes sure that the steady state of this model is identical to a flexible wage economy without monopolistic employment agencies.

Consider agency i. It faces a demand for its differentiated labor $H_{i,t} = \left(\frac{W_{i,t}}{W_t}\right)^{-\epsilon} H_t$. With probability η, it cannot reoptimize its wage $W_{i,t}$. Then, $W_{i,t} = \theta W_{i,t-1}$ and the associated value function is given by

$$V_{i,t}^S(W_{i,t}) = \lambda_t \Big((1+\tau) \frac{W_{i,t}}{P_t} - w_t \Big) H_{i,t} + \beta \lambda_{t+1} \Big(\eta V_{i,t}^S(\theta W_{i,t}) + (1-\eta) V_{i,t+1}^R \Big)$$

where V^R is the value of reoptimizing the nominal wage. It is given by:

$$V_{i,t}^R = \max_{W_i} \lambda_t \Big((1+\tau) \frac{W_i}{P_t} - w_t \Big) H_{i,t} + \beta \lambda_{t+1} \Big(\eta V_{i,t}^S(\theta W_i) + (1-\eta) V_{i,t+1}^R \Big),$$

where the maximization is subject to the demand for labor $H_{i,t}$. The optimal reset wage W_t^* is given by:

$$W_t^* = \frac{\Gamma_t^1}{\Gamma_t^0} P_t \tag{57}$$

where

$$\Gamma_t^0 = \lambda_t H_t + \beta\eta \left(\frac{W_{t+1}}{\theta W_t}\right)^{\epsilon} \frac{\theta P_t}{P_{t+1}} \Gamma_{t+1}^0,$$
$$\Gamma_t^1 = \lambda_t H_t w_t + \beta\eta \left(\frac{W_{t+1}}{\theta W_t}\right)^{\epsilon} \Gamma_{t+1}^1,$$

where we have set the subsidy to $\tau = (\epsilon - 1)^{-1}$.

Aggregating across agencies, we get the evolution of the nominal wage of composite labor W_t:

$$W_t = \left[\eta(\theta W_{t-1})^{1-\epsilon} + (1-\eta)(W_t^*)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}. \tag{58}$$

The aggregate demand for homogeneous labor by employment agencies is $\int H_{i,t} di = \Delta_t H_t$ with $\Delta_t = \int \left(\frac{W_{i,t}}{W_t}\right)^{-\epsilon} di$ capturing wage dispersion across agencies. Market clearing on the labor market yields:

$$\Delta_t H_t = h_t. \tag{59}$$

The evolution of price dispersion is given by

$$\Delta_t = \eta \Big(\frac{W_t}{\theta W_{t-1}}\Big)^{\epsilon} \Delta_{t-1} + (1-\eta) \Big(\frac{W_t^*}{W_t}\Big)^{-\epsilon}. \tag{60}$$

Finally, the aggregate dividends received by workers are given by $D_t = W_t H_t / P_t - w_t h_t$.

To close the model, the return paid by firms to investors is now $\rho_t = \alpha k_t^{\alpha-1} H_t^{1-\alpha} + (1-\delta)$ and the real wage paid by firms to employment agencies is given by

$$\frac{W_t}{P_t} = (1-\alpha)\left(\frac{k_t}{H_t}\right)^{\alpha}. \tag{61}$$

Cashless dynamics In cashless equilibria, the model can be log-linearized to get a standard New Keneysian framework. For simplicity, consider the case of autarkic investors. Denoting $\tilde{x}_t$ the log-deviation from an initial steady state associated with $\phi_t = \phi_0$, and choosing separable preferences for workers $u(c,h) = \log c - h^{1+\sigma^{-1}}/(1+\sigma^{-1})$, we get:

$$\tilde{y}_t = \tilde{y}_{t+1} - \Big[\tilde{i}_{t+1} - \tilde{\pi}_{t+1} - \underbrace{\Big(\frac{\phi_0}{1-\phi_0}\tilde{\phi}_{t-1} + \tilde{\phi}_t\Big)}_{\text{natural rate}}\Big], \tag{62a}$$

$$\tilde{\pi}^w_t = \beta\tilde{\pi}^w_{t+1} + \frac{(1-\eta)(1-\beta\eta)}{\eta}\Big[\tilde{\pi}_{t+1} + \tilde{y}_{t+1} + \frac{\sigma^{-1}}{1-\alpha}(\tilde{y}_t - \alpha\tilde{k}_t) - \frac{\alpha}{1-\alpha}(\tilde{k}_t - \tilde{y}_t)\Big], \tag{62b}$$

$$\tilde{\pi}^w_t = \tilde{\pi}_t + \frac{\alpha}{1-\alpha}[(\tilde{k}_t - \tilde{y}_t) - (\tilde{k}_{t-1} - \tilde{y}_{t-1})] \tag{62c}$$

$$\tilde{k}_{t+1} = \tilde{y}_t \tag{62d}$$

$$\tilde{\pi}_t = -(\tilde{y}_t - \tilde{y}_{t-1}), \tag{62e}$$

where $\tilde{\pi}$ is inflation, and $\tilde{\pi}^w$ nominal wage inflation. In the cashless equilibrium, the model behaves similarly to a conventional new-keynesian model: (62a), the log-linearized version of the Euler equation, is the standard new-keynesian IS curve, where the deleveraging shock shows up as a natural rate shock; (62b) is the new-keynesian Phillips curve for wages; (62c) is the relation between wage and price inflation; (62d) is the log-linear version of (15) with $m^S = 0$; and (62e) is implied by the constant money growth rate.

Replacing our assumption of a constant money growth rule by a more usual Taylor rule is straightforward. For example, it could be replaced by

$$\tilde{i}_{t+1} = \tilde{\phi}_t + \frac{\phi_0}{1-\phi_0}\tilde{\phi}_{t+1} + \psi\tilde{\pi}_t \tag{63}$$

where the first two terms represent the log-deviation of the natural rate from the initial steady state.

3.8.2 The augmented IS curve

We derive our augmented IS curve by substituting the money market equilibrium (16) into the Euler equation of savers (14):

$$\underbrace{\beta\alpha(1-\phi_{t-1})P_tY_t}_{\substack{\text{nominal demand}\\ \text{for assets by savers}}} + \underbrace{(1-\alpha)P_tY_t}_{\substack{\text{money demand}\\ \text{by workers}}} = \underbrace{(\phi_t\alpha + \bar{l}_t)\frac{P_{t+1}Y_{t+1}}{i_{t+1}}}_{\text{nominal supply of bonds}} + \underbrace{M_{t+1}}_{\substack{\text{money}\\ \text{supply}}}. \tag{64}$$

This relationship equates the total demand for assets to the total supply of assets in the economy. When $i > 0$, money and bond markets operate independently. Indeed, (16) becomes a quantity equation $M_{t+1} = (1-\alpha)P_t y_t$. The terms related to money demand and supply then drop from Equation (64), which simply becomes an equality between the demand for bonds by savers and the supply of bonds, i.e. an IS curve. Using the notation $\tilde{x}_t$ for log-deviations, it can be rewritten as the familiar IS curve of the New-Keynesian model, with the deleveraging shocks showing up as a natural rate shock:

$$\tilde{y}_t = \tilde{y}_{t+1} - \left[\tilde{\imath}_{t+1} - \tilde{\pi}_{t+1} - \underbrace{\left(\frac{\phi_0}{1-\phi_0}\tilde{\phi}_{t-1} + \frac{\alpha\phi_0}{\alpha\phi_0 + \bar{l}_0}\tilde{\phi}_t + \frac{\bar{l}_0}{\alpha\phi_0 + \bar{l}_0}\tilde{l}_t\right)}_{\text{natural rate}}\right],$$

where $\tilde{\pi}$ is log-linearized inflation and the subscript 0 in $\bar{l}_0$ and ϕ_0 refers to values in the initial steady state. As in the New Keynesian model, monetary policy is able to fully offset deleveraging shocks as long as the economy does not hit the ZLB.

When $i_{t+1} = 1$, the simple quantity equation of the cashless dynamics ceases to hold, bonds and money are perfect substitutes and their corresponding markets merge. As in the standard New Keynesian model at the ZLB, a deleveraging shock (a negative shock to the natural rate) has to be accommodated by a drop in nominal output, which decreases the demand for assets. However, there is a noticeable difference: here, money supply M_{t+1} appears explicitly as a component of the asset supply. An increase in money supply M_{t+1} can therefore make the adjustment much easier.

To see this, iterate Equation (64) forward:

$$[(\beta\alpha(1-\phi_{t-1}) + 1 - \alpha)]\, P_t Y_t = \sum_{s=0}^{\infty}\left[\prod_{j=0}^{s-1}\frac{\theta(\alpha\phi_{t+j} + \bar{l}_{t+j})}{\beta\alpha(1-\phi_{t+j}) + 1 - \alpha}\right] M_{t+1}. \tag{65}$$

With constant values of ϕ and $\bar{l}$, the ratio of money supply to nominal output quickly converges (after one period) to $M_{t+1}/(P_t Y_t) = [\beta\alpha(1-\phi) - \theta(\alpha\phi + \bar{l}) + 1 - \alpha]$. The apparent velocity of money decreases after a deleveraging shock on ϕ or $\bar{l}$. If prices are very sticky and the stock of money does not change, then a deleveraging shock is fully absorbed by a drop in real output, until prices have adjusted enough to provide the desired real money holdings. However, an appropriate increase in money supply can potentially offset this transitory effect of the shock.

Table 1: Calibration

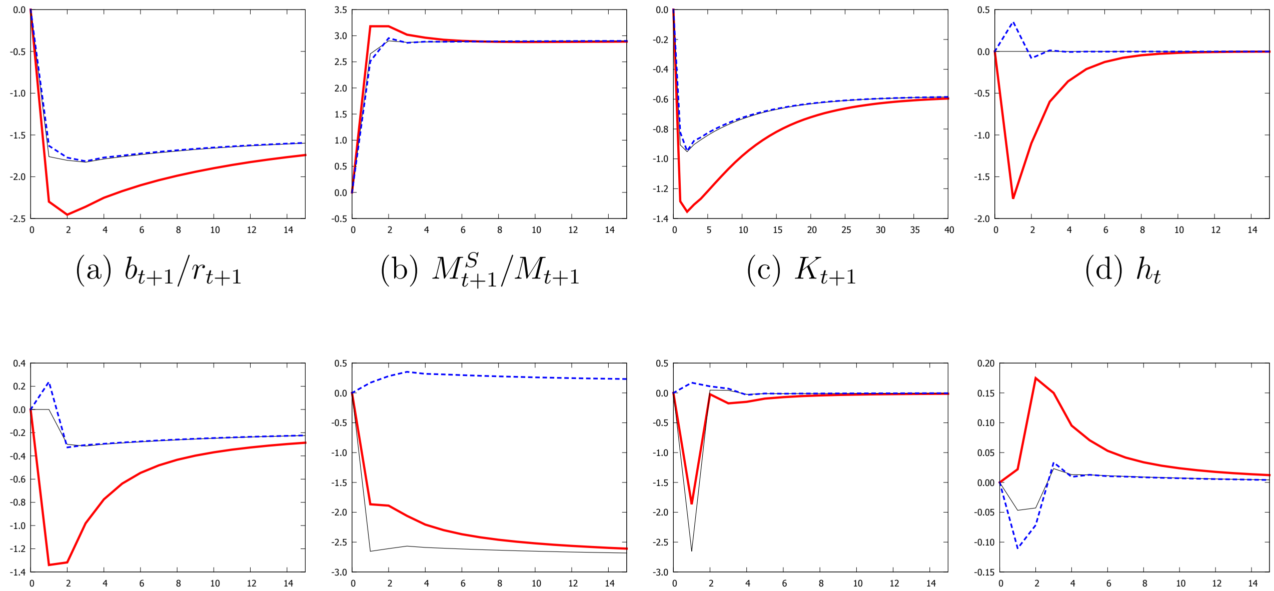
Parameter	Value
β	0.94
δ	0.10
α	0.33
η	0.75
ϵ	13
σ	1

Calibration The model is calibrated as described in Table 1. We define a period to be a year. All parameters are standard, except the discount rate β. Indeed, in the cashless autarkic steady state of this model, the inverse of the discount rate is equal to the rate of return on capital, not to the interest rate r. We set β to match a rate of return of 6 percent per year annually.

3.8.3 Adjustment at the ZLB

Figure 3 represents the response of the sticky wage model (with partial depreciation of capital $\delta < 1$) to a deleveraging shock on investors in the liquidity trap, in the case of autarkic investors. Here we assume a constant money supply, with $\theta = 1$. In period 0, the economy is in an initial steady state with $\phi = \phi_T$, that is, at the limit of the ZLB. In period 1, ϕ decreases permanently and unexpectedly by 1 percent. Outside of the ZLB, real variables would not react at all to such a shock, which would be entirely accommodated by a drop in the nominal interest rate i. The adjustment process is completely different in the liquidity trap.

Consider first the case of flexible wages, represented by the thin black line. When the shock hits, S-investors start demanding money (panel b) and the equilibrium adjustment comes from a drop in the price level (panel f). This works through a Pigou-Patinkin effect: the lower price level increases workers' real money holdings and makes them consume more, exchanging some of their money against goods to S-investors who want to hoard cash. This flexible price dynamics is different from models with a representative agent such as Krugman (1998) or models of moneyless economies such as Eggertsson and Krugman (2012): in these works, there is no Pigou effect and the price level has to decrease enough to generate inflation expectations that



(a) b_{t+1}/r_{t+1} (b) M^S_{t+1}/M_{t+1} (c) K_{t+1} (d) h_t

(e) y_t (f) P_t (g) P_{t+1}/P_t (h) r_{t+1}

Figure 3: Transitory dynamics after a permanent tightening of the borrowing constraint at the ZLB. Thick red line: sticky wages. Thin black line: flexible wages. Dashed blue line: sticky wages with a permanent monetary expansion when the shock hits. All variables are relative deviation from initial steady state, in percent, except rates of return, inflation and M^s/M which are in absolute deviation from initial steady state, in percent.

overcome the ZLB. By contrast, with heterogeneous agents and an explicitly modeled money demand, the price decrease has distributional effects which dampens the effect of the shock.

Consider now the case of sticky wages, represented by the thick red line in Figure 3. Because of staggered contracts, the nominal wage, and therefore the price level P_t, can only adjust gradually after the shock hits (panel f), triggering a long lasting deflationary process which raises the interest rate (panel g). Absent the drop in the price level, adjustment comes instead from lower employment and a lower output (panels d and e): total output falls because worker consumption cannot offset the fall in investment. With a lower production, capital accumulation drops sharply at impact (panel c). The demand for loans by I-investors is negatively affected by expected deflation, which raises the real interest rate, and by the lower expected return on capital (panel a). With a lower supply of saving instruments by I-investors, S-investors increase their demand for money even more than with flexible prices. As time goes by and prices gradually adjust, employment and output increase back to their flexible-price level and the economy converges to the liquidity trap steady state.

Therefore, with sticky wages, a deleveraging shock large enough to move the economy to the ZLB, creates a negative output gap in the short run, as in the existing New Keynesian literature. Contrary to that literature, the economy stays at the ZLB with a lower capital stock and lower level of output, even after wages have adjusted and the output gap has closed.

3.8.4 Alternative monetary policy

Consider now a monetary expansion taking the form of transfers to workers. In the simulation represented by the dashed blue line, the government increases M once and for all when the shock hits. The increase is calibrated so that the nominal wage converges back to its initial value in the new steady state. As the figure shows, the resulting dynamics of real variables is very close to the dynamics with flexible wages. By increasing money supply, monetary policy substitutes to the fall in the price level that would obtain with flexible prices. Workers receiving monetary transfers feel richer exactly as they would with a lower price level. As a result they increase their consumption and sustain a higher level of output.

This result stands in sharp contrast to existing work, for instance Krugman (1998), where money creation taking the form of transfers has no effect at the ZLB with pre-set prices (see footnote 11 of this work). It comes from the non-ricardian structure of the model, which gives rise to the Pigou-Patinkin effect described above. However, if a policy of monetary transfers can be very effective in closing the output gap in the short-run of this model, it has no effect in the long run and therefore cannot prevent the long term output losses.

Because the composition of government liabilities at the ZLB does not matter, an increase in transfers to households financed by government debt would have the same effect as a monetary expansion.

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