Technical Appendix

Gradual Portfolio Adjustment: Implications for Global Equity Portfolios and Returns

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I Computation of Optimal Portfolio

We will focus on the optimal portfolio of an agent in the Home country. As discussed in the text, with a constant fraction ζ of wealth consumed each period, the agent maximizes

$$\sum_{s=1}^{\infty} \beta^s E_t \frac{W_{H,t+s}^{1-\gamma}}{1-\gamma} \tag{1}$$

subject to

$$W_{H,t+1} = (1-\zeta)\left(R_{t+1}^{pH} + T_{H,t+1}\right) W_t + G_{H,t+1} \tag{2}$$

where

$$R_{t+1}^{pH} = z_{Ht} R_{H,t+1} + (1-z_{Ht}) R_{F,t+1} e^{-\tau_{Ht}} \tag{3}$$

and

$$\hat{R}_{t+1}^{pH} \equiv R_{t+1}^{pH} + T_{H,t+1} = z_{Ht} R_{H,t+1} + (1-z_{Ht}) R_{F,t+1} \tag{4}$$

Our objective is to compute the optimal portfolio at time t if this agent is picked at time t to choose a new portfolio. We will write this optimal portfolio as $\tilde{z}_{Ht}$. The probability that the agent chooses a new portfolio again at time $t+i$ with $i > 0$ is $p_i = p(1-p)^{i-1}$. We can then write

$$E_t W_{H,t+s}^{1-\gamma} = \sum_{i=1}^{s-1} p_i E_t W_{H,t+s}(i)^{1-\gamma} + \left(1 - \sum_{m=1}^{s-1} p_m\right) E_t \hat{W}_{H,t+s}^{1-\gamma} \tag{5}$$

Here $W_{H,t+s}(i)$ denotes wealth at $t+s$ conditional on the next portfolio change taking place at $t+i < t+s$. This means that the portfolio share $\tilde{z}_{Ht}$ is held constant until $t+i$. $\hat{W}_{H,t+s}$ denotes wealth at $t+s$ conditional on the next portfolio change taking place at $t+s$ or later. In that case the portfolio share $\tilde{z}_t$ remains constant until at least $t+s$.

The first-order condition for the optimal portfolio $\tilde{z}_{Ht}$ is then

$$\sum_{s=1}^{\infty}\sum_{i=1}^{s-1} p_i \beta^s E_t W_{H,t+s}(i)^{-\gamma} \frac{\partial W_{H,t+s}(i)}{\partial \tilde{z}_{Ht}} +$$
$$\sum_{s=1}^{\infty}\left(1-\sum_{m=1}^{s-1} p_m\right)\beta^s E_t \hat{W}_{H,t+s}^{-\gamma}\frac{\partial \hat{W}_{H,t+s}}{\partial \tilde{z}_{Ht}} = 0 \tag{6}$$

We have

$$\frac{\partial W_{H,t+s}(i)}{\partial \tilde{z}_{Ht}} = \frac{\partial W_{H,t+i}}{\partial \tilde{z}_{Ht}}\frac{\partial W_{H,t+s}}{\partial W_{H,t+i}} \tag{7}$$

where

$$\frac{\partial W_{H,t+s}}{\partial W_{H,t+i}} = (1-\zeta)^{s-i}\hat{R}^{pH}_{t+i,t+s} \tag{8}$$
$$\frac{\partial W_{H,t+i}}{\partial \tilde{z}_{Ht}} = \sum_{j=1}^{i}(1-\zeta)^{i-j+1}(R_{H,t+j}-R_{F,t+j}e^{-\tau_{Ht}})\hat{R}^{pH}_{t+j,t+i}W_{H,t+j-1} \tag{9}$$

Here $\hat{R}^{pH}_{t+i,t+s} = \prod_{j=i+1}^{s}\hat{R}^{pH}_{t+j}$ is the cumulative portfolio return from $t+i$ to $t+s$. $\partial \hat{W}_{t+s}/\partial \tilde{z}_{Ht}$ is equal to $\partial W_{t+i}/\partial \tilde{z}_{Ht}$ for $i=s$. We have assumed that the cost τ_{Ht} will remain constant during the duration that the investor holds the portfolio $\tilde{z}_t$.

Using this, the first order condition with respect to $\tilde{z}_{Ht}$ is now

$$\sum_{s=1}^{\infty}\sum_{i=1}^{s-1}\sum_{j=1}^{i} p_i\beta^s(1-\zeta)^{s-j+1}E_t\left(R_{H,t+j}-R_{F,t+j}e^{-\tau_{Ht}}\right)\hat{R}^{pH}_{t+j,t+i}\hat{R}^{pH}_{t+i,t+s}W_{H,t+j-1}W_{H,t+s}(i)^{-\gamma} +$$
$$\sum_{s=1}^{\infty}\sum_{j=1}^{s}\beta^s(1-\zeta)^{s-j+1}\left(1-\sum_{m=1}^{s-1}p_m\right)E_t\left(R_{H,t+j}-R_{F,t+j}e^{-\tau_{Ht}}\right)\hat{R}^{pH}_{t+j,t+s}W_{H,t+j-1}\hat{W}_{H,t+s}^{-\gamma} = 0$$

Now write the first order condition in terms of exponentials of logs, with lower case letters denoting logs in deviation from steady state. Using that $(1-\zeta)\bar{R} = \theta$, we have

$$\sum_{s=1}^{\infty}\sum_{i=1}^{s-1}\sum_{j=1}^{i}(\beta\theta)^s p_i\theta^{1-j}E_t e^{-\gamma w_{H,t+s}(i)+w_{H,t+j-1}+\hat{r}^{pH}_{t+j,t+i}+\hat{r}^{pH}_{t+i,t+s}+r_{H,t+j}} -$$
$$\sum_{s=1}^{\infty}\sum_{i=1}^{s-1}\sum_{j=1}^{i}(\beta\theta)^s p_i\theta^{1-j}E_t e^{-\gamma w_{H,t+s}(i)+w_{H,t+j-1}+\hat{r}^{pH}_{t+j,t+i}+\hat{r}^{pH}_{t+i,t+s}+r_{F,t+j}-\tau_{Ht}} +$$
$$\sum_{s=1}^{\infty}\sum_{j=1}^{s}(\beta\theta)^s\theta^{1-j}\left(1-\sum_{m=1}^{s-1}p_m\right)E_t e^{-\gamma\hat{w}_{t+s}+w_{H,t+j-1}+\hat{r}^{pH}_{t+j,t+s}+r_{H,t+j}} -$$
$$\sum_{s=1}^{\infty}\sum_{j=1}^{s}(\beta\theta)^s\theta^{1-j}\left(1-\sum_{m=1}^{s-1}p_m\right)E_t e^{-\gamma\hat{w}_{t+s}+w_{H,t+j-1}+\hat{r}^{pH}_{t+j,t+s}+r_{F,t+j}-\tau_{Ht}} = 0$$

Next replace $r_{H,t+j}$ with $0.5er_{t+j} + r^A_{t+j}$ and $r_{F,t+j}$ with $-0.5er_{t+j} + r^A_{t+j}$, where $r^A_{t+j} = (r_{H,t+j} + r_{F,t+j})/2$ is the average log return and $er_{t+j} = r_{H,t+j} - r_{F,t+j}$ is the excess return, and define

$$x_{i,j,s} = -\gamma w_{H,t+s}(i) + w_{H,t+j-1} + \hat{r}^{pH}_{t+j,t+i} + \hat{r}^{pH}_{t+i,t+s} + r^A_{t+j} \tag{10}$$
$$y_{j,s} = -\gamma \hat{w}_{H,t+s} + w_{H,t+j-1} + \hat{r}^{pH}_{t+j,t+s} + r^A_{t+j} \tag{11}$$

Assuming that log wealth and log returns in the exponents are normally distributed, the first order condition can be written as

$$\sum_{s=1}^{\infty}\sum_{i=1}^{s-1}\sum_{j=1}^{i}(\beta\theta)^s p_i \theta^{1-j}\left(e^{A_1} - e^{A_2}\right) +$$
$$\sum_{s=1}^{\infty}\sum_{j=1}^{s}(\beta\theta)^s \left(1 - \sum_{m=1}^{s-1} p_m\right)\theta^{1-j}\left(e^{A_3} - e^{A_4}\right) = 0$$

where

$$A_1 = E_t\left(x_{i,j,s} + 0.5er_{t+j}\right) + 0.5var_t(x_{i,j,s}) + 0.125var_t(er_{t+j}) + 0.5cov(er_{t+j}, x_{i,j,s})$$
$$A_2 = E_t\left(x_{i,j,s} - 0.5er_{t+j}\right) - \tau_{Ht} + 0.5var_t(x_{i,j,s}) + 0.125var_t(er_{t+j}) - 0.5cov(er_{t+j}, x_{i,j,s})$$
$$A_3 = E_t\left(y_{j,s} + 0.5er_{t+j}\right) + 0.5var_t(y_{j,s}) + 0.125var_t(er_{t+j}) + 0.5cov(er_{t+j}, y_{j,s})$$
$$A_4 = E_t\left(y_{j,s} - 0.5er_{t+j}\right) - \tau_{Ht} + 0.5var_t(y_{j,s}) + 0.125var_t(er_{t+j}) - 0.5cov(er_{t+j}, y_{j,s})$$

Using the approximation $e^x = 1 + x$, this becomes

$$\sum_{s=1}^{\infty}\sum_{i=1}^{s-1}\sum_{j=1}^{i}(\beta\theta)^s p_i \theta^{1-j}\left(E_t er_{t+j} + \tau_{Ht} + cov(er_{t+j}, x_{i,j,s})\right) +$$
$$\sum_{s=1}^{\infty}\sum_{j=1}^{s}(\beta\theta)^s \theta^{1-j}\left(1 - \sum_{m=1}^{s-1} p_m\right)\left(E_t er_{t+j} + \tau_{Ht} + cov(er_{t+j}, y_{j,s})\right) = 0 \tag{12}$$

We will now focus on the weighted average covariance

$$\sum_{j=1}^{i}\theta^{1-j} cov(er_{t+j}, x_{i,j,s}) \tag{13}$$

This is equal to

$$-\gamma cov(\tilde{er}_{t,t+i}, w_{H,t+s}(i)) + \sum_{j=1}^{i}\theta^{1-j} cov(er_{t+j}, w_{H,t+j-1}) +$$
$$\sum_{j=1}^{i}\theta^{1-j} cov(er_{t+j}, \hat{r}^{pH}_{t+j,t+i}) + cov(\tilde{er}_{t,t+i}, \hat{r}^{pH}_{t+i,t+s}) \tag{14}$$

where

$$\tilde{er}_{t,t+i} = \sum_{j=1}^{i} \theta^{1-j} er_{t+j} \tag{15}$$

We have used that $cov(er_{t+j}, r^A_{t+j}) = 0.5var(r_{H,t+j}) - 0.5var(r_{F,t+j}) = 0$ because of symmetry.

In order to compute these terms, we use log-linearized expressions for wealth and portfolio returns. Start with $w_{H,t+s}(i)$. When we log-linearize the wealth accumulation equation

$$W_{H,t+1} = (1-\zeta)\hat{R}^{pH}_{t+1} W_t + G_{H,t+1} \tag{16}$$

using $\bar{G}/\bar{W} = 1-\theta$, we get

$$w_{H,t+1} = \theta w_{Ht} + \theta \hat{r}^{pH}_{t+1} + (1-\theta) g_{H,t+1} \tag{17}$$

We can use this to derive

$$w_{H,t+s}(i) = \theta^{s-i} w_{H,t+i} + \theta^{s-i} \hat{\tilde{r}}^{pH}_{t+i,t+s} + \frac{1-\theta}{\theta}\theta^{s-i}\tilde{g}_{H,t+i,t+s} \tag{18}$$

In analogy to the definition of $\tilde{er}_{t,t+i}$, we define

$$\tilde{r}^{pH}_{t+i,t+s} = \sum_{j=1}^{s-i} \theta^{1-j} \hat{r}^{pH}_{t+i+j} \tag{19}$$

$$\tilde{g}_{H,t+i,t+s} = \sum_{j=1}^{s-i} \theta^{1-j} g_{H,t+i+j} \tag{20}$$

We also have

$$w_{H,t+i} = \theta^i w_{Ht} + \theta^i \tilde{r}^{pH}_{t,t+i} + \frac{1-\theta}{\theta}\theta^i \tilde{g}_{H,t,t+i} \tag{21}$$

Using that

$$\tilde{r}^{pH}_{t,t+i} = (\tilde{z}_{Ht} - 0.5)\tilde{er}_{t,t+i} + \tilde{r}^A_{t,t+i} \tag{22}$$

we have

$$\begin{aligned} cov(\tilde{er}_{t,t+i}, w_{H,t+s}(i)) &= \theta^s(\tilde{z}_{Ht} - 0.5)var_t(\tilde{er}_{t,t+i}) + \\ &+\theta^{s-i} cov(\tilde{er}_{t,t+i}, \tilde{r}^{pH}_{t+i,t+s}) + \frac{1-\theta}{\theta}\theta^s cov(\tilde{er}_{t,t+i}, \tilde{g}_{H,t,t+s}) \end{aligned} \tag{23}$$

We have again used that average and excess returns are uncorrelated due to symmetry.

We next need to focus on the term

$$\sum_{j=1}^{i} \theta^{1-j} cov(er_{t+j}, w_{H,t+j-1}) \tag{24}$$

First write an expression for $w_{H,t+j-1}$:

$$w_{H,t+j-1} = \theta^{j-1} w_{Ht} + \theta^{j-1} \tilde{r}^{pH}_{t,t+j-1} + \frac{1-\theta}{\theta} \theta^{j-1} \tilde{g}_{H,t,t+j-1} \tag{25}$$

Ignoring the term in r^A that will drop out of the covariance, this is equal to

$$w_{H,t+j-1} = \theta^{j-1} w_{Ht} + \theta^{j-1} (\tilde{z}_{Ht} - 0.5) \tilde{er}_{t,t+j-1} + \frac{1-\theta}{\theta} \theta^{j-1} \tilde{g}_{H,t,t+j-1} \tag{26}$$

so that

$$\sum_{j=1}^{i} \theta^{1-j} cov(er_{t+j}, w_{H,t+j-1}) =$$

$$(\tilde{z}_{Ht} - 0.5) \sum_{j=1}^{i} cov(\tilde{er}_{t,t+j-1}, er_{t+j}) + \frac{1-\theta}{\theta} \sum_{j=1}^{i} cov(\tilde{g}_{H,t,t+j-1}, er_{t+j}) \tag{27}$$

Next consider the term

$$\sum_{j=1}^{i} \theta^{1-j} cov(er_{t+j}, \hat{r}^{pH}_{t+j,t+i}) \tag{28}$$

This is equal to

$$(\tilde{z}_{Ht} - 0.5) \sum_{j=1}^{i} \theta^{1-j} cov(er_{t+j}, er_{t+j,t+i}) \tag{29}$$

This is equivalent to

$$(\tilde{z}_{Ht} - 0.5) \sum_{j=1}^{i} cov(\tilde{er}_{t,t+j-1}, er_{t+j}) \tag{30}$$

To summarize, we have

$$\sum_{j=1}^{i} \theta^{1-j} cov(er_{t+j}, x_{i,j,s}) = -\gamma\theta^{s} (\tilde{z}_{Ht} - 0.5) var_t(\tilde{er}_{t,t+i}) - \gamma\theta^{s-i} cov(\tilde{er}_{t,t+i}, \tilde{r}^{pH}_{t+i,t+s})$$
$$-\gamma \frac{1-\theta}{\theta} \theta^{s} cov(\tilde{er}_{t,t+i}, \tilde{g}_{H,t,t+s}) + 2(\tilde{z}_{Ht} - 0.5) \sum_{j=1}^{i} cov(\tilde{er}_{t,t+j-1}, er_{t+j})$$
$$+\frac{1-\theta}{\theta} \sum_{j=1}^{i} cov(\tilde{g}_{H,t,t+j-1}, er_{t+j}) + cov(\tilde{er}_{t,t+i}, \hat{r}^{pH}_{t+i,t+s}) \tag{31}$$

Using $i = s$ in the previous equation, we have

$$\sum_{j=1}^{s} \theta^{1-j} cov(er_{t+j}, y_{j,s}) = -\gamma\theta^{s}(\tilde{z}_{Ht} - 0.5) var_t(\tilde{er}_{t,t+s})$$
$$-\gamma\frac{1-\theta}{\theta}\theta^{s} cov(\tilde{er}_{t,t+s}, \tilde{g}_{H,t,t+s}) + 2(\tilde{z}_{Ht} - 0.5)\sum_{j=1}^{s} cov(\tilde{er}_{t,t+j-1}, er_{t+j})$$
$$+\frac{1-\theta}{\theta}\sum_{j=1}^{s} cov(\tilde{g}_{H,t,t+j-1}, er_{t+j}) \qquad (32)$$

The optimal portfolio then becomes

$$\tilde{z}_{Ht} = 0.5 + \frac{\sum_{s=1}^{\infty}(\beta\theta)^{s}\left[\sum_{i=1}^{s-1} p_i E_t\tilde{er}_{t,t+i} + (1 - \sum_{m=1}^{s-1} p_m)E_t\tilde{er}_{t,t+s}\right]}{DEN_t} + h_{Ht}^{in} \qquad (33)$$

Here h_{Ht}^{in} is a hedge portfolio (also depending on the τ_{Ht}), defined below and the denominator DEN_t is equal to

$$DEN_t = \sum_{s=1}^{\infty}\sum_{i=1}^{s-1}(\beta\theta)^{s} p_i \left(\gamma\theta^{s} var_t(\tilde{er}_{t,t+i}) - 2\sum_{j=1}^{i} cov(\tilde{er}_{t,t+j-1}, er_{t+j})\right)$$
$$+\sum_{s=1}^{\infty}(\beta\theta)^{s}\left(1 - \sum_{m=1}^{s-1} p_m\right)\left(\gamma\theta^{s} var_t(\tilde{er}_{t,t+s}) - 2\sum_{j=1}^{s} cov(\tilde{er}_{t,t+j-1}, er_{t+j})\right)$$

The hedge term h_{Ht}^{in} is equal to N_{Ht}/DEN_t, where

$$N_{Ht} = -\sum_{s=1}^{\infty}\sum_{i=1}^{s-1}(\beta\theta)^{s} p_i \gamma\theta^{s-i} cov(\tilde{er}_{t,t+i}, \tilde{r}_{t+i,t+s}^{pH})$$
$$-\sum_{s=1}^{\infty}\sum_{i=1}^{s-1}(\beta\theta)^{s} p_i \gamma\frac{1-\theta}{\theta}\theta^{s} cov(\tilde{er}_{t,t+i}, \tilde{g}_{H,t,t+s})$$
$$+\sum_{s=1}^{\infty}\sum_{i=1}^{s-1}(\beta\theta)^{s} p_i \frac{1-\theta}{\theta}\sum_{j=1}^{i} cov(\tilde{g}_{H,t,t+j-1}, er_{t+j})$$
$$+\sum_{s=1}^{\infty}\sum_{i=1}^{s-1}(\beta\theta)^{s} p_i cov(\tilde{er}_{t,t+i}, \hat{r}_{t+i,t+s}^{pH})$$
$$-\sum_{s=1}^{\infty}(\beta\theta)^{s}\left(1 - \sum_{m=1}^{s-1} p_m\right)\gamma\frac{1-\theta}{\theta}\theta^{s} cov(\tilde{er}_{t,t+s}, \tilde{g}_{H,t,t+s})$$
$$+\sum_{s=1}^{\infty}(\beta\theta)^{s}\left(1 - \sum_{m=1}^{s-1} p_m\right)\frac{1-\theta}{\theta}\sum_{j=1}^{s} cov(\tilde{g}_{H,t,t+j-1}, er_{t+j})$$

$$+\sum_{s=1}^{\infty}\sum_{i=1}^{s-1}\sum_{j=1}^{i}(\beta\theta)^s p_i \theta^{1-j}\tau_{Ht}$$
$$+\sum_{s=1}^{\infty}\sum_{j=1}^{s}(\beta\theta)^s\theta^{1-j}\left(1-\sum_{m=1}^{s-1}p_m\right)\tau_{Ht} \tag{34}$$

Writing out the numerator and denominator of the expected excess return term of (33), we get respectively

$$\frac{\beta\theta}{1-\beta\theta}\sum_{s=1}^{\infty}[\beta(1-p)]^{s-1}E_t er_{t+s}$$
$$\sum_{s=1}^{\infty}[\beta(1-p)]^{s-1}\left[\frac{\gamma\beta\theta^2}{1-\beta\theta^2}var_t(er_{t+s})+2\left(\frac{\gamma\beta\theta^2}{1-\beta\theta^2}-\frac{\beta\theta}{1-\beta\theta}\right)\sum_{i<s}\theta^{s-i}cov_t(er_{t+s},er_{t+i})\right]$$

Dividing by $\beta\theta/(1-\beta\theta)$, we can then write the portfolio as

$$\tilde{z}_{Ht}=0.5+\frac{\sum_{s=1}^{\infty}[\beta(1-p)]^{s-1}E_t er_{t+s}}{V_t}+h^{in}_{Ht} \tag{35}$$

where

$$V_t=\sum_{s=1}^{\infty}[\beta(1-p)]^{s-1}\left[\tilde{\gamma}var_t(er_{t+s})+2(\tilde{\gamma}-1)\sum_{i<s}\theta^{s-i}cov_t(er_{t+s},er_{t+i})\right] \tag{36}$$

and

$$\tilde{\gamma}=\gamma\theta\frac{1-\beta\theta}{1-\beta\theta^2} \tag{37}$$

The optimal portfolio for Foreign investors is the same, with the exception of the hedge term. The hedge term for the Foreign investors is N_{Ft}/DEN_t, where N_{Ft} replaces the superscripts and subscripts H in the expression for N_{Ht} with superscripts and subscripts F. In addition, τ_{Ht} is replaced by $-\tau_{Ft}$.

Only the average hedge term matters for the model. In the average hedge term $h_t^{A,in}=(h^{in}_{Ht}+h^{in}_{Ft})/2$ all terms other than those involving the fees τ_{Ht} and τ_{Ft} drop out. The reason for this is as follows. Consider the first covariance in the expression for N_{Ht}. When we add the corresponding term for Foreign investors, the covariance is

$$cov(\tilde{er}_{t,t+i},\tilde{r}^{pH}_{t+i,t+s}+\tilde{r}^{pH}_{t+i,t+s})=\sum_{j=1}^{s-i}\theta^{1-j}cov(\tilde{er}_{t,t+i},\hat{r}^{pH}_{t+i,t+s}+\hat{r}^{pF}_{t+i,t+s})=2\sum_{j=1}^{s-i}\theta^{1-j}cov(\tilde{er}_{t,t+i},r^{A}_{t+i,t+s})$$

where in the last equality we used that from linearization $\hat{r}^{pH}_{t+i,t+s}=\bar{z}r_{H,t+i,t+s}+(1-\bar{z})r_{F,t+i,t+s}$ and $\hat{r}^{pF}_{t+i,t+s}=(1-\bar{z})r_{H,t+i,t+s}+\bar{z}r_{F,t+i,t+s}$. Since by symmetry the covariance between excess returns and average returns is zero, this expression is zero. The same applies to the other terms in N_{Ht} that involve covariances. For example, the covariance between $\tilde{er}_{t,t+i}$ and $\tilde{g}^A_{t,t+s}$, with the latter equal to $0.5(\tilde{g}_{H,t,t+s} + \tilde{g}_{F,t,t+s})$, is zero for the same reason. Home and Foreign returns have the same covariance with variables that are averages across the two countries.

In the end we therefore have

$$h_t^{A,in} = 0.5\frac{\sum_{s=1}^{\infty}\sum_{i=1}^{s-1}\sum_{j=1}^{i}(\beta\theta)^s p_i \theta^{1-j} + \sum_{s=1}^{\infty}\sum_{j=1}^{s}(\beta\theta)^s\theta^{1-j}\left(1-\sum_{m=1}^{s-1}p_m\right)}{DEN_t}\tau_t^D \tag{38}$$

After some algebra this can be simplified to

$$h_t^{A,in} = 0.5\frac{\theta\beta}{(1-\theta\beta)(1-\beta(1-p))}\frac{1}{DEN_t}\tau_t^D \tag{39}$$

We have

$$DEN_t = \frac{\beta\theta}{1-\beta\theta}V_t \tag{40}$$

so that we can write

$$h_t^{A,in} = \frac{0.5}{V_t(1-\beta(1-p))}\tau_t^D \tag{41}$$

II Solution of the Model

It is useful to repeat the equations of the model:

$$q_t^D = 4z_t^A + (2\bar{z}-1)\tilde{w}_t^D \tag{42}$$

$$\tilde{w}_t^D = \theta\tilde{w}_{t-1}^D + \theta(2\bar{z}-1)er_t + a_t^D \tag{43}$$

$$z_t^A = f\frac{E_t er_{t+1}}{\tilde{\gamma} var_t(er_{t+1})} + (1-f)z_t + n_t \tag{44}$$

$$z_t = (1-p)z_{t-1} + \frac{p}{V_t}\sum_{s=1}^{\infty}[\beta(1-p)]^s E_t er_{t+s} \tag{45}$$

$$d_t^D = \rho_1^d d_{t-1}^D + \rho_2^d d_{t-2}^D + \varepsilon_t^d \tag{46}$$

$$n_t = \rho_1 n_{t-1} + \rho_2 n_{t-2} + \varepsilon_t^n \tag{47}$$

$$a_t^D = \rho_1^a a_{t-1}^D + \rho_2^a a_{t-2}^D + \varepsilon_t^a \tag{48}$$

$$er_{t+1} = (1-\delta)q_{t+1}^D - q_t^D + \delta d_{t+1}^D \tag{49}$$

$$V_t = \sum_{s=1}^{\infty}[\beta(1-p)]^{s-1}\left[\tilde{\gamma} var_t(er_{t+s}) + 2(\tilde{\gamma}-1)\sum_{i<s}\theta^{s-i}cov_t(er_{t+s}, er_{t+i})\right] \tag{50}$$

We first describe the solution for given values of the variances and covariances that enter these expressions. After that we will discuss the solution of these variances and covariances. For now we therefore take $var(er_{t+1})$ and D as given.

We first need to truncate the infinite sum in (45). We truncate at the horizon T, so that

$$z_t = (1-p)z_{t-1} + \frac{p}{V}\sum_{s=1}^{T}[\beta(1-p)]^{s-1}E_t er_{t+s} \tag{51}$$

In practice we set $T = 60$, which is 5 years. Setting it longer does not affect the results. Excess returns are not really predictable more than 5 years into the future. We also set V to a constant as it will be based on second moments of the excess return, which are time-invariant based on the first-order solution of the excess return.

Define

$$Y_t = \begin{pmatrix} q^D_{t+T-1} \\ \dots \\ q^D_t \\ q^D_{t-1} \\ z_{t-1} \\ n_t \\ n_{t-1} \\ a^D_t \\ a^D_{t-1} \\ d^D_t \\ d^D_{t-1} \\ \tilde{w}^D_{t-1} \end{pmatrix} \tag{52}$$

Also introduce

$$b_1 = 4(1-f) \tag{53}$$
$$b_2 = \theta(2\bar{z}-1)(1-\delta) \tag{54}$$
$$b_3 = \theta(2\bar{z}-1)\delta \tag{55}$$
$$b_4 = \theta(2\bar{z}-1) \tag{56}$$
$$b_5 = 4f\lambda_2\delta\rho^d_1 \tag{57}$$
$$b_6 = 4f\lambda_2\delta\rho^d_2 \tag{58}$$
$$b_7 = 4f\lambda_2(1-\delta) \tag{59}$$

$$b_8 = 1 + 4f\lambda_2 \tag{60}$$

where

$$\lambda_2 = \frac{1}{\tilde{\gamma} var(er_{t+1})} \tag{61}$$

After substituting (44), the market equilibrium condition (42) becomes

$$q_t^D = 4f\lambda_2 E_t er_{t+1} + 4(1-f)z_t + 4n_t + (2\bar{z}-1)\tilde{w}_t^D \tag{62}$$

Substituting the excess return expression (49) and the dividend process (46), we can write this as

$$b_8 q_t^D = (2\bar{z}-1)\tilde{w}_t^D + b_1 z_t + b_7 E_t q_{t+1}^D + b_5 d_t^D + b_6 d_{t-1}^D + 4n_t \tag{63}$$

Substituting the expression for the excess return, the wealth accumulation equation (43) can be written as

$$\tilde{w}_t^D = a_t^D + \theta\tilde{w}_{t-1}^D + \theta(2\bar{z}-1)\left((1-\delta)q_t^D - q_{t-1}^D + \delta d_t^D\right) \tag{64}$$

This becomes

$$\tilde{w}_t^D = a_t^D + \theta\tilde{w}_{t-1}^D + b_2 q_t^D - b_4 q_{t-1}^D + b_3 d_t^D \tag{65}$$

We need to do more work for the expression (45) for z_t. We have

$$z_t = (1-p)z_{t-1} + p\lambda_1 \sum_{s=1}^{T} [\beta(1-p)]^s E_t er_{t+s} \tag{66}$$

where

$$\lambda_1 = \frac{1}{V} \tag{67}$$

We have

$$er_{t+s} = (1-\delta)q_{t+s}^D - q_{t+s-1}^D + \delta d_{t+s}^D \tag{68}$$

Using the process for d_t^D, we can write

$$E_t d_{t+s}^D = a_1(s) d_t^D + a_2(s) d_{t-1}^D \tag{69}$$

where $a_1(s)$ and $a_2(s)$ depend on ρ_1^d and ρ_2^d.

Using these results, and a bit of algebra, we get

$$\sum_{s=1}^{T} [\beta(1-p)]^s E_t er_{t+s} = \sum_{s=0}^{T} \upsilon_s E_t q_{t+s}^D + a_1 d_t^D + a_2 d_{t-1}^D \tag{70}$$

where

$$v_0 = -\beta(1-p) \tag{71}$$

$$v_T = [\beta(1-p)]^T(1-\delta) \tag{72}$$

$$v_s = (1-\delta)[\beta(1-p)]^s - [\beta(1-p)]^{s+1} \quad s = 1, ..., T-1 \tag{73}$$

$$a_1 = \delta \sum_{s=1}^{T} [\beta(1-p)]^s a_1(s) \tag{74}$$

$$a_2 = \delta \sum_{s=1}^{T} [\beta(1-p)]^s a_2(s) \tag{75}$$

We then have

$$z_t = (1-p)z_{t-1} + p\lambda_1 \sum_{s=0}^{T} v_s E_t q_{t+s}^D + b_9 d_t^D + b_{10} d_{t-1}^D \tag{76}$$

where $b_9 = p\lambda_1 a_1$ and $b_{10} = p\lambda_1 a_2$.

We can now write the model in the form

$$AE_tY_{t+1} = BY_t \tag{77}$$

where the matrices A and B are:

$$A = \begin{pmatrix}
0 & ... & 0 & b_7 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 2\bar{z}-1 \\
0 & 0 & ... & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
p\lambda_1 v_T & p\lambda_1 v_{T-1} & ... & p\lambda_1 v_1 & p\lambda_1 v_0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & ... & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & \\
0 & 0 & ... & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & ... & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & ... & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & ... & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & ... & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & ... & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & ... & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

$$B = \begin{pmatrix} 0 & ... & 0 & b_8 & 0 & 0 & -4 & 0 & 0 & 0 & -b_5 & -b_6 & 0 \\ 0 & ... & 0 & b_2 & -b_4 & 0 & 0 & 0 & 1 & 0 & b_3 & 0 & \theta \\ 0 & 0 & ... & 0 & 0 & p-1 & 0 & 0 & 0 & 0 & -b_9 & -b_{10} & 0 \\ 1 & 0 & 0 & ... & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & \\ 0 & ... & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & \rho_1 & \rho_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 & \rho_1^a & \rho_2^a & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 & 0 & 0 & \rho_1^d & \rho_d^2 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The control variables in Y_t are

$$CV_t = \begin{pmatrix} q^D_{t+T-1} \\ ... \\ q^D_t \end{pmatrix} \tag{78}$$

The vector of state variables is

$$SV_t = \begin{pmatrix} q^D_{t-1} \\ z_{t-1} \\ n_t \\ n_{t-1} \\ a^D_t \\ a^D_{t-1} \\ d^D_t \\ d^D_{t-1} \\ \tilde{w}^D_{t-1} \end{pmatrix} \tag{79}$$

Using the solab.m code by Paul Klein (see http://paulklein.ca/newsite/codes/codes.phpcode) to solve for the system $AE_tY_{t+1} = BY_t$, we obtain the matrices F and P for the solution of the control variables as a function of the state variables and the accumulation equation for the state variables:

$$CV_t = FSV_t \tag{80}$$

$$E_tSV_{t+1} = PSV_t \tag{81}$$

We can do impulse response analysis using that

$$y_t = Gy_{t-1} + Q\varepsilon_t \tag{82}$$

$$y_t^* = My_t \tag{83}$$

Here

$$y_t = \begin{pmatrix} q_t^D \\ z_t \\ n_t \\ n_{t-1} \\ a_t^D \\ a_{t-1}^D \\ d_t^D \\ d_{t-1}^D \\ \tilde{w}_t^D \end{pmatrix} \tag{84}$$

and

$$y_t^* = \begin{pmatrix} q_t^D \\ z_t^A \\ d_t^D \end{pmatrix} \tag{85}$$

M is defined as:

$$M = \begin{pmatrix} 1 & 0 & ... & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & ... & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2\bar{z}-1}{4} \\ 0 & 0 & ... & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{86}$$

ε_t is the vector of shocks:

$$\varepsilon_t = \begin{pmatrix} \varepsilon_t^n \\ \varepsilon_t^a \\ \varepsilon_t^d \end{pmatrix} \tag{87}$$

The matrices G and Q follow from (80)-(81). Let f be the last row of F, p the second row of P and h the last row of P. Then

$$G = \begin{pmatrix} f_1 & f_2 & f_3\rho_1 + f_4 & f_3\rho_2 & f_5\rho_1^a + f_6 & f_5\rho_2^a & f_7\rho_1^d + f_8 & f_7\rho_2^d & f_9 \\ p_1 & p_2 & p_3\rho_1 + p_4 & p_3\rho_2 & p_5\rho_1^a + p_6 & p_5\rho_2^a & p_7\rho_1^d + p_8 & p_7\rho_2^d & p_9 \\ 0 & 0 & \rho_1 & \rho_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_1^a & \rho^a - 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_1^d & \rho_2^d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ h_1 & h_2 & h_3\rho_1 + h_4 & h_3\rho_2 & h_5\rho_1^a + h_6 & h_5\rho_2^a & h_7\rho_1^d + h_8 & h_7\rho_2^d & h_9 \end{pmatrix}$$

Finally, the matrix Q multiplying the shocks is

$$Q = \begin{pmatrix} f_3 & f_5 & f_7 \\ p_3 & p_5 & p_7 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ h_3 & h_5 & h_7 \end{pmatrix}$$

The solution is conditional on λ_1 and λ_2, which depend on $var(er_{t+1})$ and V. In order to solve for $var(er_{t+1})$ and V, one approach is to iterate on their values. Start with $var(er_{t+1}) = 0.025^2$ and

$$V = \sum_{s=1}^{\infty} [\beta(1-p)]^s \tilde{\gamma}(0.025^2) = \frac{\beta(1-p)}{1-\beta(1-p)} \tilde{\gamma}(0.025^2) \tag{88}$$

We can solve the model conditional on this and use the result to compute the implied theoretical values of $var(er_{t+1})$ and V. We can then solve again conditional on these values and keep iterating until $var(er_{t+1})$ and V no longer change.

The procedure described in the last paragraph is computationally very intensive when we estimate the parameters and we therefore follow a different approach. The model allows us to estimate

$$(1-f)\lambda_1 = \frac{1-f}{V} \tag{89}$$

$$f\lambda_2 = \frac{f}{\tilde{\gamma} var(er_{t+1})} \tag{90}$$

These multiply expected excess returns in the expression for z_t^A. We could estimate $(1-f)\lambda_1$ and $f\lambda_1$. Alternatively we could also set the variances and covariances at some level and then estimate f and $\tilde{\gamma}$. Both will yield the same values of $(1-f)\lambda_1$ and $f\lambda_1$. Consider setting all the variances of future excess returns equal to 0.025^2 and the covariances that enter V equal to 0. Conditional on doing so, the estimates of f and $\tilde{\gamma}$ are denoted $\bar{f}$ and $\bar{\gamma}$.

We can now use the following procedure to find the correct estimates $\hat{f}$ and $\hat{\gamma}$ that are consistent with the theoretical solutions for the variances and covariances. We rescale f and $\tilde{\gamma}$ to be consistent with the theoretical values of the variances and covariances, while keeping $(1-f)\lambda_1$ and $f\lambda_1$ unchanged to the level found when setting the variances equal to 0.025^2 and the covariances equal to zero. The theoretical variances and covariances can be computed based on the solution where the variances are 0.025^2 and the covariances are zero because the solution, and therefore the impulse responses, depends only on the values of $(1-f)\lambda_1$ and $f\lambda_1$.

We then have

$$\frac{1-\hat{f}}{\sum_{s=1}^{\infty}[\beta(1-p)]^{s-1}\left[\hat{\gamma} var_t(er_{t+s}) + 2(\hat{\gamma}-1)\sum_{i<s}\theta^{s-i}cov_t(er_{t+s}, er_{t+i})\right]} = \frac{1-\bar{f}}{\sum_{s=1}^{\infty}[\beta(1-p)]^{s-1}\left[\bar{\gamma}(0.025^2)\right]} \tag{91}$$

$$\frac{\hat{f}}{\hat{\gamma} var(er_{t+1})} = \frac{\bar{f}}{\bar{\gamma}(0.025^2)} \tag{92}$$

Solving this yields $\hat{\gamma} = A_1/A_2$, where

$$\begin{aligned} A_1 &= \frac{1}{1-\bar{f}}\sum_{s=1}^{\infty}[\beta(1-p)]^{s-1}\left[\bar{\gamma}(0.025^2)\right] + \\ &\quad 2\sum_{s=1}^{\infty}[\beta(1-p)]^{s-1}\sum_{i<s}\theta^{s-i}cov_t(er_{t+s}, er_{t+i}) \end{aligned} \tag{93}$$

and

$$\begin{aligned} A_2 &= \sum_{s=1}^{\infty}[\beta(1-p)]^{s-1}\left[var_t(er_{t+s}) + 2\sum_{i<s}\theta^{s-i}cov_t(er_{t+s}, er_{t+i})\right] + \\ &\quad \frac{\bar{f} var_t(er_{t+1})}{(1-\bar{f})}\sum_{s=1}^{\infty}[\beta(1-p)]^{s-1} \end{aligned} \tag{94}$$

In addition

$$\hat{f} = \frac{\bar{f}\hat{\gamma} var(er_{t+1})}{\bar{\gamma}(0.025^2)} \tag{95}$$