Appendix

Money and Capital in a Persistent Liquidity Trap

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1 Decomposing cash holdings in the US

This section investigates the rise in cash holdings occurring at the ZLB by focusing on the US. First, we use the balance-sheet tables of the Households and Nonprofit Organizations (B.101), of the Nonfinancial Corporate Business (B.103) and of the Nonfinancial Noncorporate Business (B.104) from the Financial Accounts of the US. The series used are the Checkable Deposits and Currency (FL153020005, FL103020005 and FL113020005). These are deflated by the CPI, obtained from the International Financial Statistics. The results are shown in Figure 1. The figure shows that both households and the nonfinancial businesses increase their cash holdings at the end of 2008, when the Fed funds rate started approaching zero. Among the nonfinancial businesses, the corporate sector accounts for most of the increase. This is consistent with our model, as the demand for money for saving purposes arises for the *less* constrained agents in the economy.

Second, we use the Survey of Consumer Finances (SCF) to decompose cash holdings by households into different household categories. We consider checking account holdings only,

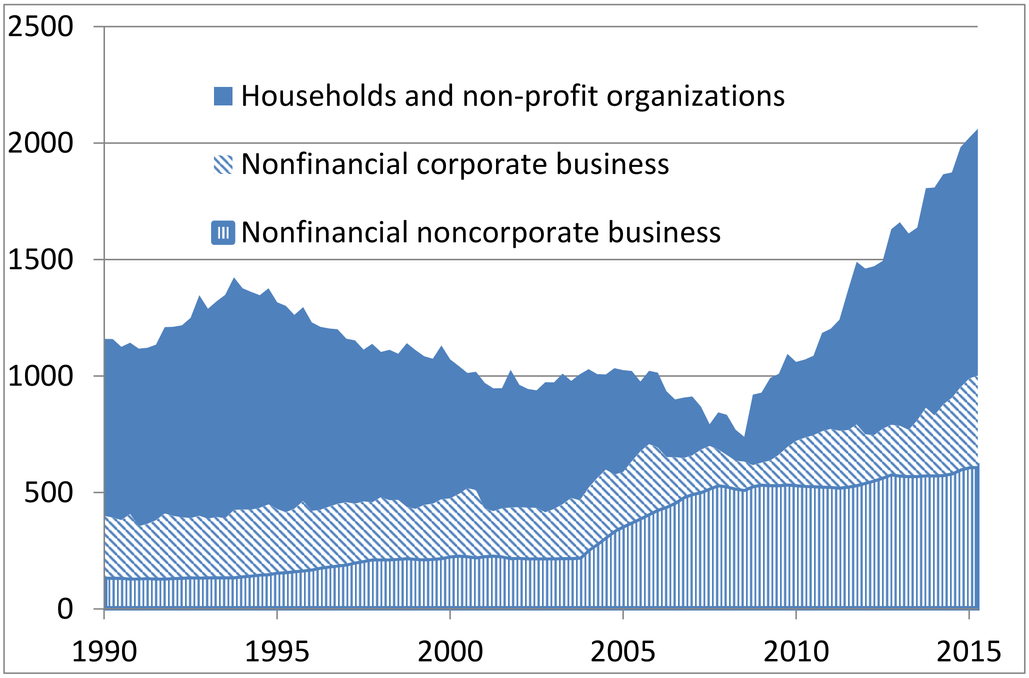


Figure 1: Checkable Deposits and Currency, billions of 2010 USD

Source: Financial accounts of the US, Federal Reserve, International Financial Statistics, authors' calculations.

as currency is not available in the survey. We aggregate this variable within some categories of households, trying to reflect the split between households who participate in financial markets (our investors) and households who do not participate in financial markets (our hand-to-mouth workers). We use several proxies: households above the 90^{th} percentile of net worth versus those below; households above the 90^{th} percentile of liquid wealth versus those below; households owning a business or some stocks versus those owning neither; non hand-to-mouth households versus hand-to-mouth households (HTM). Aggregated holdings are deflated by the CPI, obtained from the International Financial Statistics.

The definition of liquid wealth follows Kaplan et al. (2014): it consists of checking, saving, money market, and call accounts as well as directly held mutual funds, stocks, corporate bonds, and government bonds, minus the sum of all credit card balances that accrue interest, after the most recent payment. The definition of HTM households also follows Kaplan et al. (2014), but is even more inclusive. HTM households are those whose average liquid wealth balances are positive (to capture the fact they are not borrowing), but are equal to or less than half their earnings per pay period. as a pay period, we use a month instead of two weeks, to obtain a

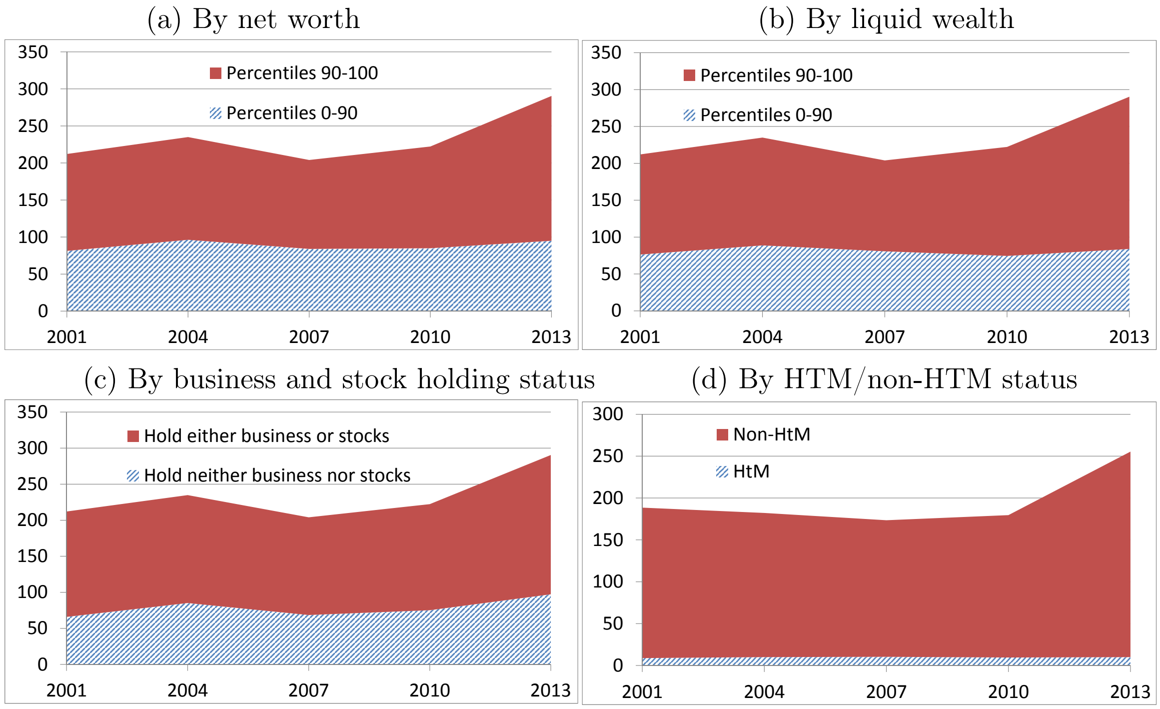


Figure 2: Households' checking account holdings, billions of 2010 USD

Source: Survey of Consumer Finances, Federal Reserve System, authors' calculations.

higher share of HTM households. The proportion of non-HTM is therefore more conservative. We obtain a proportion of HTM that is around 0.36 on average between 2001 and 2013.

The results are represented in Figure 2. The figure shows that the bulk of the increase in cash holdings between the 2007 and 2013 surveys, among households, comes mainly from households with a high net worth, liquid wealth, households owning a business or stocks and especially, non-HTM households.

2 Calibration

2.1 Net position of investors

To evaluate the net position of investors, we first use the balance-sheet table for Nonfinancial Corporate Business (B.103) from the Financial Accounts of the US. A simple way to map the net position to the data is to follow the capital accumulation equation, (14), taken in the normal equilibrium with zero money holdings. The net position is the difference between investors' net worth (the right-hand side of the equation) and capital (the first term on the left-hand side). Accordingly, we can define the net position in the data as the difference between net worth (FL10209005) and Nonfinancial Assets (LM102010005). The resulting net position is very close to 0 in pre-crisis years, moving from -2% of US GDP in 2000 (about -300 billions USD) to 6% of US GDP in 2006 (about 800 billions USD). By contrast, there was a large and negative net position earlier in time, at about -20% of GDP from the mid-nineteen seventies to the beginning of the nineteen nineties.

A more detailed mapping can be done by focusing on interest-bearing assets. We compute a second measure of the net position as: Time and savings deposits (FL103030003) + Money market fund shares (FL103034003) + Security repurchase agreements (FL102051003) + Credit market instruments (FL104004005) + Trade receivables (FL103070005) + Miscellaneous assets (FL103090005) on the asset side − Credit market instruments (FL104104005) − Trade payables (FL103170005) − Miscellaneous liabilities (FL103190005) on the liability side. The difference with the previous measure is that we now exclude deposits and currency, mutual fund shares and foreign direct investment. This second measure is also close to 0 pre-crisis, at -9% of GDP in 2000 and -2% of GDP in 2006.

Overall, the data suggests that investors were close to autarky in pre-crisis years.

2.2 Supply of assets by other agents

To calibrate the balance sheet parameters $\bar{l}^g$ and $\bar{l}^w$, we measure the net supply of available interest-bearing instruments by the Government in the Financial Accounts of the US in 2006.

We start by constructing the net position in interest-bearing instruments of the Government (including the monetary authority) from tables L.105 and L.109 as:

- Government: Time and savings deposits (FL363030005) + Money market fund shares (FL213034003) + Security repurchase agreements (FL212051003) + Debt securities (FL364022005) + Loans (FL364023005) + Trade receivables (FL363070005) on the asset side − Debt securities (FL364122005) − Loans (FL364123005) − Trade payables (FL363170005) on the liability side;

- monetary authority: Security repurchase agreements (FL712051000) + Debt securities

(FL714022005) + Loans (FL713068005) on the asset side − Security repurchase agreements (FL712151003) on the liability side.

We find a net position approximately equal to −40% of GDP in 2006, implying a net supply of 40% of GDP. However, not all assets were available to domestic savers since the rest of the world is also a large consumer of US assets. We then compute the net position of the rest of the world in interest-bearing assets from table L.133 as: US time deposits (FL263030005) + Money market fund shares (FL263034003) + Security repurchase agreements (FL262051003) + Debt securities (FL264022005) + Loans to US corporate business (FL263069500) + Trade receivables (LM263070003) on the asset side − Security repurchase agreements (FL262151003) − Debt securities (FL264122005) − Loans (FL264123005) − Trade payables (LM263170003) on the liability side. The resulting net position was also close to 40% in 2006. As a result, the supply of interest-bearing assets by the Government net of the demand by the rest of the world was approximately 0. We therefore calibrate $\bar{l}^g = 0$.

The assumption of autarkic investors requires $\bar{l}^g + \bar{l}^w = 0$. Therefore, we set $\bar{l}^w = 0$.

3 Model

3.1 Equilibrium conditions of the benchmark model

The equilibrium conditions of the benchmark model presented in Section 2 are the following.

$$\frac{1}{c_t^S} = \beta r_{t+1} \frac{1}{c_{t+1}^I} \tag{21a}$$

$$c_t^I = (1-\beta)(a_{t+1} + m_{t+1}^S) \tag{21b}$$

$$M_{t+1}^S \left(r_{t+1} - \frac{P_t}{P_{t+1}} \right) = 0, \quad r_{t+1} \geq \frac{P_t}{P_{t+1}}, \quad M_{t+1}^S \geq 0, \tag{21c}$$

$$\frac{b_{t+1}}{r_{t+1}} + a_t + \frac{M_t^S}{P_t} = c_t^I + k_{t+1}, \tag{21d}$$

$$\rho_t k_t = c_t^S + b_t + \frac{a_{t+1}}{r_{t+1}} + \frac{M_{t+1}^S}{P_t}, \tag{21e}$$

$$b_{t+1} = \phi_t \rho_{t+1} k_{t+1}, \tag{21f}$$

$$c_t^w + \frac{M_{t+1}^w}{P_t} + l_t^w = w_t + \frac{T_t^w}{P_t} + \frac{M_t^w}{P_t} + \frac{l_{t+1}^w}{r_{t+1}}, \tag{21g}$$

$$l_{t+1}^w = \bar{l}_t^w y_{t+1}, \tag{21h}$$

$$M_{t+1}^w = (1-\alpha) P_t y_t, \tag{21i}$$

$$y_t = k_t^\alpha, \tag{21j}$$

$$\rho_t k_t = \alpha y_t, \tag{21k}$$

$$w_t = (1-\alpha) y_t, \tag{21l}$$

$$\frac{M_{t+1}}{P_t} + \frac{l_{t+1}^g}{r_{t+1}} = \frac{M_t + T_t^w}{P_t} + l_t^g, \tag{21m}$$

$$l_{t+1}^g = \bar{l}_t^g y_{t+1}, \qquad M_{t+1}/M_t = \theta_{t+1}, \tag{21n}$$

$$b_{t+1} + l_{t+1}^w + l_{t+1}^g = a_{t+1}, \tag{21o}$$

$$M_{t+1}^S + M_{t+1}^w = M_{t+1}. \tag{21p}$$

Equations (21a) to (21c) are the optimal consumption and portfolio choice of investors, under budget and (binding) borrowing constraints (21d) to (21f), as discussed in Section (2.2). Equations (21g) to (21i) characterize consumption, borrowing, and money demand of workers given the budget and (binding) borrowing and CIA constraints. Equations (21j) to (21l) are the production function, the equilibrium return on capital and the equilibrium wage, (21m) and (21n) describe government policy, and (21o) and (21p) are market clearing conditions for bonds and money.

3.2 Extended model used in simulations

The model that is simulated is a stochastic version of the model presented in Section 2, with partial capital depreciation and, in some cases, downwardly-rigid nominal wages. In this model, leverage ϕ_t is a stochastic that can take two values, $\phi^H > \phi^L$. We simulate an economy that starts initially in a steady state where $\phi_t = \phi^H$. In period 1, ϕ_t unexpectedly drops to ϕ^H. In each period t, ϕ_t switches back to ϕ^L with probability λ, and stays there. Because of aggregate uncertainty, nominal bonds are not equivalent to real bonds, as the future price level is uncertain, given rise to risk premia.

3.2.1 The extended setup

Investors The I-investors' budget constraint (1) is then replaced with:

$$\frac{B_{t+1}}{P_t i_{t+1}} + \frac{A_t}{P_t} + \frac{M_t^S}{P_t} = c_t^I + k_{t+1}.$$

and the S-investors' budget constraint (2) is replaced with:

$$\rho_t k_t = c_t^S + \frac{B_t}{P_t} + \frac{A_{t+1}}{P_t i_{t+1}} + \frac{M_{t+1}^S}{P_t}$$

The borrowing constraints (3) must also be adapted to a stochastic context, as they involve uncertain future variables. In the case of investors, we assume that it is of the following type:

$$\mathrm{E}_t \left[\frac{u'(c_{t+1}^I) B_{t+1}}{P_{t+1}} \right] \leq \phi_t \, \mathrm{E}_t \left[u'(c_{t+1}^I) \rho_{t+1} k_{t+1} \right]$$

Borrowers can renegotiate with their creditors (S-investors) the value of their debt repayment, using the threat to default. The decision to renegotiate the debt is taken at the beginning of period $t+1$, before learning the macro shocks. In case of default, creditors can only get the value of collateral $\phi_t \rho_{t+1} k_{t+1}$. In order to avoid renegotiation, creditors limit their lending ex ante so that to the expected value of debt repayment does not exceed that of collateral. Those expected values are computed using the stochastic discount factor of creditors (c_{t+1}^I is the consumption of S-investors in the next period).

Workers Similarly, the workers's budget constraint (4) now must be written as:

$$c_t^w + \frac{M_{t+1}^w}{P_t} + \frac{L_t^w}{P_t} = \frac{W_t}{P_t} + \frac{T_t^w}{P_t} + \frac{M_t^w}{P_t} + \frac{L_{t+1}^w}{P_t i_{t+1}}$$

where L_{t+1}^w is the workers' nominal debt and $W_t = P_t w_t$ the nominal wage. They face the same CIA constraint (5) as in the benchmark model. As for investors, their borrowing constraint (6) is replaced by:

$$\mathrm{E}_t \left[\frac{u'(c_{t+1}^I) L_{t+1}^w}{P_{t+1}} \right] \leq \bar{l}_t^w \, \mathrm{E}_t \left[u'(c_{t+1}^I) y_{t+1} \right].$$

Downward wage rigidities We suppose that the nominal wage cannot decrease too much from period to period:

$$W_t = \max\{\gamma W_{t-1}, P_t(1-\alpha)k_t^\alpha\} \tag{22}$$

where $\gamma \in (0, \theta)$ is the degree of nominal rigidities. The level of employment h_t is determined by

$$h_t = \min\left\{1, \left(\frac{(1-\alpha)P_t}{\gamma W_{t-1}}\right)^{1/\alpha} k_t\right\}. \tag{23}$$

Government The budget constraint of the government is now

$$M_{t+1} + \frac{L_{t+1}^g}{i_{t+1}} = M_t + T_t^w + L_t^g.$$

We assume that the fiscal authority provides a supply of nominal bonds that sets the expected value of repayment proportional to that of next period output:

$$\mathrm{E}_t\left[\frac{u'(c_{t+1}^I)L_{t+1}^g}{P_{t+1}}\right] = \bar{l}_t^g\, \mathrm{E}_t\left[u'(c_{t+1}^I)y_{t+1}\right].$$

This assumption is a way to normalize the supply of government bonds that will make our computations easier. The stochastic discount factor used in this expression is the one of S-investors as these are the only agents who can arbitrage in equilibrium between government debt and consumption and hence who will be pricing this debt.

3.2.2 Equilibrium conditions

Before deriving equilibrium conditions it is useful to notice that all the nominal income of I-investors in period $t+1$ comes from their saving in nominal assets, either in bonds or in money, $A_{t+1} + M_{t+1}^S$, and is known in period t. With logarithmic utility, their consumption c_{t+1}^I is proportional to $(A_{t+1} + M_{t+1}^S)/P_{t+1}$. As the numerator is known in advance, the stochastic discount factor $u'(c_{t+1}^I) = 1/c_{t+1}^I$ essentially only depends on the price level P_{t+1}. In particular, $P_{t+1}c_{t+1}^I$ is known in period t. This simplifies a lot the equilibrium conditions of the model. For instance, the borrowing constraint of investors simplifies to $B_{t+1} \leq \phi_t\, \mathrm{E}_t[P_{t+1}\rho_{t+1}]k_{t+1}$.

In some simulations, borrowing constraints may temporarily stop binding for 1 or 2 periods after the shock hits. Therefore, we derive equilibrium conditions under the general case where borrowing constraints are not always binding, even though they will be binding in the steady state and during most of transition dynamics. The equilibrium conditions are the following.

$$\frac{1}{c_t^S} = \beta i_{t+1} P_t \, \mathrm{E}_t \left[\frac{1}{P_{t+1} c_{t+1}^I} \right] \tag{24a}$$

$$c_t^I = (1-\beta)(a_{t+1} + m_{t+1}^S) \tag{24b}$$

$$M_{t+1}^S (i_{t+1} - 1) = 0, \quad i_{t+1} \geq 1, \quad M_{t+1}^S \geq 0, \tag{24c}$$

$$\frac{B_{t+1}}{P_t i_{t+1}} + \frac{A_t}{P_t} + \frac{M_t^S}{P_t} = c_t^I + k_{t+1}, \tag{24d}$$

$$\rho_t k_t = c_t^S + \frac{B_t}{P_t} + \frac{A_{t+1}}{P_t i_{t+1}} + \frac{M_{t+1}^S}{P_t}, \tag{24e}$$

$$\begin{gathered} \big[B_{t+1} - \phi_t \, \mathrm{E}_t[P_{t+1}\rho_{t+1}] k_{t+1}\big] \big[\mathrm{E}_t(P_{t+1}\rho_{t+1}) - P_t i_{t+1}\big] = 0, \\ B_{t+1} \leq \phi_t \, \mathrm{E}_t[P_{t+1}\rho_{t+1}] k_{t+1}, \quad \mathrm{E}_t(P_{t+1}\rho_{t+1}) \geq P_t i_{t+1}, \end{gathered} \tag{24f}$$

$$\frac{1}{c_t^w} = \gamma_t + \beta P_t i_{t+1} \, \mathrm{E}_t \left(\frac{1}{P_{t+1} c_{t+1}^w} \right), \tag{24g}$$

$$c_t^w + \frac{M_{t+1}^w}{P_t} + \frac{L_t^w}{P_t} = \frac{W_t}{P_t} + \frac{T_t^w}{P_t} + \frac{M_t^w}{P_t} + \frac{L_{t+1}^w}{P_t i_{t+1}}, \tag{24h}$$

$$\begin{gathered} \gamma_t \big[L_{t+1}^w - \bar{l}_t^w \, \mathrm{E}_t(P_{t+1} y_{t+1})\big] = 0, \\ \gamma_t \geq 0, L_{t+1}^w \leq \bar{l}_t^w \, \mathrm{E}_t(P_{t+1} y_{t+1}), \end{gathered} \tag{24i}$$

$$M_{t+1}^w = (1-\alpha) P_t y_t, \tag{24j}$$

$$y_t = k_t^\alpha h_t^{1-\alpha}, \tag{24k}$$

$$\rho_t k_t = \alpha y_t + (1-\delta) k_t, \tag{24l}$$

$$W_t h_t = P_t (1-\alpha) y_t, \tag{24m}$$

$$h_t = \min \left\{ 1, \left(\frac{(1-\alpha) P_t}{\gamma W_{t-1}} \right)^{1/\alpha} k_t \right\}, \tag{24n}$$

$$M_{t+1} + \frac{L_{t+1}^g}{i_{t+1}} = M_t + T_t^w + L_t^g, \tag{24o}$$

$$L_{t+1}^g = \bar{l}_t^g \, \mathrm{E}_t(P_{t+1} y_{t+1}), \qquad M_{t+1}/M_t = \theta_{t+1}, \tag{24p}$$

$$B_{t+1} + L_{t+1}^w + L_{t+1}^g = a_{t+1}, \tag{24q}$$

$$M_{t+1}^S + M_{t+1}^w = M_{t+1}. \tag{24r}$$

Equations (24a) to (24f) solve the optimization problems of both groups of entrepreneurs, under budget constraints (24d) and (24e). As before, (24a) to (24c) are the optimal consumption and portfolio choice of investors. Equation (24f) is the optimal choice of leverage determined either by the binding borrowing constraint or by the arbitrage condition between the expected real return on bonds and capital.

Equations (24g) to (24j) characterize consumption, borrowing, and money demand of workers given the budget, borrowing and CIA constraints. In particular, (24g) is the first-order condition with respect to consumption, where γ_t is the Lagrange multiplier of the borrowing constraint, and (24i) is the complementary slackness condition of the borrowing constraint. The Lagrange multiplier of the CIA constraint can be shown to equal

$$\gamma_t + (i_{t+1} - 1)P_t \beta \,\mathrm{E}_t\left(\frac{1}{(P_{t+1}c^w_{t+1}}\right).$$

The CIA is then strictly binding when either the borrowing constraint is binding or the interest rate is away from the ZLB after the shock hits. This is in general the case. However, as mentioned earlier, in some simulations the borrowing constraint stops binding for 1 or 2 periods at the ZLB. Then the CIA constraint does not bind and the allocation of workers' saving across bonds and money is indeterminate during that period. As this only lasts for a very short period of time, such indeterminacy has little material consequences, and so we simply assume that households always hold the minimum possible amount of money, hence the binding CIA (24j).

Equations (24k) to (24m) are the production function, the equilibrium return on capital and the equilibrium wage, (24n) is the labor supply curve implied by the downward-wage rigidity, (24o) and (24p) describe government policy, and (24q) and (24r) are market clearing conditions for bonds and money.

When borrowing constraints for investors and workers are binding, we get the equivalent of the Euler equation (13) and the aggregate budget constraint (14) of the simple model:

$$\beta\left(1 - \phi_{t-1}\frac{\mathrm{E}_{t-1}(P_t\rho_t)}{P_t\rho_t}\right)\rho_t k_t = \frac{1}{P_t i_{t+1}}\left[\phi_t \,\mathrm{E}_t(P_{t+1}\rho_{t+1})k_{t+1} + \bar{l}_t \,\mathrm{E}_t(P_{t+1}y_{t+1}) + M^S_{t+1}\right],$$

$$k_{t+1} + \frac{\bar{l}_t \,\mathrm{E}_t(P_{t+1}y_{t+1})}{P_t i_{t+1}} + \frac{M^S_{t+1}}{P_t} = \beta\left[\rho_t k_t + \bar{l}_{t-1}\frac{\mathrm{E}_{t-1}(P_t y_t)}{P_t y_t}y_t + \frac{M^S_t}{P_t}\right].$$

The main difference with the simple model is that (i) the future nominal output and the future nominal return on capital are now uncertain, and (ii) real debt repayments depend on surprises on both variables.

3.2.3 Simulations

We simulate a particular realization of the sequence of leverage $\{\phi_t\}_{t\geq 0}$ where ϕ_t switches back to the high value ϕ^H in period $t = 11$. We proceed in the following way.

We first construct an "unlucky" path where ϕ_t never switches back—a 0-probability event. Along this path, agents do expect ϕ_t to switch back with probability λ in every period. We keep periods 0 to 10 of this path. Then, using as initial condition the vector of state variables of the unlucky path at the end of period $t = 10$, we simulate a second path where $\phi_t = \phi^H$. Pasting this second path to periods 0 to 10 of the unlucky path, we get the full equilibrium path corresponding to the realization of $\{\phi_t\}_{t\geq 0}$.

We use Dynare (version 4.4.3) to simulate each of these two paths. The second one is a straightforward deterministic simulation. The first one is more complex to simulate as it requires to keep track in every period of expectations of future variables in the state where ϕ_t switches back to ϕ^H in the following period. In particular, we need to compute $\mathrm{E}_t[P_{t+1}|\phi_{t+1} = \phi^H]$. Once $\mathrm{E}_t[P_{t+1}|\phi_{t+1} = \phi^H]$ is known, it is easy to compute all other needed expected variables using relevant equilibrium conditions. The difficulty comes from the fact that $\mathrm{E}_t[P_{t+1}|\phi_{t+1} = \phi^H]$ itself may depend on $\mathrm{E}_t[P_{t+2}|\phi_{t+1} = \phi^H]$ and so on. Fortunately, the forward iteration is finite. Eventually, the economy exits the ZLB after the deleveraging shock stops. When that happens, $\mathrm{E}_t[P_{t+s}|\phi_{t+1} = \phi^H]$ is only a function of predetermined variables. Indeed, it is pinned down by the CIA constraint when investors' money holdings is 0, together with the labor supply curve:

$$M_{t+s} = (1-\alpha)\,\mathrm{E}_t[P_{t+s}k_{t+s}^{\alpha}h_{t+s}^{1-\alpha}|\phi_{t+1} = \phi^H],$$
$$\mathrm{E}_t[h_{t+s}|\phi_{t+1} = \phi^H] = \min\left\{1, \mathrm{E}_t\left[\left(\frac{(1-\alpha)P_{t+s}}{\gamma W_{t+s-1}}\right)^{1-\alpha} k_{t+s}\middle|\phi_{t+1} = \phi^H\right]\right\}.$$

These two equations uniquely determine $\mathrm{E}_t[P_{t+s}|\phi_{t+1} = \phi^H]$ and $\mathrm{E}_t[h_{t+s}|\phi_{t+1} = \phi^H]$ as a function of M_{t+s} and expected variables determined in previous periods.

To simulate the unlucky path, we then complement the model with conditional expectations

$\mathrm{E}_t[\cdot|\phi_{t+1} = \phi^H]$ of equilibrium conditions 1, 2,..., T periods ahead, assuming that the economy has exited the ZLB in period $t+T$ when ϕ_{t+1} switches back to ϕ^H. In practice, $T = 2$ is enough in most cases, as the economy usually exits the ZLB one period after the deleveraging shock stops. When we simulate quantitative easing with late exit, the economy stays longer at the ZLB and we need $T = 5$.

In some cases (especially when we simulate large shocks), the solving algorithm does not converge with the model that allows for occasionally non-binding constraints. In those few cases, we assume binding borrowing constraints in the model, and check that the computed equilibrium satisfies $\gamma_t > 0$ (the borrowing constraints of workers is binding) and $\mathrm{E}_t(P_{t+1}\rho_{t+1}) > P_t i_{t+1}$ (the borrowing constraints of investors is binding).

4 Proof of Proposition 1

We establish first the following Lemma:

Lemma 1 *The normal and liquidity trap steady states are characterized as follows:*

(i) *In a normal steady state,*

$$r^* = \frac{\alpha\phi + \bar{l}}{\beta\alpha(1-\phi)}, \qquad k^* = \left[\beta\alpha - \bar{l}(1/r^* - \beta)\right]^{\frac{1}{1-\alpha}}, \qquad m^{S*} = 0.$$

(ii) *In a liquidity-trap steady state,*

$$\hat{r} = 1/\theta, \qquad \hat{k} = \left(\frac{\beta^2 + \phi(\theta^2 - \beta^2)}{\theta/\alpha}\right)^{\frac{1}{1-\alpha}}, \qquad \hat{m}^S = \alpha\left[(1-\phi)\frac{\beta}{\theta} - \phi - \bar{l}/\alpha\right]\hat{k}^\alpha.$$

Proof. In a steady state, the money market equilibrium implies that $P_{t+1}/P_t = \theta$. As a result, $i = r\theta$.

In a steady state with $i^* > 1$, (13) and (14) are satisfied with $M^S = 0$. Equation (13) taken at the steady state gives r^*. Besides, (14) in the steady state gives:

$$k^*/y^* = \beta\alpha - \bar{l}(1/r^* - \beta)$$

which yields our result for k^*. This proves result (i).

In a steady state with $i = 1$, (13) and (14) are satisfied with $r = \hat{r} = \theta^{-1}$, which yields $\hat{k}/\hat{y} = \frac{\beta^2+\phi(\theta^2-\beta^2)}{\theta\alpha^{-1}}$, from which we derive $\hat{k}$, and $\hat{m}^S = \alpha\left[(1-\phi)\frac{\beta}{\theta} - \phi\right]\hat{k}^\alpha - \bar{l}\hat{k}^\alpha$. This proves result (ii). ■

We now prove Proposition 1.

Proof. Consider a normal steady state with $\bar{l} = 0$. According to Lemma 1, $r^* = \phi/[\beta(1-\phi)]$. We check that $0 < \beta r^* < 1$ as $\phi < \phi_{max}$ and that $i^* = \theta r^* \geq 1$ as $\phi \geq \phi_T$, which insures that the normal steady state exists and is locally constrained. This proves result (i).

If $\phi < \phi_T$, then the steady state without money does not exist, as the implied nominal interest rate i^* would be below one. If there exists a steady state with $i = 1$, then it is a liquidity trap described by Lemma 1. According to Lemma 1, when $\bar{l} = 0$, $\hat{m}^S = \alpha\left[(1-\phi)\frac{\beta}{\theta} - \phi\right]\hat{k}^\alpha$, which is strictly positive when $\phi < \phi_T$. Besides, $\hat{r} = \theta$, which implies that $0 < \beta\hat{r} < 1$ under Assumption 1, and $\hat{r} > r^*$ for $\phi < \phi^T$. We also check that $\hat{k} = \left(\frac{\beta^2+\phi(\theta^2-\beta^2)}{\theta/\alpha}\right)^{\frac{1}{1-\alpha}} < k^* = (\beta\alpha)^{\frac{1}{1-\alpha}}$ for $\phi < \phi^T$. This proves result (ii). Results (iii) and (iv) derive naturally from Lemma 1. ■

5 Extensions

5.1 Investors with a net debt position

When $\bar{l} < 0$, investors have a negative net position in assets. In that case, changes in the interest rate have a redistributive effect between investors and workers. Capital accumulation in the normal steady state becomes:

$$k = \beta\alpha y - \left(\frac{1}{r} - \beta\right)\bar{l}y. \tag{14'}$$

Since the economy is liquidity-scarce, the price of liquidity—here $1/r$—is still larger than the propensity to save β. With a lower interest rate, the price of liquidity increases even further, but now investors are *net suppliers* of liquidity ($\bar{l} < 0$), so asset scarcity generates net resources that increase the capital stock. Besides, as shown by the normal steady-state Euler equation:

$$r = \frac{\phi + \bar{l}/\alpha}{\beta(1-\phi)}, \tag{20}$$

the interest rate falls after a deleveraging shock in the normal economy as before. Therefore, a deleveraging shock should *increase* the long-run capital stock in the normal economy.

In a liquidity trap however, a deleveraging shock still has a negative long-run effect on capital. In that case, as money and bonds are perfect substitutes, capital accumulation is not affected by the net supply of bonds $\bar{l}$ per se, but by the total amount of net liquidity $s = m^S + \bar{l}y$:

$$k = \beta\alpha y - (\theta - \beta)s. \tag{25}$$

where s is determined by the steady-state Euler equation taken in a liquidity trap, independently from the net supply of bonds $\bar{l}$:

$$s = \alpha\left[(1-\phi)\frac{\beta}{\theta} - \phi\right]y. \tag{26}$$

This equation is similar to (18), with net liquidity s replacing cash holdings m^S. After a deleveraging shock on investors, the price of liquidity remains fixed at θ, whereas liquidity s increases. Since s has the same price as money in a liquidity trap, an increase in s takes resources away from investment as in the case of autarkic investors. Notice that we still have $m^S = \kappa\Delta y$, where the shadow rate is now defined by the right-hand side of (20). We therefore refer to $\bar{l}y$ as the shadow liquidity. Indeed, $\bar{l}y$ would be the liquidity available to agents in the absence of liquidity trap, as $s = \bar{l}y$ when $\Delta = 0$.

The main results are summarized in the following Proposition:

Proposition 2 (Steady state when entrepreneurs are net debtors) *Define* $\phi_{min}(\bar{l}) = -\bar{l}/\alpha$, $\phi_{max}(\bar{l}) = (1-\bar{l}/\alpha)/2$ *and* $\phi_T(\bar{l}) = (\beta - \theta\bar{l}/\alpha)/(\theta+\beta)$. *If* $\phi_{min} < \phi < \phi_{max}(\bar{l})$, *then there exists a locally constrained steady state with* $r < 1/\beta$.

(i) *If, additionally,* $\phi \geq \phi_T(\bar{l})$, *then the steady state is normal.*

(ii) *If* $\phi < \phi_T(\bar{l})$, *then the steady state is a liquidity trap.*

(iii) *In the normal steady state, the real interest rate* r *and the nominal interest rate* i *are increasing in* ϕ, $m^S = 0$ *and if* $\bar{l} < 0$ ($\bar{l} > 0$), *then* k *is decreasing (increasing) in* ϕ.

(iv) *In the liquidity-trap steady state, the real interest rate r and the nominal interest rate i are invariant in ϕ, m^S/y is decreasing in ϕ and k is increasing in ϕ.*

Proof.

Consider a normal steady state. Using Lemma 1, we check that $0 < \beta r^* < 1$ as $\phi_{min}(\bar{l}) < \phi < \phi_{max}(\bar{l})$ and that $i^* = \theta r^* > 1$ as $\phi > \phi_T(\bar{l})$, which insures that the normal steady state exists and is locally constrained. This proves result (i).

If $\phi < \phi_T(\bar{l})$, then the steady state without money does not exist, as the implied nominal interest rate i^* would be below one. If there exists a steady state with $i = 1$, then it is a liquidity trap described by Lemma 1. According to Lemma 1, $\hat{m}^S = \alpha \left[(1-\phi)\frac{\beta}{\theta} - \phi - \bar{l}/\alpha\right] \hat{k}^\alpha$, which is strictly positive when $\phi < \phi_T(\bar{l})$. Besides, $\hat{r} = \theta$, which implies that $0 < \beta\hat{r} < 1$ under Assumption 1, and $\hat{r} > r^*$ for $\phi < \phi^T$. We also check that $\hat{k} = \left(\frac{\beta^2+\phi(\theta^2-\beta^2)}{\theta/\alpha}\right)^{\frac{1}{1-\alpha}} < k^* = k^* = \left[\beta\alpha - \bar{l}(1/r^* - \beta)\right]^{\frac{1}{1-\alpha}}$ for $\phi < \phi^T$. This proves result (ii).

Regarding the properties of r, i and m^S, results (iii) and (iv) derive directly from Lemma 1. To derive the properties of k, we replace r^* in k^* to obtain

$$k^* = \left(\alpha\beta - \bar{l}\left[\frac{\alpha\beta(1-\phi)}{\alpha\phi + \bar{l}} - \beta\right]\right)^{1/(1-\alpha)} \tag{27}$$

We can see that k^* is increasing in ϕ for $\bar{l} > 0$, decreasing for $\bar{l} < 0$.

■

Figure 3 represents the effect of ϕ on the steady state with a net supply of bonds from the rest of the economy ($\bar{l} < 0$). The solid lines show the effective values of k, r, and s as a function of ϕ, while the broken lines show their values if the ZLB were not binding. When ϕ is above ϕ_T, the steady state is normal, so $s = \bar{l}y$. When ϕ decreases while staying above ϕ_T, the equilibrium interest rate decreases. Since investors are net debtors, this has a positive effect on the investors' income, which increases the long-run capital stock. When ϕ falls below ϕ_T, then the steady state is a liquidity trap. As a result, the interest rate does not fall as a response to a deleveraging shock, thus not reestablishing the financing capacities of investors. Instead, investors start increasing their liquidity s by holding money, which has a negative effect on capital accumulation.[1] As a result, an economy that experiences a drop in ϕ that brings the

[1]When investors are net creditors ($\bar{l} > 0$), the capital stock decreases in ϕ both in the normal and liquidity

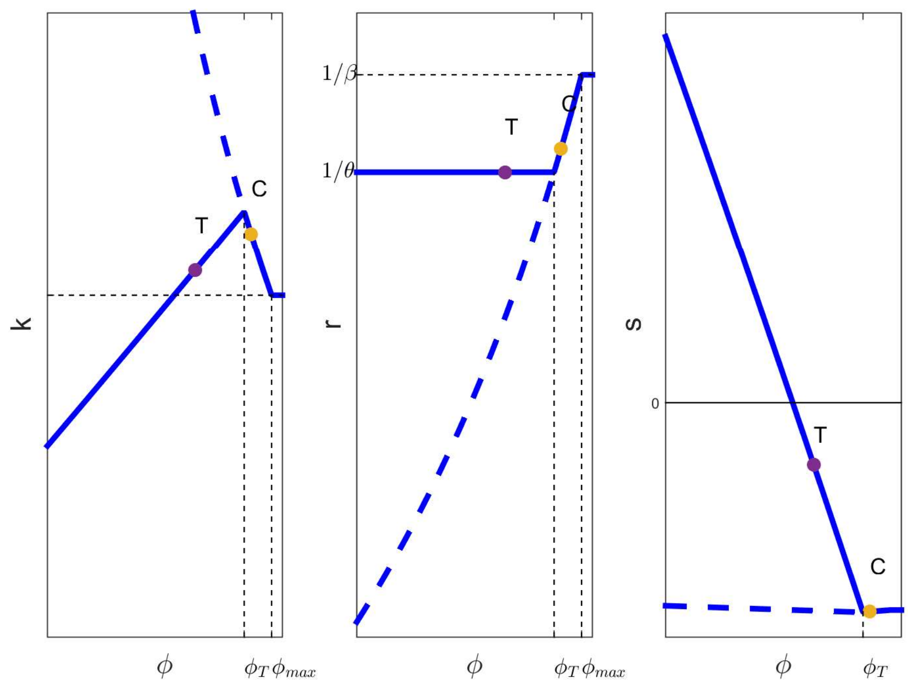


Figure 3: Steady states - Comparative statics w.r.t. ϕ, with $\bar{l} < 0$

equilibrium from C to T as in Figure 3 has less capital in the long run.

5.2 The effect of liquidity injections

Next, we study what happens when the Government implements a liquidity injection resulting in a higher $\bar{l}$ when investors are initially net debtors or in autarky. The effect on capital accumulation is ambiguous. It depends on private leverage ϕ and on $\bar{l}$ as the economy exits the trap. Indeed, the following Proposition shows that, in general, capital in a normal equilibrium is a U-shaped function of $\bar{l}$, and defines the corresponding threshold $\bar{l}_0$ above which it becomes increasing as well as the level of liquidity $\bar{l}_T$ necessary to get out of the ZLB.

Proposition 3 (Effect of $\bar{l}$) *Define* $\bar{l}_0(\phi) = \alpha\sqrt{\phi}(\sqrt{1-\phi} - \sqrt{\phi})$, $\bar{l}_{min}(\phi) = -\alpha\phi$, $\bar{l}_{max}(\phi) = \alpha(1-2\phi)$ *and* $\bar{l}_T(\phi) = \alpha\beta(1-\phi)/\theta - \alpha\phi$. *We have* $\bar{l}_{min} < \bar{l}_0 < \bar{l}_{max}$ *iif* $0 < \phi < 1/2$. *For* $\bar{l}_{min}(\phi) < \bar{l} < \bar{l}_{max}(\phi)$, *then there exists a locally constrained steady state with* $r < 1/\beta$.

(i) *If, additionally,* $\bar{l}_T(\phi) \leq \bar{l}$, *the steady state is normal.*

(ii) *If* $\bar{l} < \bar{l}_T(\phi)$, *the steady state is a liquidity trap.*

trap steady state. However, this case is less realistic.

(iii) *In the normal steady state, the real interest rate r and the nominal interest rate i are increasing in $\bar{l}$, $m^S = 0$ and k is decreasing (increasing) in $\bar{l}$ for $\bar{l} < \bar{l}_0$ ($\bar{l} > \bar{l}_0$).*

(iv) *In the liquidity-trap state, the real interest rate r and the nominal interest rate i are invariant in $\bar{l}$, m^S/y is decreasing one for one in $\bar{l}$ and k is invariant in $\bar{l}$.*

(v) *if $\phi > \beta/(\beta+\theta)$, then $\bar{l}_T < 0$, so there exists normal steady states with $\bar{l} < 0$. In that case, $\bar{l}_0 > \bar{l}_T$ so k is decreasing in $\bar{l}$ in the right neighborhood of $\bar{l}_T$.*

Proof.

Results (i) and (ii) derive directly from Lemma 1. Regarding the properties of r, i and m^S, results (iii) and (iv) derive directly from Lemma 1. To derive the properties of k, we use (27) and take the derivative of k with respect to $\bar{l}$. We find that k is decreasing in $\bar{l}$ whenever $P(\bar{l}) \geq 0$ with

$$P(\bar{l}) = \bar{l}^2 + 2\alpha\phi\bar{l} - \alpha^2\phi(1 - 2\phi)$$

This second-order polynomial admits two roots: $\bar{l}_{00} = -\alpha\phi - \alpha\sqrt{\phi}\sqrt{1-\phi}$ and $\bar{l}_0 = -\alpha\phi + \alpha\sqrt{\phi}\sqrt{1-\phi}$. As $\bar{l}_{00} < \bar{l}_{min}$, $\bar{l}_0$ is the only relevant solution. As a result, k is decreasing in $\bar{l}$ for $\bar{l}_{min} \leq \bar{l} \leq \bar{l}_0$ and increasing for $\bar{l}_0 \leq \bar{l} \leq \bar{l}_{max}$.

To show (iv), note that there exists normal steady states with $\bar{l} < 0$ iif $\bar{l}_T(\phi) < 0$, which is the case when $\phi > \beta/(\beta+\theta)$. Besides, k is decreasing in $\bar{l}$ in the right neighborhood of $\bar{l}_T(\phi)$ iif $\bar{l}_0 > \bar{l}_T$, which is the case when $\phi > \beta^2/(\beta^2+\theta^2)$. Since $\theta/\beta > 1$ by assumption, we have $\phi > \beta/(\beta+\theta)$ implies $\phi > \beta^2/(\beta^2+\theta^2)$, hence the result.

■

Hence, if $\bar{l}_T$ is lower than $\bar{l}_0$, then exiting the liquidity trap through a higher public debt would have a negative effect on capital. We can show that this happens if the deleveraging shock is not too large, leaving $\phi > \beta^2/(\beta^2+\theta^2)$. Indeed, in that case, the level of liquidity necessary to get out of the ZLB is low. If on the opposite the deleveraging shock is large, so that $\phi < \beta^2/(\beta^2+\theta^2)$, then $\bar{l}_T$ is higher than $\bar{l}_0$, leading to a positive effect on capital. This is illustrated in Figure 4. The left panel consider the case with a high ϕ, where at the exit of the ZLB capital starts to decrease. The right panel considers the case with a low ϕ, where at the exit of the ZLB capital starts to increase.

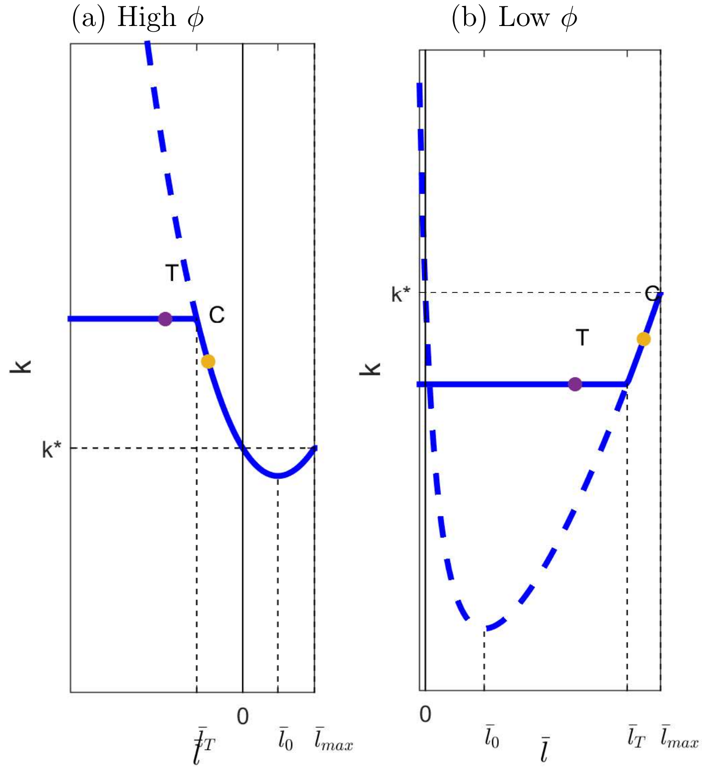


Figure 4: Steady states - Comparative statics w.r.t. $\bar{l}$, with high and low ϕ

When investors are initially in a constrained autarkic steady state, our benchmark case, a sufficient increase in public debt always leads to higher capital, even if lower capital can obtain on an intermediate range of public debt.

Workers' deleveraging Consider a deleveraging shock on workers, that is, a fall in $\bar{l} = \bar{l}^g + \bar{l}^w$ through a fall in $\bar{l}^w$. According to (ii), workers' deleveraging can also lead to the zero lower bound if $\bar{l} < \bar{l}_T$. Indeed, the effect on r is similar to a deleveraging shock on investors, as r declines with the size of the deleveraging shock, as stated in (iii), leading to the zero lower bound.

However, once the economy is in a liquidity trap, changes in $\bar{l}$ have no effect, as stated in (iv). Indeed, since the interest rate cannot adjust in a liquidity trap, the net demand for assets s is constant, so any decrease in the supply of assets to investors through $\bar{l}$ is matched by an increase through m^S. As before, higher real holdings of money obtain through a downward shift in the path of prices.

Finally, since investors are initially autarkic, they become net debtors following the workers' deleveraging ($\bar{l} < 0$), so we are in the situation described by (v), namely with $\bar{l}_0 > \bar{l}_T$, so k is decreasing in $\bar{l}$ in the right neighborhood of $\bar{l}_T$. Therefore, if the workers' deleveraging brings the economy to the liquidity trap, capital increases before falling in the trap. Thus, in the case where investors are initially in autarky, only an investors' deleveraging that brings the economy to a liquidity trap can decrease capital in the long run.

5.3 Increasing inflation

The effect of inflation on equilibrium is described in details in the following proposition:

Proposition 4 (Effect of steady-state inflation) *Define $\theta_0(\phi) = (1/\phi - 1)^{\frac{1}{2}}\beta$, $\theta_T(\bar{l}, \phi) = \beta\alpha(1-\phi)/(\alpha\phi + \bar{l})$ and assume $\bar{l}_{min}(\phi) < \bar{l} \leq \bar{l}_{max}(\phi)$ as in Proposition 3. Then $\beta < \theta_T(\bar{l}, \phi)$. If $\theta \geq \theta_T(\bar{l}, \phi)$, then the steady state is normal. If $\beta < \theta \leq \theta_T(\bar{l}, \phi)$, then the steady state is a liquidity trap and has the following properties:*

(i) *the real interest rate r is decreasing in θ;*

(ii) *if $\phi < 1/2$, the capital stock is U-shaped in θ, decreasing for $\beta < \theta < \theta_0(\phi)$ and increasing for $\theta_0(\phi) \leq \theta \leq \theta_T(\bar{l}, \phi)$; if $\phi \geq 1/2$, it is always increasing in θ;*

(iii) *still, an increase in θ from a value below $\theta_T(\bar{l}, \phi)$ to a value above $\theta_T(\bar{l}, \phi)$ necessarily increases the capital stock if $\bar{l} \leq 0$.*

Proof.

The proof derives from Lemma 1, with a threshold $\theta_T(\bar{l})$ defined such that $\phi = \phi_T(\bar{l})$ when $\theta = \theta_T(\bar{l})$. To derive result (ii), we take the derivative of k with respect to θ and show that it is negative for $\beta < \theta < (1/\phi - 1)^{\frac{1}{2}}\beta$ and positive for $(1/\phi - 1)^{\frac{1}{2}}\beta \geq \theta \geq \theta_T(\bar{l})$. To show (iii), it is enough to show that capital with $\theta = \beta$ is lower than capital with $\theta = \theta_T(\bar{l})$. With $\theta = \beta$, we have $k = (\alpha\beta)^{\frac{1}{1-\alpha}}$. With $\theta = \theta_T(\bar{l})$, the economy becomes normal so we have $k = k^*$. Using the definition of k^* as given by Lemma 1, we know that $k^* \geq (\alpha\beta)^{\frac{1}{1-\alpha}}$ whenever $\bar{l} \leq 0$.

■

The inequality $\phi < 1/2$ is natural when considering a liquidity trap: it is a necessary condition to get a liquidity trap with autarkic investors. The situation where $\theta < \theta_0(\phi)$ occurs when both ϕ and θ are low, generating high real money holdings m, which makes higher inflation more costly. However, if θ is increased to a level that is higher than $\theta_T(\phi)$, then the economy gets out of the liquidity trap, money holdings disappear, and inflation has a positive effect on capital.[2]

5.4 Addressing the short-term output gap with monetary transfers

This section shows that an appropriate monetary policy is able to address the negative output gap after a deleveraging shock when there are downward wage rigidities, but not the medium run effects studied in the paper.

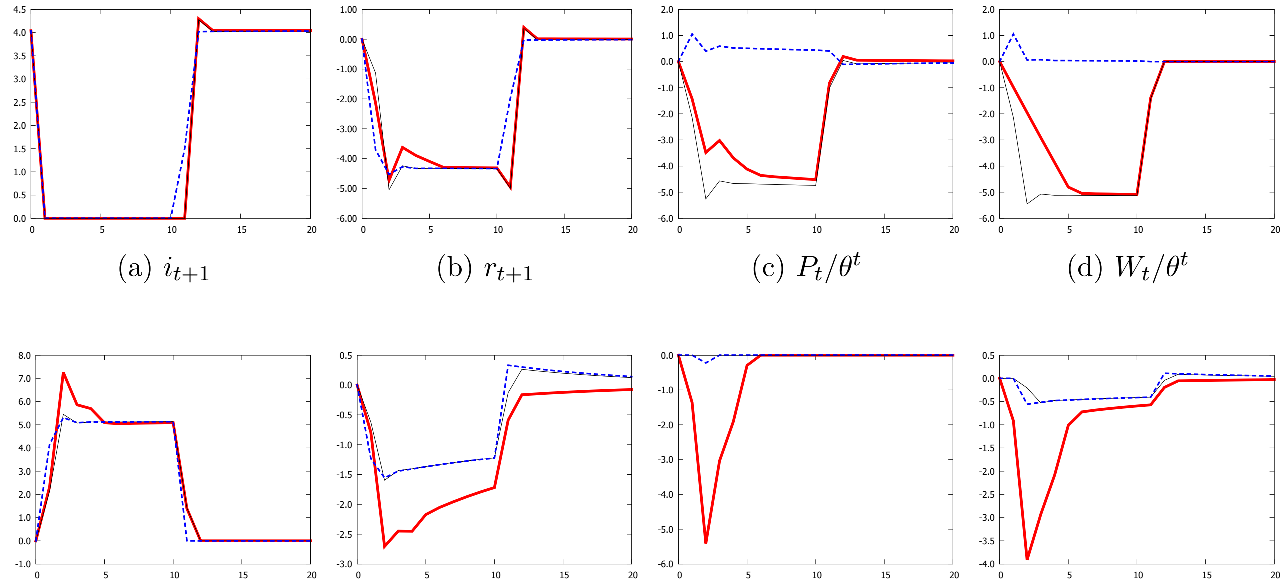
The thick red line in Figure 5 represents the case of downwardly-rigid wages, and the thin black line is the case of flexible wages. Consider now a monetary expansion taking the form of transfers to workers. In the simulation represented by the dashed blue line, the government increases M when the shock hits. The increase is calibrated so that the nominal wage converges back to its initial value as time goes by. When the shock is reversed, the monetary expansion is reversed as well.

As the figure shows, the resulting dynamics of real variables is very close to the dynamics with flexible wages. By increasing money supply, monetary policy substitutes to the fall in the price level that would obtain with flexible prices.

This result stands in sharp contrast to existing work, for instance Krugman (1998), where money creation taking the form of transfers has no effect at the ZLB with pre-set prices (see footnote 11 of this work). It comes from the non-ricardian structure of the model, which gives rise to the Pigou-Patinkin effect described in the main text. However, if a policy of monetary transfers can be very effective in closing the output gap in the short-run of this model, it has no effect in the medium run and therefore cannot prevent the medium term output losses.

Because the composition of government liabilities at the ZLB does not matter, an increase in transfers to households financed by government debt would have the same effect as a monetary expansion.

[2]Assumption 1 implies that the threshold ϕ_T of Proposition 1 is strictly lower than 1/2.



(a) i_{t+1} (b) r_{t+1} (c) P_t/θ^t (d) W_t/θ^t

(e) M^S_{t+1}/M_{t+1} (f) K_{t+1} (g) h_t (h) y_t

Figure 5: Transitory dynamics after a deleveraging shock. The shock hits in period 1 and lasts for 10 years. Thick red line: downwardly-rigid wages. Dashed blue line: downwardly-rigid wages with a monetary expansion when the shock hits. Thin black line: flexible wages. All variables are in relative deviation from initial steady state, in percent, except interest rates and M^s/M which are in absolute deviation from initial steady state, in percent.

5.5 Bubbles

Consider an infinitely-lived asset in fixed unitary supply with no intrinsic value—a bubble. Denote z_t its relative price in terms of consumption goods. The real return of the bubble as of time t is z_{t+1}/z_t. For the bubble to be traded, this rate of return must be equal to the real interest rate: $z_{t+1}/z_t = r_{t+1}$. With r_{t+1} different from 1, the bubble would either asymptotically disappear or diverge to an infinite value. Then, a bubbly steady state necessarily has a zero *real* interest rate: $r = 1$. With positive long run inflation, $1 > 1/\theta$, the bubble strictly dominates money as a saving instrument. Therefore, S-investors would hold the bubble and would not hold money.[3] In the case of autarkic investors, such a bubbly steady state is described by:

$$z = \alpha[(1-\phi)\beta - \phi]y \tag{28}$$

$$k = \beta\alpha y - (1-\beta)z \tag{29}$$

[3] With negative long-run inflation, bubbles would be dominated by money and could never arise in equilibrium.

where (28) is the Euler equation of savers and (29) the aggregate budget constraint of investors. As can be seen from equations (18) and (19), the bubbly steady state is formally equivalent to a liquidity trap steady state with $m^S = z$ and $\theta = 1$. The bubble plays the same role as investor-held money in the liquidity trap, but offers a higher real return. This allows us to derive the following Proposition.

Proposition 5 (Bubbly steady state with autarkic investors) *Suppose* $0 < \phi < \phi_{max}$ *and* $\theta > 1$. *Define* $\phi_B = \beta/(1+\beta)$ *and* $\phi_K = \beta^2/(\theta + \beta^2)$. *We have* $\phi_B > \phi_T > \phi_K$.

(i) *If* $\phi \geq \phi_B$, *there is a unique normal steady state as described by Proposition 1.*

(ii) *If* $\phi_T \leq \phi < \phi_B$, *there is a normal steady state with* $r = \phi/[\beta(1-\phi)] < 1$ *and a bubbly steady state with* $r = 1$.

(iii) *If* $\phi < \phi_T$, *there is a liquidity-trap steady state with* $r = 1/\theta < 1$ *as described in Proposition 1 and a bubbly steady state with* $r = 1$.

(iv) *In the bubbly steady state, the real (nominal) interest rate is given by* $r = 1$ *(*$i = \theta$*),* z/y *is decreasing in* ϕ *and* k *is increasing in* ϕ.

(v) *Capital and output are strictly lower in the bubbly steady state than in the normal steady state. They are lower in the bubbly steady state than in the liquidity-trap steady state when* $\phi_K \leq \phi < \phi_T$ *and larger in the bubbly steady state than in the liquidity-trap steady state when* $\phi < \phi_K$.

Proof. Points (i) to (iv) directly follow from Proposition 1 using the formal equivalence between bubbly steady states and liquidity-trap steady states mentioned in the text. From (28) and (29), we get $k^{1-\alpha} = \alpha\big(\beta - (1-\beta)[(1-\phi)\beta - \phi]\big)$ in the bubbly steady state. Comparing this with the normal and liquidity trap steady states, we get point (v). ■

As the Proposition shows, a bubble can help the economy exit the liquidity trap if $\theta > 1$. The bubble raises the nominal interest rate from $i = 1$ to $i = \theta$. S-investors then substitute the bubble for money in their portfolio. For a given money supply, this also reflates the economy as the price level increases to accomodate the lower money demand.

However, the bubbly steady state is qualitatively similar to a liquidity trap. As with money, holding the bubble takes out resources from investment and output is lower in the bubbly equilibrium than in the normal steady state. In the intermediate case where $\phi_T \leq \phi < \phi_B$, a bubble prevents the downward interest rate adjustment that would restore the normal level of capital and output. In the case of low leverage $\phi < \phi_T$, bubbles increase the real interest rate, which may or may not increase capital and output compared to the liquidity trap. A higher real interest rate decreases the price of liquidity but increases the net liquidity of investors, with an ambiguous total effect on investment depending on the level of net liquidity. This is similar to the ambiguous effect of inflation described in Proposition 4.

5.6 Preference and Growth Shocks

With study the effect of β on output in the case of autarkic investors ($\bar{l} = 0$).[4] We derive the following Proposition:

Proposition 6 (Effect of β on the steady state with autarkic investors) *Define $\beta_T = \theta\phi/(1-\phi)$ and $\phi_{max} = 1/2$.*

If $0 < \phi < \phi_{max}$, then there exists a locally constrained steady state with $r < 1/\beta$.

(i) *If, additionally, $\beta \leq \beta_T$, then the steady state is normal.*

(ii) *If $\beta > \beta_T$, then the steady state is a liquidity trap.*

(iii) *In the normal steady state, the real interest rate r and the nominal interest rate i are decreasing in β, $m^S = 0$ and k is increasing in β.*

(iv) *In the liquidity-trap steady state, the real interest rate r is invariant in β, m^S/y is increasing in β and k is increasing in β.*

Proof. The proof derives from Lemma 1. Note simply that $\beta > \beta_T$ is equivalent to $\phi < \phi_T$, which defines the liquidity trap steady state. We then derive r, k and m^S with respect to β in the normal and liquidity trap steady states. ■

[4] This is without loss of generality as the investors' net debt matters only in the normal economy.

An increase in β makes the long-run interest rate fall, and eventually hit the zero-lower bound. In both the normal and liquidity-trap steady states, an increase in β increases the investors' propensity to save, which increases the capital stock in the long run. As a result, whereas an increase in β can explain the emergence of a liquidity trap, it cannot explain the slowdown in investment. In the presence of trend growth, the same conclusions would hold in case of a growth slowdown. In particular, with lower trend growth, less investment is required to keep the capital stock on its trend. Therefore a given amount of saving leads to an upward shift in the capital intensity of production, and hence in the investment rate.

5.7 Partial Capital Depreciation

We assume here that $\delta < 1$, so that capital depreciates only partially from period to period. For consistency, we focus on the case where investors are net debtors or in autarky $\bar{l} \leq 0$. All our results generalize provided some mild condition on $\bar{l}$, as shown in the following Proposition:

Proposition 7 (Steady state when entrepreneurs are net debtors) *Define* $\phi_{max}(\bar{l}) = (1-[1-\beta(1-\delta)]\bar{l}/\alpha)/2$ *and* $\phi_T(\bar{l}) = (\beta - [\theta - \beta^2(1-\delta)]\bar{l}/\alpha)/[\theta + \beta - (\theta^2 - \beta^2)(1-\delta)\bar{l}/\alpha]$. *If* $\bar{l} \leq 0$ *and* $0 < \phi < \phi_{max}$, *then there exists a locally constrained steady state with* $0 < r < 1/\beta$.

(i) *If, additionally,* $\phi \geq \phi_T$, *then the steady state is normal.*

(ii) *If* $\phi < \phi_T$, *then the steady state is a liquidity trap.*

(iii) *In the normal steady state, the real interest rate* r *and the nominal interest rate* i *are increasing in* ϕ *if* $\bar{l} > -1/\beta(1-\delta)$, *and increasing in* $\bar{l}$ *if* $\bar{l} > -1/[1+\beta(1-\delta)]$, $m^S = 0$, k *is decreasing in* ϕ *and decreasing in* $\bar{l}$ *in the neighborhood of* $\bar{l} = 0$.

(iv) *In the liquidity-trap steady state, the real interest rate* r *and the nominal interest rate* i *are invariant in* ϕ *and* $\bar{l}$, $m^S/\rho k$ *is decreasing in* ϕ *and* $\bar{l}$ *and* k *is increasing in* ϕ *and independent of* $\bar{l}$.

Proof. With partial depreciation, using $f(k) = [\rho(k) - (1-\delta)]k/\alpha$, we can show that the dynamic system at the normal steady state satisfies

$$r\beta(1-\phi)\rho(k) = \phi\rho(k) + \frac{\bar{l}}{\alpha}[\rho(k) - (1-\delta)] \tag{30}$$

$$r = \beta r \left[\rho(k) + \frac{\bar{l}}{\alpha}\left(1 - \frac{1}{\beta r}\right)[\rho(k) - (1-\delta)]\right] \tag{31}$$

We derive r^* as follows. We use (31) to express ρ as a function of r and replace in (30). This gives $P(r) = 0$ where P is a second-order polynomial defined by

$$P(r) = \beta(1-\phi)\left[1 + \beta(1-\delta)\frac{\bar{l}}{\alpha}\right] r^2 - \left(\phi + \frac{\bar{l}}{\alpha}\right) r + \phi(1-\delta)\frac{\bar{l}}{\alpha}$$

This polynomial admits two roots. We have $P(0) = \phi(1-\delta)\frac{\bar{l}}{\alpha} \leq 0$ as $\bar{l} \leq 0$ and $\phi > 0$, so it admits only one positive root, which we then take as our solution for r.

This solution is lower than $1/\beta$ as long as $P(1/\beta) > 0$. This is equivalent to $\phi < \phi_{max}$. Finally, $i = r\theta > 0$ is guaranteed by $P(1/\theta) < 0$, which implies $\phi > \phi_T$. In that case, the economy is normal and follows (30) and (31). This proves result (i). Otherwise, the economy is in a liquidity trap and follows

$$\frac{\beta(1-\phi)}{\theta}\rho = \phi\rho + \frac{\bar{l}}{\alpha}[\rho - (1-\delta)] + \frac{m^S}{k} \tag{32}$$

$$1 = \beta\rho - (\theta - \beta)\left(\frac{\bar{l}}{\alpha}[\rho - (1-\delta)] + \frac{m^S}{k}\right) \tag{33}$$

This proves result(ii).

To establish result (iii), we totally differentiate P with respect to ϕ . Using the fact that r is the upper root of P so that $P'(r) > 0$, we can show that r is increasing in ϕ if and only if

$$\beta\left[1 + \beta(1-\delta)\frac{\bar{l}}{\alpha}\right] r^2 + r - (1-\delta)\frac{\bar{l}}{\alpha} > 0$$

This is the case for $-1/\beta(1-\delta) \leq \bar{l} \leq 0$ as r is positive.

Similarly, we totally differentiate P with respect to $\bar{l}$. Using the fact that r is the upper root of P so that $P'(r) > 0$, we can show that r is increasing in $\bar{l}$ if and only if

$$r - (1-\phi)(1-\delta)\beta^2 r^2 - \phi(1-\delta) > 0$$

As $\beta^2 r^2 < 1$, a sufficient condition is $r > 1 - \delta$. This is the case for $\bar{l} > -1/[1 + \beta(1-\delta)]$, as this guarantees $P(1-\delta) < 0$.

We then express r as a function of ρ using (30) and replace it in (31). We find that $Q(\rho) = 0$ where Q is a second-order polynomial defined by

$$Q(\rho) = \beta\left[\left(\phi + \frac{\bar{l}}{\alpha}\right)\left(1 + \frac{\bar{l}}{\alpha}\right) - (1-\phi)\frac{\bar{l}}{\alpha}\right]\rho^2 - \left(\phi + \frac{\bar{l}}{\alpha}\right)\left[1 + 2\beta(1-\delta)\frac{\bar{l}}{\alpha}\right]\rho + (1-\delta)\frac{\bar{l}}{\alpha}\left[1 + \beta(1-\delta)\frac{\bar{l}}{\alpha}\right]$$

We select the upper root of this polynomial for similar reasons. We thus have

$$\rho = \frac{\left(\phi + \frac{\bar{l}}{\alpha}\right)\left[1 + 2\beta(1-\delta)\frac{\bar{l}}{\alpha}\right] + \sqrt{\left(\phi + \frac{\bar{l}}{\alpha}\right)^2 - \phi(1-\phi)\beta(1-\delta)\left[1 + \beta(1-\delta)\frac{\bar{l}}{\alpha}\right]}}{2\beta\left[\left(\phi + \frac{\bar{l}}{\alpha}\right)\left(1 + \frac{\bar{l}}{\alpha}\right) - (1-\phi)\frac{\bar{l}}{\alpha}\right]}$$

We compute $Q(1/\beta)$ and show

$$Q(1/\beta) = -[1 - \beta(1-\delta)]\frac{\bar{l}}{\alpha}\left[1 - 2\phi - [1 - \beta(1-\delta)]\frac{\bar{l}}{\alpha}\right]$$

This is positive if $\bar{l} < 0$, which implies that $\rho < 1/\beta$.

To study the effect of ϕ on k, we totally differentiate Q with respect to ϕ. Using the fact that ρ is the upper root of Q so that $Q'(\rho) > 0$, we can show that ρ is increasing in ϕ if and only if

$$(\beta\rho - 1) + 2\frac{\bar{l}}{\alpha}\beta[\rho - (1-\delta)] < 0$$

For $\bar{l} < 0$, $\beta\rho < 1$. Besides, as the non-negativity on k imposes $\rho \geq 1 - \delta$, then the second term is also negative in that case. As a result, ρ is increasing in ϕ, which implies that k is decreasing in ϕ.

Similarly, to study the effect of $\bar{l}$ on k, we totally differentiate Q with respect to $\bar{l}$. Using the fact that ρ is the upper root of Q so that $Q'(\rho) > 0$, we can show that ρ is increasing in $\bar{l}$ if and only if

$$\left[1 + 2\beta(1-\delta)\left(\phi + 2\frac{\bar{l}}{\alpha}\right)\right]\rho - 2\beta\left(\phi + \frac{\bar{l}}{\alpha}\right)\rho^2 - (1-\delta)\left[1 + 2\beta(1-\delta)\frac{\bar{l}}{\alpha}\right] > 0$$

This is the case both for $\bar{l} = 0$, for which $\rho = 1/\beta$. Therefore, ρ is increasing in $\bar{l}$ in the neighborhood of $\bar{l} = 0$. Since k is inversely related to ρ, k is decreasing in $\bar{l}$ in the neighborhood of $\bar{l} = 0$.

To derive result (iv), consider Equations (32) and (33), which describe the liquidity-trap steady state. They yield

$$\begin{aligned}
\rho &= \tfrac{\theta}{\beta^2+(\theta^2-\beta^2)\phi} \\
\tfrac{m^S}{k} &= \left[\tfrac{\beta}{\theta}(1-\phi)-\phi-\tfrac{\bar{l}}{\alpha}\right]\rho+(1-\delta)\tfrac{\bar{l}}{\alpha}
\end{aligned}$$

As $\theta > \beta$, ρ is decreasing in ϕ, which implies that k is increasing in ϕ. We can also see that ρ and hence k are independent of $\bar{l}$. Similarly, as $i = 1$ and $r = 1/\theta$ in a liquidity trap, i and r are independent of ϕ and $\bar{l}$. Regarding m^S/k, since ρ is decreasing in ϕ, then m^S/k is decreasing in ϕ. Finally, since ρ is independent of $\bar{l}$ and $\rho > 1-\delta$, then m^S/k is decreasing in $\bar{l}$. ■

5.8 Financial Intermediation

In the benchmark model, money is modeled as *outside money* directly supplied by the government. However, in practice, cash holdings usually take the form of deposits, which are a liability of banks, and could in principle be intermediated to capital investment. This extension shows that this is not the case. At the ZLB, banks are unable to channel deposits to credit-constrained I-investors for the same reason that savers are unable to do so in the benchmark model. Instead, banks increase their excess reserves at the central bank.

Consider a simple model of *endogenous money*. The monetary authority now only controls base money M^0_{t+1}, which is assumed to be made entirely of banks' reserves. Total money M_{t+1} is made of deposits endogenously supplied by banks. In Equations (8) and (9) of the benchmark model, money supply M_{t+1} has then to be replaced by base money M^0_{t+1}.

There is a unit measure of banks owned by the representative worker. Banks receive a charter from the government which allows them to issue deposits M_{t+1}, a zero nominal interest liability that can be used for transactions in the cash-in-advance constraint of workers. On their asset side, banks buy central bank reserves M^0_{t+1} and bonds for a nominal amount $M_{t+1} - M^0_{t+1}$. Banks maximize next-period profits, which they rebate (period by period) to households. In order to limit money creation, the bank charter subjects them to a reserve requirement: their buying of bonds cannot exceed a fraction μ of the net present value of deposits:[5]

$$M_{t+1} - M^0_{t+1} \leq \mu \frac{M_{t+1}}{i_{t+1}}.$$

The market clearing condition for bonds, given by Equation (10) in the benchmark model, has to be modified to account for bond demand by banks:

$$b_{t+1} + l^w_{t+1} + l^g_{t+1} = a_{t+1} + r_{t+1} \frac{M_{t+1} - M^0_{t+1}}{P_t}. \tag{34}$$

It is useful to define $\tilde{M}^0_{t+1}$, an indicator of *excess* reserves of banks, by:

$$\tilde{M}^0_{t+1} = M^0_{t+1} - \big(1 - \frac{\mu}{i_{t+1}}\big) P_t (1-\alpha) Y_t.$$

We obviously have $\tilde{M}^0_{t+1} = 0$ in the normal equilibrium.[6] In the general case, the bond market equilibrium can be rewritten

$$b_{t+1} + l^w_{t+1} + l^g_{t+1} = a_{t+1} + \mu \frac{P_t}{P_{t+1}} (1-\alpha) Y_t + \frac{M^S_{t+1} - \tilde{M}^0_{t+1}}{P_{t+1}}.$$

Note that a fraction μ of workers's money holdings for transaction purposes is channeled by banks to the bond market. At the zero-lower bound, banks are indifferent between buying bonds or reserves, the reserve requirement does not bind, and excess reserves $\tilde{M}^0 \geq 0$.

We can now rewrite the main equations of the benchmark model, the Euler equation (13) and the aggregate budget constraint (14) as

$$\beta\alpha(1-\phi_{t-1}) y_t = \frac{1}{r_{t+1}} \left[(\phi_t \alpha + \bar{l}_t) y_{t+1} - \mu \frac{P_t}{P_{t+1}} (1-\alpha) y_t + \frac{\tilde{M}^0_{t+1}}{P_{t+1}} \right], \tag{35}$$

$$k_{t+1} + \frac{\tilde{M}^0_{t+1}}{P_t} + \bar{l}_t \frac{y_{t+1}}{r_{t+1}} - \mu \frac{P_t}{P_{t+1}} \frac{y_t}{r_{t+1}} = \beta \left[(\alpha + \bar{l}_{t-1}) y_t - \mu \frac{P_{t-1}}{P_t} (1-\alpha) y_{t-1} + \frac{\tilde{M}^0_t}{P_t} \right]. \tag{36}$$

There are only two changes compared to the benchmark model. First, the net supply of bonds

[5] While the precise form of the reserve requirement does not matter, this expression yields tractable results.

[6] When $i_{t+1} > 1$, banks issue as much money and buy as little reserves as they can and the reserve requirement is binding. Banks' reserves are then equal to $M^0_{t+1} = (1 - \mu/i_{t+1}) M_{t+1} = (1 - \mu/i_{t+1})(1-\alpha) P_t y_t$.

from the rest of the economy is decreased by the share μ of workers' deposits lent by banks to investors: $\bar{l}_t y_{t+1}$ has to be replaced by $\bar{l}_t y_{t+1} - \mu P_t / P_{t+1}(1-\alpha) y_t$. Second, money holdings by investors M^S is replaced by excess reserves $\tilde{M}^0$ at the Central Bank. In this extended model, the increase in cash holdings by investors at the zero lower bound shows up as an increase in excess reserves at the Central Bank. Results on the steady state of the benchmark model extend to the case of endogenous money with the simple change of parameter $\bar{l} \to \bar{l} - \mu(1-\alpha)/\theta$.

5.9 Inefficient saving technology

Suppose there is an inefficient storage technology available to savers with return $\sigma \in (\theta^{-1}, \beta^{-1})$. This technology provides an alternative saving instrument to bonds and money holdings. There is an installation cost: investing a fraction x of saving in this technology only yields a fraction $\Psi(x)$ that is actually stored, with Ψ twice differentiable, $\Psi(0) = 0$, $\Psi'(0) = 1$, $\Psi'(1) > 0$, $\Psi'(x) > 0$, and $\Psi''(x) < 0$. For simplicity, we focus on the baseline case of autarkic investors.

Investors in their saving phase choose x to maximize the total return on their saving $\rho^S_{t+1} = (1-x_t) r_{t+1} + \sigma \Psi(x_t)$. When ϕ is large enough so that $r_{t+1} \geq \sigma$, the storage technology is too inefficient to be used. For lower values of ϕ, the storage technology starts being used and the first-order condition with respect to x is $r_{t+1} = \sigma \Psi'(x_t)$. The real interest rate decreases with the use of the inefficient technology. The normal steady state is described by the following equations:

$$\beta\sigma(1-\phi)(1-x)\Psi'(x) = \phi, \tag{37}$$

$$k = \beta\alpha y - \beta\alpha y(1-\phi)[x - \beta\sigma\Psi(x)], \tag{38}$$

where (37) replaces the Euler equation and (38) is the aggregate budget constraint of investors. From (37), it is clear that a lower leverage ϕ is associated with a higher use x of the inefficient storage technology, and therefore with a lower interest rate. From (38), this crowds out investment k in the efficient production technology. It is easy to check that the average productivity of capital invested in both technologies is decreasing in ϕ. This negative reallocative effect of low interest rates on aggregate productivity is similar to the one studied by Buera and Nicolini (2016).

In a liquidity trap equilibrium, the use of the inefficient technology is pinned down by inflation: $\theta\sigma\Psi'(x) = 1$. Then, deleveraging shocks are adjusted by higher real money holdings which crowd out good capital as in the benchmark case, while leaving investment in inefficient storage unaffected. The liquidity trap equilibrium is indeed described by:

$$m^S = \alpha \left[(1-x)(1-\phi)\frac{\beta}{\theta} - \phi \right] y, \tag{39}$$

$$k = \beta\alpha y - \beta\alpha y(1-\phi)[x - \beta\sigma\Psi(x)] - (\theta - \beta)m^S. \tag{40}$$

The key result of the benchmark model remains valid: m^S/y (k/y) decreases (increases) with ϕ.

Note that the storage technology puts a strictly positive lower bound to the shadow rate, contrary to the benchmark model where the shadow rate went to 0 in the limit $\phi \to 0$. Indeed, setting ϕ to 0 in (37), we get $x = 1$, with a corresponding shadow rate $r^S = \sigma\Psi'(1) > 0$.

These results are summarized by the following Proposition.

Proposition 8 (Inefficient storage technology) *Suppose $\theta < 1/[\sigma\Psi'(1)]$. Define $\phi_E = \beta\sigma/(1+\beta\sigma)$ and $\phi_{TE} = \dfrac{\beta\big(1-\Psi'^{-1}(1/\sigma\theta)\big)}{\theta + \beta\big(1-\Psi'^{-1}(1/\sigma\theta)\big)}$. We have $\phi_{max} > \phi_E > \phi_{TE} > 0$.*

(i) *If $\phi_E \le \phi < \phi_{max}$, there is a unique normal steady state with $x = 0$ similar to the one described by Proposition 1.*

(ii) *If $\phi_{TE} \le \phi < \phi_E$, there is a unique normal steady state with $x > 0$, where r and k are increasing in ϕ, and x is decreasing in ϕ.*

(iii) *If $0 \le \phi < \phi_{TE}$, there is a unique liquidity-trap steady state with $r = 1/\theta < 1$ and $x > 0$, where x is invariant in ϕ, m^S/y is decreasing in ϕ, and k is increasing in ϕ.*

(iv) *The shadow rate r^S is increasing in ϕ. When ϕ goes to 0, the shadow rate goes to a lower bound $\sigma\Psi'(1)$ corresponding to $x = 1$.*

Proof. We start by deriving Equations (37) to (40). The optimization problem of investors is the same as in the benchmark model, with the total return ρ^S replacing the interest rate r. With log utility, investors still choose to save a fraction β of their wealth. The demand for bonds and money by saving investors is a fraction $(1-x)$ of their saving $\beta(1-\phi)\alpha y$. In the normal steady state, it has to be equal to the supply of bonds by investors $\phi\alpha y/r$. Using the first-order condition with respect to x, $r = \sigma\Psi'(x)$, we get (37). In the liquidity trap steady state, the demand for bond and money has to be equal to the supply of bonds $\theta\alpha y$ plus real money holdings θm^S, which gives (39).

To get the aggregate budget constraint of investors, note that their aggregate wealth is equal to $\alpha y + \sigma\Psi(x)\beta(1-\phi)\alpha y + m^S$. The first term is profits from the efficient sector, the second term is the return of the inefficient storage technology, and the last term is money holdings. They save a fraction β of this wealth to buy capital k, invest $x\beta(1-\phi)\alpha y$ in the storage technology, and acquire money θm^S. This gives Equations (38) and (40).

The storage technology is not used as long as the first-order condition with respect to x is a corner solution: $-r + \sigma\Psi'(0) \leq 0$. Then, we are in the normal steady state of the benchmark model with $r = \phi/[\beta(1-\phi)]$. With $\Psi'(0) = 1$, the first-order condition becomes $\phi \geq \phi_E$. This proves Point (i). The comparative statics of Point (ii) directly derive from Equations (37) and (38), together with the first-order condition $r = \sigma\Psi'(x)$. Note in particular that $x - \beta\sigma\Psi(x)$ on the right-hand side of (38) is strictly increasing in x. Indeed, its derivative is given by $1 - \beta\sigma\Psi'(x) > 1 - \beta\sigma > 0$ since $\Psi'(x) < \Psi'(0) = 1$.

When the inefficient technology is in use, the shadow rate is the one that solves

$$\beta r^S(1-\phi)\big(1 - \Psi'^{-1}(r^S/\sigma)\big) = \phi$$

where we have substituted the first-order condition with respect to x in (37). It is decreasing in ϕ. For $\phi = 0$, we have $x = 1$ from (37) and the shadow rate is then $r^S = \sigma\Psi'(1)$, which proves Point (iv). The steady state is normal as long as $r^S\theta > 1$. This obtains for $\phi > \phi_{TE}$, which ends proving Point (ii).

The comparative statics of Point (iii) are straightforward given Equations (39) and (40) when $x = \Psi'^{-1}(1/\theta\sigma)$. ■

5.10 Idiosyncratic Uncertainty

In this Appendix we examine a stochastic transition between saving and investing phases. We assume the following 2-state Markov process for individual investors:

- an investor with no investment opportunity at time $t-1$ receives an investment opportunity at time t with probability $\omega \in (0,1]$,
- an investor with an investment opportunity at time $t-1$ receives no investment opportunity at time t.

While investors face some risk at the individual level, they do not face risk in the aggregate, as the fraction of investors with investment opportunity is always $\omega/(1+\omega)$.

A modified aggregate Euler equation of savers Consider an investor i, and denote Ω_t^i her wealth at the beginning of period t. With log utility, her consumption c_t^i is a fraction $1-\beta$ of wealth Ω_t^i, and the Euler equation of an (unconstrained) saver is $1/c_t^i = \beta r_{t+1} \mathrm{E}_t[1/c_{t+1}^i]$, which implies $1/\Omega_t^i = \beta r_{t+1} \mathrm{E}_t[1/\Omega_{t+1}^i]$. For an investor in her saving phase in period t, wealth in period $t+1$ is given by $\Omega_{t+1}^i = a_{t+1}^i + M_{t+1}^i/P_{t+1}$. As there is no aggregate risk, P_{t+1} is known in t, so Ω_{t+1}^i is known in t and we have $\beta\Omega_t^i = \Omega_{t+1}^i/r_{t+1}$. Aggregating over saving investors, we get

$$\beta \int S_t(i)\Omega_t^i di = \frac{1}{r_{t+1}} \int S_t(i)[a_{t+1}^i + M_{t+1}^i/P_t]di = \frac{1}{r_{t+1}}\left(a_{t+1} + \frac{M_{t+1}^S}{P_{t+1}}\right) \tag{41}$$

where $S_t(i)$ is an indicator equal to 1 if investor i has no investment opportunity at time 1 and 0 if she has, and a and M^S denote aggregate bond and money holdings by savers, as in the benchmark model. To compute the left-hand side of (41), note that investors in their saving phase at time t are made of a fraction $1-\omega$ of investors in their saving phase at time $t-1$ and all investors in their investment phase at time $t-1$. The latter enter period t with wealth $\Omega_t^i = \rho_t k_t^i - b_t^i$. This implies:

$$\begin{aligned}\int S_t(i)\Omega_t^i di = (1-\omega)\int S_{t-1}(i)\Omega_t^i di + \int [1 - S_{t-1}(i)]\Omega_t^i di \\ = (1-\omega)\left(a_t + \frac{M_t^S}{P_t}\right) + \rho_t k_t - b_t,\end{aligned}$$

where k and b are aggregate capital and aggregate debt of borrowers. As long as $\rho_t > r_t$, which will be the case in equilibrium, investors with an investment opportunity will leverage up as much as possible until they hit their borrowing constraint. Thus, we have $b_t^i = \phi_{t-1}\rho_t k_t^i$, which aggregates to $b_t = \phi_{t-1}\rho_t k_t = \phi_{t-1}\alpha y_t$. Substituting these expressions back into Equation (41), and using the market-clearing condition (10), we find:

$$\beta(1-\omega)\left[(\phi_{t-1}\alpha+\bar{l}_{t-1})y_t+\frac{M_t^S}{P_t}\right]+\beta\alpha(1-\phi_{t-1})y_t=\frac{1}{r_{t+1}}\left[(\phi_t\alpha+\bar{l}_t)y_{t+1}+\frac{M_{t+1}^S}{P_{t+1}}\right]. \quad (42)$$

This equation extends Equation (13) from the benchmark model to the case of idiosyncratic uncertainty. It only differs by the first term on the left hand side. This term represents demand for saving instruments at time t from savers that were already savers at time $t-1$. The lower ω, the larger the share of savers, the higher this additional demand for saving instruments compared to the benchmark model. The term vanishes when $\omega = 1$.

This is the only difference between the extended model and the benchmark. Indeed, we can aggregate the budget constraints of all investors, regardless of whether they save or borrow, to get the same aggregate budget constraint (14) as in the benchmark model.

Steady state with autarkic equilibrium This extended model behaves quite similarly to the benchmark model. Consider for example the case of autarkic investors ($\bar{l} = 0$) treated in Proposition 1 for the benchmark model. In the extended model, the steady state is determined by:

$$\beta(1-\omega)(\phi\alpha y+m^S)+\beta\alpha(1-\phi)y=\frac{1}{r}(\phi\alpha y+m^S),$$
$$k+(\theta-\beta)m^S=\beta\alpha y.$$

When $\beta/(\theta+\omega\beta) \leq \phi < 1/(1+\omega)$, the steady state is normal with $m^S = 0$, a constant capital stock $k = (\beta\alpha)^{1/(1-\alpha)}$ as in the benchmark model, and

$$r=\frac{\phi}{\beta(1-\omega\phi)}.$$

A lower ω is associated with a lower interest rate: because there are more savers, channeling saving to investment is more difficult and requires a lower interest rate compared to the benchmark model. The interest rate is still strictly increasing in ϕ, but $dr/d\phi$ is increasing in ω: with a larger share of savers (i.e. a lower ω), the interest rate is lower but less responsive to ϕ. Note also that the upper bound on ϕ in the normal equilibrium is larger than $\phi_{\max} = 1/2$: it is easier to have binding borrowing constraints when there are more savers. Likewise, the lower bound is larger than ϕ_T: it is easier to be in the liquidity trap equilibrium when there are more savers.

When $0 < \phi < \beta/(\theta + \omega\beta)$, the steady state is a liquidity trap with $r = 1/\theta$, and

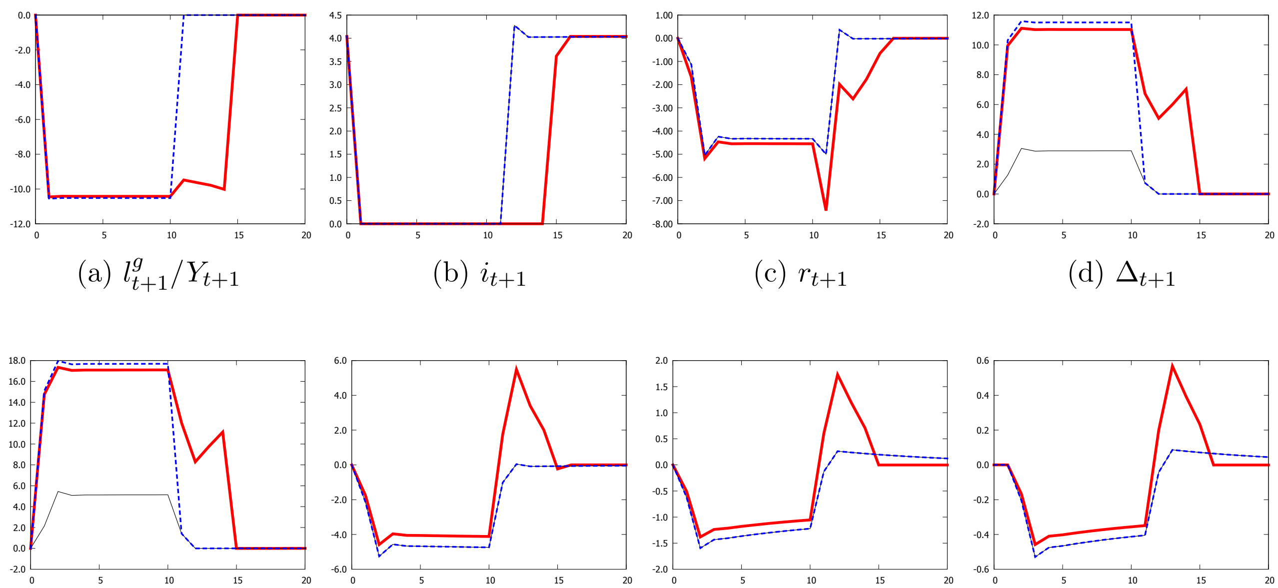
$$
\begin{aligned}
k^{1-\alpha} &= \alpha \, \frac{\omega\beta^2 + \phi\big(\theta^2 - \beta[\omega\beta + (1-\omega)\theta]\big)}{\theta - (1-\omega)\beta}, \\
m^S &= \alpha \left[\frac{(1-\omega\phi)\beta - \phi\theta}{\theta - (1-\omega)\beta} \right] y.
\end{aligned}
$$

A lower ω, that is, a higher share of savers, leads to a stronger demand for money m^S/y and a lower stock of capital k. In the liquidity trap, we get the unusual result that more saving actually leads to less investment. As in Proposition 1, k is strictly increasing in ϕ, and m^S/y is strictly decreasing in ϕ. In addition, $dk/d\phi$ is decreasing in ω and $d(m^S/y)/d\phi$ is increasing in ω. A larger share of savers (i.e. a lower ω) implies steeper slopes of k and m^S/y with respect to ϕ.

Overall, the results we have in the benchmark model become stronger when investment opportunity arrive randomly to saving investors instead of in deterministic way.

5.11 QE easing with expected late exit

In the simulation of quantitative easing with late exit presented in Figure 5 of the main text, the late exit came as a surprise to agents in the model, as it was only announced when the deleveraging shock stops. In Figure 6, late exit is announced from the start instead. As the Figure shows, expectation of a late exit has a slight expansionary effect during the deleveraging shock, supporting capital and output.



(a) l^g_{t+1}/Y_{t+1} (b) i_{t+1} (c) r_{t+1} (d) Δ_{t+1}

(e) M^S_{t+1}/M_{t+1} (f) P_t/θ^t (g) K_{t+1} (h) y_t

Figure 6: Transitory dynamics after an unexpected deleveraging shock with quantitative easing. The shock hits in period 1 and lasts for 10 years. Thick red line: quantitative easing with late exit. The policy of late exit is announced in period 1. Dashed blue line: quantitative easing with early exit. Thin black line: no quantitative easing. All variables are relative deviation from initial steady state, in percent, except interest rates, l/Y and M^s/M which are in absolute deviation from initial steady state, in percent.

6 Welfare

6.1 Intertemporal utility of investors

In a steady state, the utility of investors in their investing phase is given by

$$U = \frac{\log(c^I) + \beta \log(c^S)}{1-\beta^2} = \frac{\log(\beta r) + (1+\beta)\log\big((1-\beta)(1-\phi)\alpha y\big)}{1-\beta^2}.$$

The first term on the numerator reflects consumption smoothing and depends positively on the real interest rate. The second term reflects the level of the whole consumption path and depends on output. In the case of autarkic investors, the difference of utility in the liquidity trap and the normal steady state is then given by:

$$\begin{aligned}(1-\beta^2)(U^{\text{LT}} - U^{\text{normal}}) &= -\log(\theta r^S) + (1+\beta)\frac{\alpha}{1-\alpha}\log((1-\phi)\beta/\theta + \phi\theta/\beta) \\ &= -\log(1-\theta\Delta) + (1+\beta)\frac{\alpha}{1-\alpha}\log\Big(1 - \frac{(\theta-\beta)\Delta}{1+\beta/\theta - \beta\Delta}\Big)\end{aligned}$$

where Δ is the interest rate gap. Both logarithms are strictly negative when $\Delta > 0$. The first term is therefore positive, reflecting better consumption smoothing in the liquidity trap. The second term is negative, due to lower capital and output. However, the second term is first order in $\theta - \beta$, and is likely to be small for any realistic calibration of these parameters, since they are both close to 1. The first term can be rewritten $-\log(1-\beta\Delta) - \log\big(1 - \frac{(\theta-\beta)\Delta}{1-\beta\Delta}\big)$ and is 0-order in $\theta - \beta$. Therefore, unless α is very close to 1, the first term should be strictly larger than the second term.

A similar reasoning applies to the utility of investors in their saving phase.

6.2 Efficient allocations

The following proposition characterizes efficient allocations.

Proposition 9 (Efficient allocations) *An allocation $\{k_{t+1}, c_t^I, c_t^S, c_t^w\}$ satisfying the resource constraint $y_t = k_{t+1} + c_t^I + c_t^S + c_t^w$ is Pareto efficient if and only if $k_{t+1} = \beta\alpha y_t$ and $c_{t+1}^w/c_t^w = c_{t+1}^I/c_t^S = c_{t+1}^S/c_t^I = \beta\rho_{t+1} = y_{t+1}/y_t$.*

Proof. Denote by the superscript 1 (2) the group of investors in their saving (investing) phase in even (odd) periods. A Pareto-efficient allocation maximizes $\sum_{t=0}^{\infty}\beta^t(\lambda^1\log c_t^1+\lambda^2\log c_t^2+\lambda^w\log c_t^w)$ under the resource constraint $y_t=k_{t+1}+c_t^1+c_t^2+c_t^w$, where $\lambda^i, i=1,2,w$ are Pareto weights associated to both groups of investors and workers which sum to 1. The maximization can be carried out in two steps. First, maximize $(c_t^1)^{\lambda^1}(c_t^2)^{\lambda^2}(c_t^w)^{\lambda^w}$ s.t. $c_t^1+c_t^2+c_t^3=C_t$ in any period, which gives constant shares of aggregate consumption $c_t^i=\lambda^i C_t$. Then, maximize $\sum_{t=0}^{\infty}\beta^t\log C_t$ s.t. $y_t=k_{t+1}+C_t$. This well-known maximization problem has the first-order condition $C_{t+1}/C_t=\beta\rho_{t+1}=\beta\alpha k_{t+1}^{\alpha-1}$ and is solved by $k_{t+1}=\beta\alpha y_t$. Having individual consumptions equal to constant shares of aggregate consumption is equivalent to having all individual consumption grow at the same rate, which is also the rate of output growth. ■

We can check that for $\bar{l}\geq\bar{l}_{max}$, the steady state is Pareto-efficient and satisfies the characteristics described in Proposition 9.

6.3 Pareto-improving Policy with Additional Policy Instruments

Consider the following additional policy instruments: a capital subsidy γ_t (such that one unit of capital is paid $1-\gamma_t$ by investors), a consumption tax η_t (such that one unit of consumption costs $1+\eta_t$ to consumers), and a corporate tax τ_t^k paid by S-investors on their profits. With these additional policy instruments, the (binding) borrowing constraint (3) becomes $b_{t+1}=\phi_t(1-\tau_{t+1}^k)\rho_{t+1}k_{t+1}=\phi_t(1-\tau_{t+1}^k)\alpha y_{t+1}$, and Equations (13) and (14) respectively become:

$$\beta\alpha(1-\tau_t^k)(1-\phi_{t-1})y_t=\frac{1}{r_{t+1}}\left[\left((1-\tau_{t+1}^k)\phi_t\alpha+\bar{l}_t\right)y_{t+1}+m_{t+1}^S\right], \tag{43}$$

$$(1-\gamma_t)k_{t+1}+\pi_{t+1}m_{t+1}^S+\frac{1}{r_{t+1}}\bar{l}_t y_{t+1}=\beta\left[\left((1-\tau_t^k)\alpha+\bar{l}_{t-1}\right)y_t+m_t^S\right]. \tag{44}$$

Consumption of all three agents follows:

$$c_t^S=\frac{1}{1+\eta_t}(1-\beta)[\alpha(1-\tau_t^k)y_t-b_t], \tag{45}$$

$$c_t^I=\frac{1}{1+\eta_t}(1-\beta)\left[b_t+\bar{l}_{t-1}y_t+m_t^S\right], \tag{46}$$

$$c_t^w=\frac{1}{1+\eta_t}\left[\frac{T_t^w}{P_t}+\frac{M^w}{P_t}+\frac{l_{t+1}^w}{r_{t+1}}-l_t^w\right]. \tag{47}$$

Finally, the government budget constraint is now:

$$\frac{M_{t+1}}{P_t} + \frac{l_{t+1}^g}{r_{t+1}} + \tau_t^k \alpha y_t + \eta_t (c_t^w + c_t^I + c_t^S) = \frac{M_t}{P_t} + \frac{T^w}{P_t} + l_t^g + \gamma_t k_{t+1}. \tag{48}$$

Consider an economy in a liquidity trap steady state at $t = -1$. Suppose that the government has already implemented an open-market operation to increase debt to the limit of the normal equilibrium, such that $\bar{l}_{-1} = \bar{l}_T(\phi)$ (using the definitions of Proposition 3) and $m_0^S = 0$ in period $t = -1$. The following Proposition shows that an appropriate debt policy, together with the three policy instruments mentioned above, can lead to a Pareto-improving and Pareto-efficient equilibrium from $t = 0$ on.

Intuitively, this policy consists in increasing debt sufficiently to be in the normal equilibrium in all periods. Getting the right level of investment during the transition is done with the capital subsidy. Engineering transfers from investors to workers is done through a consumption tax (together with the lump-sum transfers to workers already assumed in the baseline model). Finally, smoothing investors' consumption during the transition is done through the tax on corporate profits.

Proposition 10 (Pareto-efficient policy) *Consider constant leverage $\{\phi, \bar{l}^w\}$. Suppose the economy is initially in a liquidity trap steady state at $t = -1$ with $\bar{l}_{-1}^g + \bar{l}^w = \bar{l}_T(\phi)$, $m_0^S = 0$ and zero taxes and subsidies: $\gamma_{-1} = \eta_{-1} = \tau_{-1}^k = 0$. Define a policy by a sequence $\{\bar{l}_t^g, \gamma_t, \eta_t, \tau_t^k\}$ for $t \geq 0$ and suppose that $M_{t+1} = \theta M_t$ and that transfers T_t^w adjust the government budget constraint* (48)*. There is a policy such that the associated equilibrium:*

- *is not a liquidity trap ($i_{t+1} > 1$ for all $t \geq 0$),*
- *is Pareto-efficient as described in Proposition 9,*
- *Pareto-improves on the initial steady state.*

Proof. We provide a proof by construction. Consider arbitrary $\lambda^1, \lambda^2, \lambda^w$ in $(0,1)$ with $\lambda^1 + \lambda^2 + \lambda^w = 1$.

Consider now the candidate policy defined in the following way.

$$1 + \eta_t = \frac{\alpha(1-\beta)}{1-\alpha\beta} \frac{\bar{l}_{-1} + \alpha\phi}{\alpha\lambda^2} \qquad t \geq 0,$$

$$
\begin{aligned}
1-\tau_0^k &= \phi + \frac{\lambda^1}{\lambda^2}\Big(\frac{\bar{l}_{-1}}{\alpha} + \phi\Big),\\
1-\tau_1^k &= (1-\tau_0^k)\frac{\phi}{1-\phi} + \frac{1}{1-\phi}\frac{\bar{l}_{-1}}{\alpha},\\
\tau_t^k &= \tau_{t-2}^k \qquad t \geq 2,\\
\gamma_t &= \tau_{t+1}^k \qquad t \geq 0,\\
\bar{l}_0^g &= \alpha(1-\tau_0^k)\frac{1-2\phi}{1-\phi} - \frac{\phi}{1-\phi}\bar{l}_{-1} - \bar{l}^w,\\
\bar{l}_t^g &= \bar{l}_{t-2}^g \qquad t \geq 1.
\end{aligned}
$$

We now show all three statements of the Proposition in turn. First, the equilibrium is normal. Indeed, start looking for a normal equilibrium with $m_t^s = 0$. Plugging the candidate policy into (43) we get $r_{t+1} = y_{t+1}/(\beta y_t)$. From the money market equilibrium (15), we have $P_{t+1}/P_t = \theta y_t/y_{t+1}$. As a result, we get a gross nominal rate $i_{t+1} = P_{t+1}r_{t+1}/P_t = \theta/\beta > 1$ from Assumption 1, which confirms that we are indeed in a normal equilibrium.

Second, plugging the candidate policy together with the equilibrium real interest rate into (44), we get $k_{t+1} = \beta\alpha y_t$. Plugging the policy in (45) and (46), we get $c_{t+1}^I/c_t^S = c_{t+1}^S/c_t^I = y_{t+1}/y_t$. Consumption growth of workers $c_{t+1}^w/c_t^w = y_{t+1}/y_t$ follows from the aggregate resource constraint $y_t = k_{t+1} + c_t^I + c_t^S + c_t^w$. From Proposition 9, this implies that the equilibrium is Pareto efficient.

Finally, plugging the policy into (45) and (46) at $t = 0$ with $b_0 = \phi\alpha y_0$, we get $c_0^S = \lambda^1(y_0 - k_1) = \lambda^1(c_0^S + c_0^I + c_0^w)$ and $c_0^I = \lambda^2(c_0^S + c_0^I + c_0^w)$. This implies that $\lambda^1, \lambda^2, \lambda^w$ are the Pareto weights associated to both groups of investors and workers, where the superscript 1 (2) denotes the group of investors in their saving (investing) phase in even (odd) periods, as in the proof of Proposition 9. As the choice of these weights was not constrained, it is always possible to choose them in such a way that all agents get at least the utility they had in the initial steady state. Therefore, the equilibrium Pareto-improves on the initial steady state. ■

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