Asymmetry in Tail Dependence in Equity Portfolios (Technical Appendix)

Eric Jondeau[*]

Contents

[*]Swiss Finance Institute and University of Lausanne, Faculty of Business and Economics, CH 1015 Lausanne, Switzerland. Tel: +41 21 692 33 49. E-mail: eric.jondeau@unil.ch.

1 Additional details on asymmetric distributions

The random vector X is distributed as a *GH asymmetric t distribution* if its joint density is given by (McNeil, Frey, and Embrechts, 2005):

$$f(x) = c \, \frac{K_{\frac{\nu+n}{2}}\left(\sqrt{(\nu+\phi(x))(\gamma'\Sigma^{-1}\gamma)}\right)\exp\left((x-m)'\Sigma^{-1}\gamma\right)}{\left(\sqrt{(\nu+\phi(x))(\gamma'\Sigma^{-1}\gamma)}\right)^{-\frac{\nu+n}{2}}(1+\phi(x)/\nu)^{\frac{\nu+n}{2}}}, \tag{1}$$

where $\phi(x) = (x-m)'\Sigma^{-1}(x-m)$, $K_\lambda(x)$ is the modified Bessel function of the third kind, and the normalizing constant is given by: $c = (2^{1-\frac{\nu+n}{2}})/((\nu\pi)^{n/2}\Gamma(\nu/2)|\Sigma|^{1/2})$. This distribution is obtained from the GH distribution when the mixing variable W is drawn from an inverse gamma distribution, $Ig\,(\nu/2, \nu/2)$.

The random vector X is distributed as a *non-central t (NCT) distribution* if its joint density is given by (Kshirsagar, 1961; Kotz and Nadarahjah, 2004):

$$\begin{aligned} f(x;\xi) &= \frac{\Gamma((\nu+n)/2)}{(\nu\pi)^{n/2}\Gamma(\nu/2)|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}\gamma'\Sigma^{-1}\gamma\right)\left(\frac{\nu}{\nu+\phi(x)}\right)^{(\nu+n)/2} \\ &\times \sum_{k=0}^{\infty}\frac{2^{k/2}\Gamma((\nu+n+2)/2)}{k!\Gamma\left((\nu+n)/2\right)}\left(\frac{(x-m)'\,\Sigma^{-1}\gamma}{\sqrt{\nu+\phi(x)}}\right)^k. \end{aligned}$$

We denote this process by $X \sim nct_n\,(m,\gamma,\nu,\Sigma)$. The marginal distributions of X are given by $X_i \sim nct_1\,(m_i,\gamma_i,\nu,\sigma_i^2)$, where $\sigma_i^2 = \Sigma_{ii}$. The k-th non-central moment of $\tilde{X}_i = (X_i - m_i)$ is $E[\tilde{X}_i^k] = \nu^{k/2}E[\chi_\nu^{-k}]E[(\gamma_i + Z_i)^k]$, where $Z_i \sim N(0,\sigma_i^2)$.

The moments of a χ_ν distributed variable are given by $E[\chi_\nu^k] = 2^{k/2}\Gamma\left((\nu+k)/2\right)/\Gamma\left(\nu/2\right)$, so that:

$$\begin{aligned} E[\chi_\nu^{-1}] &= \frac{1}{2^{1/2}}\frac{\Gamma\left((\nu-1)/2\right)}{\Gamma\left(\nu/2\right)} \\ E[\chi_\nu^{-2}] &= \frac{1}{2}\frac{\Gamma\left((\nu-2)/2\right)}{\Gamma\left(\nu/2\right)} = \frac{1}{\nu-2} \\ E[\chi_\nu^{-3}] &= \frac{1}{2^{3/2}}\frac{\Gamma\left((\nu-3)/2\right)}{\Gamma\left(\nu/2\right)} \\ E[\chi_\nu^{-4}] &= \frac{1}{2^2}\frac{\Gamma\left((\nu-4)/2\right)}{\Gamma\left(\nu/2\right)} = \frac{1}{(\nu-2)(\nu-4)}. \end{aligned}$$

The first four non-central moments of $\tilde{X}_i$ are given by:

$$
\begin{aligned}
M_1[\tilde{X}_i] &= E[\tilde{X}_i] = \left(\frac{\nu}{2}\right)^{1/2} \frac{\Gamma\left(\left(\nu-1\right)/2\right)}{\Gamma\left(\nu/2\right)}\gamma_i, \\
M_2[\tilde{X}_i] &= E[\tilde{X}_i^2] = \frac{\nu}{\nu-2}(\sigma_i^2+\gamma_i^2), \\
M_3[\tilde{X}_i] &= E[\tilde{X}_i^3] = \left(\frac{\nu}{2}\right)^{3/2} \frac{\Gamma\left(\left(\nu-3\right)/2\right)}{\Gamma\left(\nu/2\right)}\left(\gamma_i^3+3\sigma_i^2\gamma_i\right), \\
M_4[\tilde{X}_i] &= E[\tilde{X}_i^4] = \frac{\nu^2}{\left(\nu-2\right)\left(\nu-4\right)}\left(\gamma_i^4+6\gamma_i^2\sigma_i^2+3\sigma_i^4\right).
\end{aligned}
$$

The variance, skewness, and kurtosis are then defined as:

$$
\begin{aligned}
V[\tilde{X}_i] &= M_2[\tilde{X}_i] - M_1[\tilde{X}_i]^2, \\
S[\tilde{X}_i] &= E\left[\frac{\mu_3[\tilde{X}_i]}{V[\tilde{X}_i]^{3/2}}\right] = E[\tilde{X}_i]\left[\frac{\nu\left(2\nu\sigma_i^2-3\sigma_i^2+\gamma_i^2\right)}{\left(\nu-2\right)\left(\nu-3\right)} - 2V[\tilde{X}_i]\right]/V[\tilde{X}_i]^{3/2}, \\
K[\tilde{X}_i] &= E\left[\frac{\mu_4[\tilde{X}_i]}{V[\tilde{X}_i]^2}\right] = \left[\frac{\nu^2\left(3\sigma_i^4+6\gamma_i^2\sigma_i^2+\gamma_i^4\right)}{\left(\nu-2\right)\left(\nu-4\right)}\right. \\
&\quad \left. -E[\tilde{X}_i]^2\left(\frac{\nu\left[\left(\nu+1\right)\gamma_i^2+3\left(3\nu-5\right)\sigma_i^2\right]}{\left(\nu-2\right)\left(\nu-3\right)} - 3V[\tilde{X}_i]\right)\right]/V[\tilde{X}_i]^2,
\end{aligned}
$$

where $\mu_k[\tilde{X}_i] = E[(\tilde{X}_i - M_1[\tilde{X}_i])^k]$ denotes the k-th central moments of $\tilde{X}_i$. Co-moments can be obtained using the following relations. The (r,s)-th non-central co-moment of $\tilde{X}_i$ is given by:

$$
E[\tilde{X}_i^r\tilde{X}_j^s] = \nu^{(r+s)/2}E[\chi_\nu^{-(r+s)}]E[(\gamma_i+Z_i)^r\left(\gamma_j+Z_j\right)^s].
$$

From the joint normality of Z, we know that $E[Z_iZ_j]=\sigma_{ij}$, $E[Z_iZ_jZ_k]=0$, and $E[Z_iZ_jZ_kZ_l] = \sigma_{ij}\sigma_{kl}+\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk}$. Therefore, the last term on the right-hand-side of $E[\tilde{X}_i^r\tilde{X}_j^s]$ is defined as follows for $r+s=2$, 3, and 4:

$$
\begin{aligned}
& E[(\gamma_i+Z_i)\left(\gamma_j+Z_j\right)] = \gamma_i\gamma_j+\sigma_{ij}, \\
& E[(\gamma_i+Z_i)\left(\gamma_j+Z_j\right)\left(\gamma_k+Z_k\right)] = \gamma_i\gamma_j\gamma_k+\gamma_i\sigma_{jk}+\gamma_j\sigma_{ik}+\gamma_k\sigma_{ij}, \\
& E[(\gamma_i+Z_i)\left(\gamma_j+Z_j\right)\left(\gamma_k+Z_k\right)\left(\gamma_l+Z_l\right)] = \gamma_i\gamma_j\gamma_k\gamma_l+\gamma_i\gamma_j\sigma_{kl}+\gamma_i\gamma_k\sigma_{jl}+\gamma_i\gamma_l\sigma_{jk} \\
& \qquad +\gamma_j\gamma_k\sigma_{il}+\gamma_j\gamma_l\sigma_{ik}+\gamma_k\gamma_l\sigma_{ij}+\sigma_{ij}\sigma_{kl}+\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk}.
\end{aligned}
$$

The first non-central co-moments of $\tilde{X}$ are then given by:

$$
\begin{aligned}
E[\tilde{X}_i\tilde{X}_j] &= \frac{\nu}{\nu-2}(\gamma_i\gamma_j+\sigma_{ij}),\\
E[\tilde{X}_i\tilde{X}_j\tilde{X}_k] &= \left(\frac{\nu}{2}\right)^{3/2}\frac{\Gamma\left(\left(\nu-3\right)/2\right)}{\Gamma\left(\nu/2\right)}\left(\gamma_i\gamma_j\gamma_k+\gamma_i\sigma_{jk}+\gamma_j\sigma_{ik}+\gamma_k\sigma_{ij}\right),\\
E[\tilde{X}_i\tilde{X}_j\tilde{X}_k\tilde{X}_l] &= \frac{\nu^2}{\left(\nu-2\right)\left(\nu-4\right)}(\gamma_i\gamma_j\gamma_k\gamma_l+\gamma_i\gamma_j\sigma_{kl}+\gamma_i\gamma_k\sigma_{jl}+\gamma_i\gamma_l\sigma_{jk}+\gamma_j\gamma_k\sigma_{il}\\
&\quad +\gamma_j\gamma_l\sigma_{ik}+\gamma_k\gamma_l\sigma_{ij}+\sigma_{ij}\sigma_{kl}+\sigma_{ik}\sigma_{jl}+\sigma_{il}\sigma_{jk}),
\end{aligned}
$$

from which we can easily deduce the covariance, co-skewness, and co-kurtosis.

The NCT is able to capture rather large asymmetry in asset returns. Figure 1 shows the range of skewness and kurtosis that can be captured by the NCT. The maximal range is given by $S^2 < K - 1$. For the NCT, the minimum kurtosis is 3, corresponding to normality. Then as the kurtosis increases, larger and larger levels of skewness can be reached.

Second, the NCT is able to generate rather general patterns describing the dependence between the processes. As an illustration, Figure 2 compares the contour plot for the NCT with the GH asymmetric t distribution studied by Mencia and Sentana (2009). For purposes of comparison, the same parameters are used for both distributions: $\nu = 5$, $\rho = 0.5$, together with $\gamma = (-1, -1)'$ and $\gamma = (1, -1)'$. We observe that the NCT has a less extreme behavior in the tails than the asymmetric t distribution. As it appears clearly from top figures, the NCT distribution generates less lower-tail dependence and more upper-tail dependence, such that it produces more balanced levels of tail dependence. As it is shown below, the NCT allows calibrating the magnitude of the tail dependence on both sides of the distribution.

Figure 1: Domain of definition of the NCT

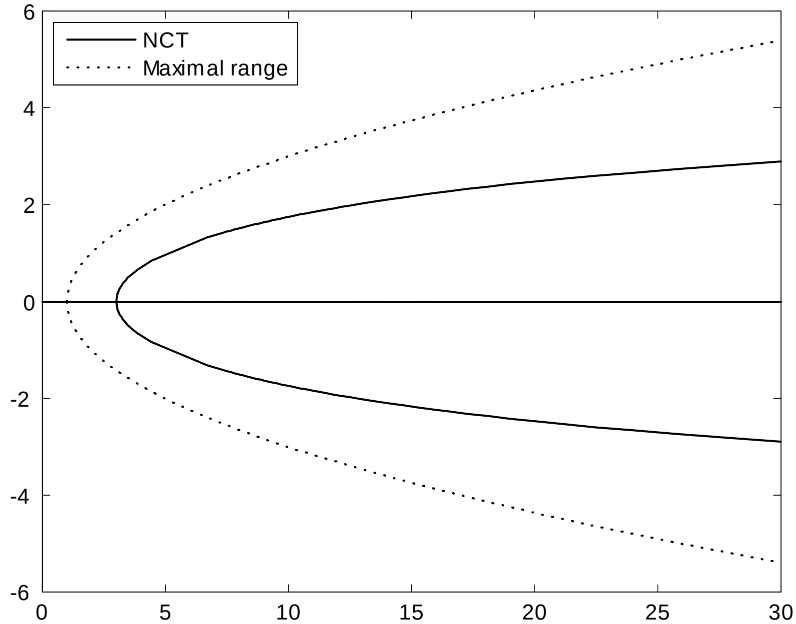
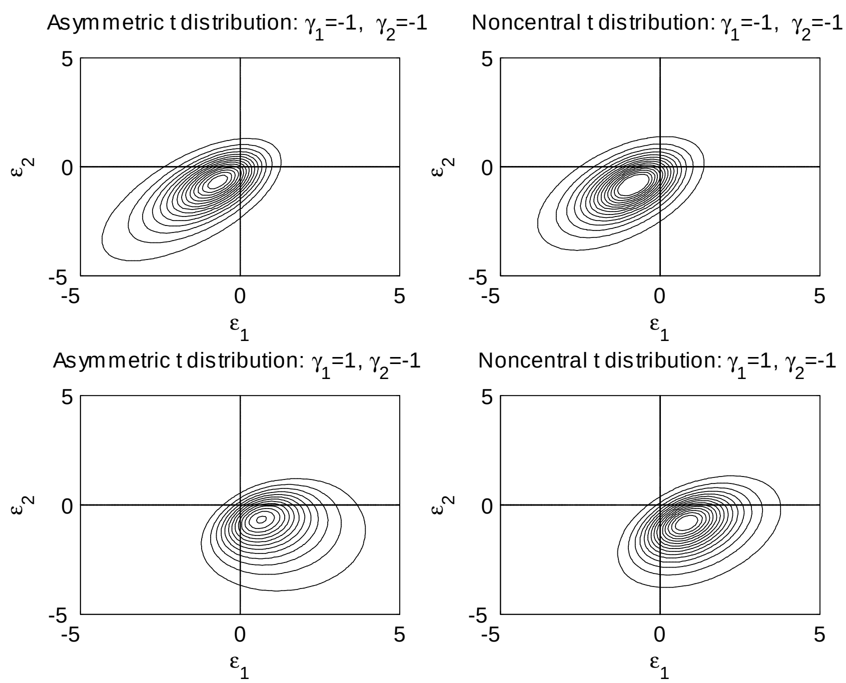


Figure 2: Countour plot of the GH Asymmetric t and the NCT distributions



2 Proof of Theorem 2

This theorem is presented in Gudendorf (2008). We adopt notations close to those already used in Banachewicz and van der Vaart (2008). F_1 and F_2 denote the marginal cdfs of X_1 and X_2, F_1^{-1} and F_2^{-1} the corresponding generalized inverse functions, f_1 and f_2 the pdfs, and $f_{1,2}$ the joint pdf of (X_1, X_2). To determine the TDC of the NCT distribution, we consider the following random variables:

$$X_1 = \frac{\gamma_1 + Z_1}{S/\sqrt{\nu}} \quad \text{and} \quad X_2 = \frac{\gamma_2 + Z_2}{S/\sqrt{\nu}}, \tag{2}$$

where (Z_1, Z_2) is a bivariate normal vector with means 0, variances 1, and correlation ρ, and S^2 is a random variable distributed as a χ^2_ν. The conditional distribution of Z_1 given $Z_2 = z_2$ is normal with mean ρz_2 and variance $1 - \rho^2$. It can be seen as the distribution of the random variable $(\rho z_2 + \sqrt{1-\rho^2}\, Y)$, where Y is a standard normal variable, independent of Z_2 and S. Last, we use the notation: $f(x) \asymp g(x)$ for $x \to a$ if $f(x)/g(x) \to 1$ as $x \to a$.

Defining $g(x) = F_1^{-1} \circ F_2(x)$, using the notation $x = F_2^{-1}(1-u)$, and relying on l'Hôpital's rule, we can rewrite the right tail coefficient as:

$$\begin{aligned}
\lambda^+ &= \lim_{u \downarrow 0} \Pr[F_1(X_1) \geq 1-u \,|F_2(X_2) \geq 1-u] = \lim_{x\to+\infty} \Pr[X_1 \geq F_1^{-1}\circ F_2(x)|X_2 \geq x] \\
&= \lim_{x\to+\infty} \frac{\Pr[X_1 \geq g(x), X_2 \geq x]}{\Pr[X_2 \geq x]} = \lim_{x\to+\infty} \frac{\int_{g(x)}^{\infty}\int_x^{\infty} f_{1,2}(s,t)dt\; ds}{\Pr[X_2 \geq x]} \\
&= \lim_{x\to+\infty} \frac{\int_{g(x)}^{\infty} f_{1,2}(s,x)ds + \int_x^{\infty} f_{1,2}(g(x),t)dt\; g'(x)}{f_2(x)} \\
&= \lim_{x\to+\infty} \left(\Pr[X_1 \geq g(x)|X_2 = x] + \Pr[X_2 \geq x|X_1 = g(x)] \frac{f_1(g(x))g'(x)}{f_2(x)} \right) \\
&= \lim_{x\to+\infty} \left(\Pr[X_1 \geq g(x)|X_2 = x] + \Pr[X_2 \geq x|X_1 = g(x)] \right). \qquad (3)
\end{aligned}$$

The last equality comes from the fact that, using the formula for the differentials of inverse functions and the chain rule, we have:

$$g'(x) = (F^{-1}\circ F_2)'(x) = (F_1^{-1})' \circ F_2(x) F_2'(x) = \frac{1}{F_1'(F_1^{-1}\circ F_2(x))} f_2(x) = \frac{f_2(x)}{f_1(g(x))}.$$

It is easier to work with expression (3) than with the original definition of the right tail coefficient, as we can work immediately with the conditional distributions of X_1 and X_2. We now exploit the fact that the random variables X_1 and X_2 are the quotient of a normal variable over a chi-square variable.

Preliminary results.

For the sake of comprehension, the proof is divided into three preparing lemmas and the main proof. All three lemmas given below are concerned with the asymptotic expansion of the function $g(\cdot)$. The first lemma prepares the field for the following ones and is taken from Banachewicz and van der Vaart (2008).

Lemma 1. *Let F be a cdf of a distribution on $[0,\infty)$ and F^{-1} its quantile function.*

1. *If $1-F(x)\asymp cx^{-k}$ as $x\to\infty$, then $F^{-1}(1-u)\asymp (c/u)^{1/k}$ as $u\downarrow 0$.*
2. *If $F(x)\asymp cx^{k}$ as $x\to 0$ for some $k>0$, then $F^{-1}(u)\asymp (u/c)^{1/k}$ as $u\downarrow 0$.*
3. *If $1-F(x)\asymp cx^{k}e^{-\gamma x}$ as $x\to\infty$ for some $\gamma>0$, then $F^{-1}(1-u)\asymp -\log(u)/\gamma$ as $u\downarrow 0$.*

Proof: See Banachewicz and van der Vaart (2008).

Lemma 2. *As $x\to+\infty$:*

$g(x)\asymp c_{\gamma_1,\gamma_2}\ x,$

with $c_{\gamma_1,\gamma_2}=\left(\frac{d_{\gamma_1}}{d_{\gamma_2}}\right)^{1/\nu}$ *and* $d_\gamma=\frac{\nu^{\nu/2-1}e^{-\gamma^2/2}}{\sqrt{\pi}\Gamma(\nu/2)}\sum_{k=0}^{\infty}\Gamma\left(\frac{\nu+k+2}{2}\right)\frac{(\sqrt{2}\gamma)^k}{k!}$.

Proof. Let define $c=\frac{v^{\nu/2}e^{-\gamma^2/2}}{\sqrt{\pi}\Gamma(\nu/2)}$. Then we have from the NCT distribution:

$$\begin{aligned}
1-F_1(x) &= c\int_x^\infty(\nu+t^2)^{-(\nu+1)/2}\sum_{k=0}^{\infty}\Gamma\left(\frac{\nu+k+2}{2}\right)\frac{(\sqrt{2}\gamma_1)^k}{k!}\left(\frac{t^2}{\nu+t^2}\right)^{k/2}dt\\
&\asymp c\left(\sum_{k=0}^{\infty}\Gamma\left(\frac{\nu+k+2}{2}\right)\frac{(\sqrt{2}\gamma_1)^k}{k!}\right)\int_x^\infty t^{-(\nu+1)}dt \quad \text{as} \quad x\to\infty\\
&= d_{\gamma_1}\left[-\frac{t^{-\nu}}{\nu}\right]_x^\infty = d_{\gamma_1}\ x^{-\nu}.
\end{aligned}$$

Using point (1) of Lemma 1 above, we find that $F_1^{-1}(1-u)=d_{\gamma_1}^{1/\nu}\ u^{-1/\nu},\quad$ as $u\downarrow 0$. We can now calculate the asymptotic expression for $g(x)$:

$$g(x)=F_1^{-1}\circ F_2(x)\asymp d_{\gamma_1}^{1/\nu}(d_{\gamma_2}x^{-\nu})^{-1/\nu}=\left(\frac{d_{\gamma_1}}{d_{\gamma_2}}\right)^{1/\nu}x=c_{\gamma_1,\gamma_2}\ x.\ \square$$

Lemma 3. *As $x\to+\infty$, the asymptotic distribution for $\tilde{S}^2=X_2^2S^2/\nu$ conditional on $X_2=x$ is given by:*

$$p_{\tilde{S}^2|X_2=x}(t)\propto t^{\frac{\nu+1}{2}-1}\exp\left(-\frac{1}{2}\left(t^{1/2}-\gamma_2\right)^2\right). \tag{4}$$

Proof. By Bayes' rule, the conditional density of $S^2|X_2 = x$ is: $p_{S^2|X_2=x}(t) \propto p_{X_2|S^2=t}(x) \times p_{S^2}(t)$. As $Z_2 = X_2\sqrt{S^2/\nu} - \gamma_2$ is distributed $N(0,1)$, we obtain that, conditionally on $S^2 = t$, X_2 is normally distributed with mean $\gamma_2\sqrt{\nu/t}$ and variance ν/t. It follows that:

$$\begin{aligned} p_{S^2|X_2=x}(t) &\propto \sqrt{\frac{t}{\nu}} \exp\left(-\frac{1}{2}\frac{t}{\nu}\left(x - \gamma_2\sqrt{\nu/t}\right)^2\right) t^{\nu/2-1} \exp\left(-\frac{t}{2}\right) \\ &\propto t^{\frac{\nu+1}{2}-1} \exp\left(-\frac{1}{2}\frac{t}{\nu}(x^2+\nu)\right) \exp\left(\gamma_2\nu^{-1/2}t^{1/2}x\right) \exp\left(-\frac{1}{2}\gamma_2^2\right). \end{aligned}$$

Using the relation $\tilde{S}^2 = X_2^2 S^2/\nu$, letting x tend to $+\infty$, and regrouping terms, we deduce that the density of $\tilde{S}^2$ conditional on $X_2 = x$ is asymptotically proportional to:

$$p_{\tilde{S}^2|X_2=x}(\tilde{t}) \propto \tilde{t}^{\frac{\nu+1}{2}-1} \exp\left(-\frac{1}{2}\left(\tilde{t}^{1/2} - \gamma_2\right)^2\right),$$

as $x \to +\infty$. In the following, we denote this distribution by $g_{\nu+1,\gamma_2}(t) \equiv p_{\tilde{S}^2|X_2=x}(t)$. □

In the main proof, we also use the following approximations. Let x be a realization of X_2 and s^2 be a realization of S^2. Then, the equation $x^2s^2/\nu = t$ has the unique positive solution $s = \nu^{1/2}t^{1/2}x^{-1}$, for $x > 0$. Let z_2 be a realization of Z_2 and x be a realization of X_2. We deduce from the definition of X_2 that $z_2 = \nu^{-1/2}xs - \gamma_2$.

Proof of Theorem 2. Let z_2 and s be realizations of Z_2 and S. For the right tail coefficient, we start with the first element on the right-hand-side of equation (3):

$$\begin{aligned} &\lim_{x\to+\infty} \Pr[X_1 \geq g(x)|X_2 = x] = \\ &= \lim_{x\to+\infty} \int \Pr\left[\frac{\gamma_1 + Z_1}{s\nu^{-1/2}} \geq g(x)|X_2 = x, \tilde{S}^2 = t\right] g_{\nu+1,\gamma_2}(t)dt \\ &= \lim_{x\to+\infty} \int \Pr\left[\rho z_2 + \sqrt{1-\rho^2}Y \geq \nu^{-1/2}g(x)s - \gamma_1|X_2 = x, \tilde{S}^2 = t\right] g_{\nu+1,\gamma_2}(t)dt \\ &= \lim_{x\to+\infty} \int \Pr\left[Y \geq \frac{\nu^{-1/2}g(x)s - \gamma_1 - \rho(\nu^{-1/2}xs - \gamma_2)}{\sqrt{1-\rho^2}}\right] g_{\nu+1,\gamma_2}(t)dt \\ &= 1 - \lim_{x\to+\infty} \int_0^\infty \Phi\left(\frac{\nu^{-1/2}s(g(x) - \rho x) - \gamma_1 + \rho\gamma_2}{\sqrt{1-\rho^2}}\right) g_{\nu+1,\gamma_2}(t)dt. \end{aligned}$$

We can exchange the order of limit and integral and replace the asymptotic approximations found above in the normal cdf, so that we obtain:

$$\lim_{x\to+\infty} \Pr[X_1 \geq g(x)|X_2 = x] = 1 - \int_0^\infty \Phi\left(\frac{\nu^{-1/2}sx(c_{\gamma_1,\gamma_2} - \rho) - \gamma_1 + \rho\gamma_2}{\sqrt{1-\rho^2}}\right) g_{\nu+1,\gamma_2}(t)dt$$
$$= 1 - \int_0^\infty \Phi\left(\frac{t^{1/2}(c_{\gamma_1,\gamma_2} - \rho) - \gamma_1 + \rho\gamma_2}{\sqrt{1-\rho^2}}\right) g_{\nu+1,\gamma_2}(t)dt.$$

The second part of equation (3) is obtained in a similar way. It is easy to establish that, for large values of x, an asymptotic approximation for s is $s = \nu^{1/2}t^{1/2}g(x)^{-1}$ and that $z_1 = \nu^{-1/2}g(x)s - \gamma_1$. The density appearing in the integral has the asymptotic form:

$$p_{X_1^2 S^2/\nu|X_1=g(x)}(\tilde{t}) \propto \tilde{t}^{\frac{\nu+1}{2}-1} \exp\left(-\frac{1}{2}(\tilde{t}^{1/2} - \gamma_1)^2\right).$$

Therefore, we deduce:

$$\lim_{x\to+\infty} \Pr[X_2 \geq x|X_1 = g(x)] = 1 - \lim_{x\to+\infty} \int_0^\infty \Phi\left(\frac{\nu^{-1/2}xs - \gamma_2 - \rho z_1)}{\sqrt{1-\rho^2}}\right) g_{\nu+1,\gamma_1}(t)dt$$
$$= 1 - \int_0^\infty \Phi\left(\frac{t^{1/2}(c_{\gamma_1,\gamma_2}^{-1} - \rho) - \gamma_2 + \rho\gamma_1}{\sqrt{1-\rho^2}}\right) g_{\nu+1,\gamma_1}(t)dt.$$

Summing the two terms of equation (3) gives the right TDC given in equation (10).□

3 Additional empirical results

3.1 Model with unfiltered returns

For unfiltered size portfolio return, the NCT provides a good fit of the tail dependence coefficients, but does not pass the goodness-of-fit test for the complete distribution. In contrast, once returns are filtered for AR-GARCH effects, the NCT reproduces the asymmetry in tail dependence found in actual data, while at the same time it passes the goodness-of-fit test.

Table 1 reports the parameter estimates of the NCT for unfiltered returns. For each column, index 1 corresponds to the market return and index 2 to the portfolio return. As the table reveals, almost all of the asymmetry parameters γ_i are negative and highly significant. The asymmetry of the market return, γ_1, is in the range $(-0.09; -0.05)$. For the portfolio returns, the asymmetry is more pronounced for small firms and past winners and is less pronounced for large firms and past losers. We also observe that the degree of freedom, ν, is low for all pairs, ranging from 2.5 to 3.7. This result suggests that the estimated joint distribution is not consistent with a finite kurtosis and (in some cases) a finite skewness, due to unaccounted GARCH effects. Last, the correlations with the market innovation are between 0.78 and 0.99. They are close to the sample correlations reported in Table 1.

The table reports goodness-of-fit test statistics based on hit regressions proposed by Christoffersen (1998) and extended by Patton (2006). The hit test compares the theoretical and empirical number of realizations of pairs of innovations in a set of regions in the unit square. To test if the hits are time independent, they are regressed on past hits (one day, one week, and one month past) using ML estimation. Under the null of time independence, all of the coefficients of past hits are jointly zero for all regions. See Patton (2006) for details. The results reported in the table clearly demonstrate that the estimated models are not able to fit the data. In all cases, the p-value of the hit test is equal to 0.

Finally, the table reports the sample and estimated TDC. It reveals that the NCT performs well in capturing the level of the sample TDC and the sign of the asymmetry between λ^- and λ^+. However, the NCT fails at estimating the size of the observed asymmetry. As already suggested, this failure presumably comes from the inability of the model to capture the time dependency found in actual returns.

The inability to fit the actual characteristics of the data is illustrated in Figure 3, which presents the theoretical and empirical number of realizations for some regions of interest in the unit square, corresponding to the goodness-of-fit test. These regions correspond to $[u_i < F_1(r_{1,t}) < u_{i+1}, u_i < F_2(r_{2,t}) < u_{i+1}]$, with $u \in [0; 0.05; \cdots; 0.95; 1]$. The figure reveals that the model is unable to reproduce the number of realizations for regions in the center of the distribution for most portfolios, in particular for large firms and growth firms.

3.2 Unconditional coverage

In this section, we compare the relative coverage of the various models. The models are the model with unfiltered returns, the model with AR-GARCH filtered returns, and the model with time-varying correlations. We consider the regions of the bivariate distributions with some

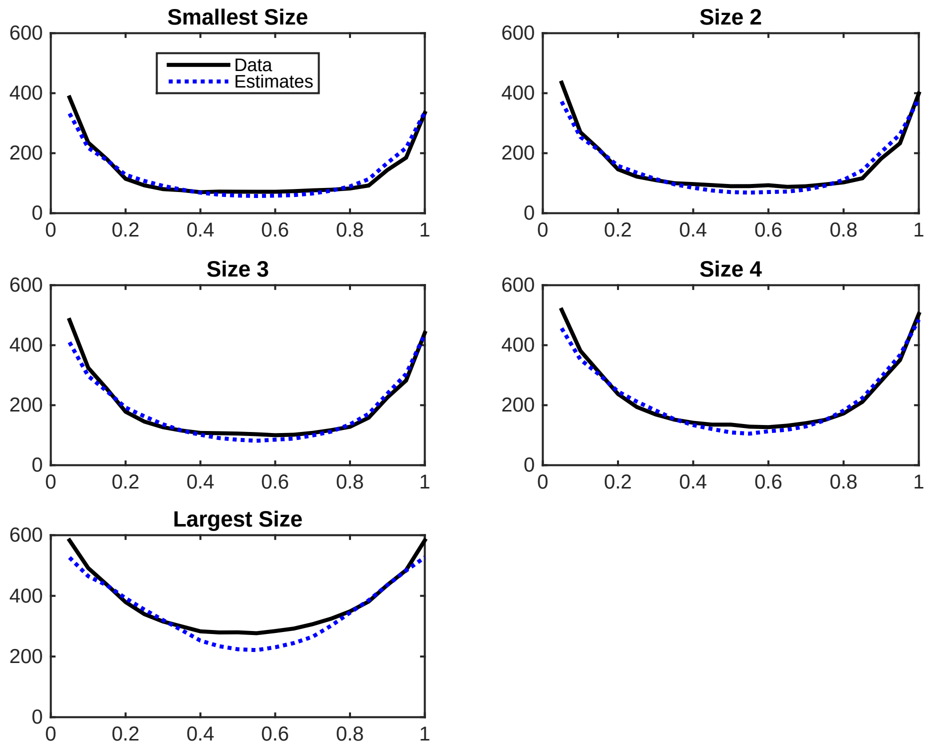
Table 1: Parameter estimates of the NCT (Unfiltered returns)

Note: This table reports parameter estimates of the NCT distribution and measures of goodness-of-fit and of tail dependence for unfiltered returns. These measures include: the log-likelihood of the model; the Hit test statistic of goodness of fit (with p-value in parentheses); the sample and estimated measures of tail asymptotic dependence, $\lambda^{(-)}$ and $\lambda^{(+)}$; and the t-stat for the difference $\hat{\lambda}^{(-)} - \hat{\lambda}^{(+)}$.

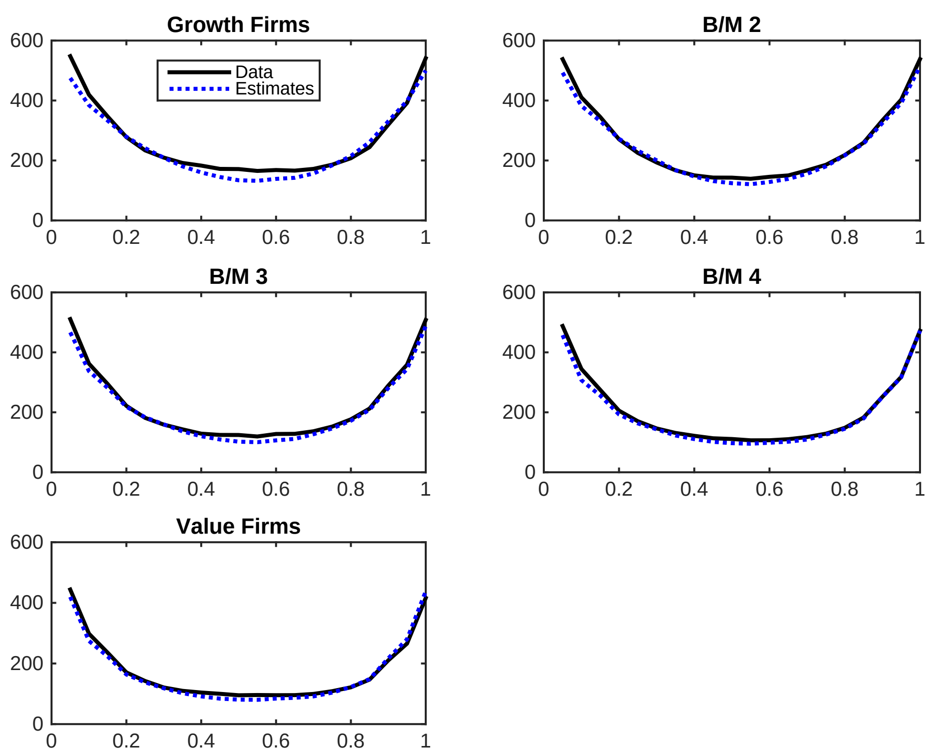
	Smallest		Size 2		Size 3		Size 4		Largest	
γ_1	-0.074	(0.012)	-0.073	(0.015)	-0.080	(0.017)	-0.075	(0.017)	-0.094	(0.015)
γ_2	-0.148	(0.012)	-0.134	(0.016)	-0.134	(0.017)	-0.098	(0.017)	-0.068	(0.015)
ν	2.904	(0.065)	3.050	(0.071)	3.250	(0.078)	3.300	(0.079)	3.205	(0.076)
σ_1	0.352	(0.007)	0.366	(0.007)	0.386	(0.008)	0.393	(0.008)	0.381	(0.008)
σ_2	0.315	(0.007)	0.409	(0.008)	0.396	(0.008)	0.390	(0.008)	0.396	(0.008)
ρ	0.798	(0.004)	0.870	(0.003)	0.917	(0.002)	0.955	(0.001)	0.990	(0.000)
$-\log L$	24903		23792		20702		16753		7140	
Hit test	660.0	(0.000)	740.1	(0.000)	801.6	(0.000)	728.8	(0.000)	961.9	(0.000)
	$\lambda^{(-)}$	$\lambda^{(+)}$	$\lambda^{(-)}$	$\lambda^{(+)}$	$\lambda^{(-)}$	$\lambda^{(+)}$	$\lambda^{(-)}$	$\lambda^{(+)}$	$\lambda^{(-)}$	$\lambda^{(+)}$
sample	0.619	0.472	0.665	0.611	0.758	0.712	0.797	0.805	0.905	0.921
estim.	0.557	0.531	0.633	0.612	0.696	0.679	0.772	0.762	0.895	0.890
$t(\lambda^{(-)} - \lambda^{(+)})$	43.707		20.377		12.760		4.572		0.986	
	Growth		B/M 2		B/M 3		B/M 4		Value	
γ_1	-0.068	(0.022)	-0.072	(0.026)	-0.064	(0.023)	-0.059	(0.018)	-0.061	(0.043)
γ_2	-0.060	(0.025)	-0.058	(0.025)	-0.051	(0.023)	-0.051	(0.017)	-0.054	(0.045)
ν	3.439	(0.084)	3.322	(0.078)	3.077	(0.070)	2.954	(0.065)	2.982	(0.067)
σ_1	0.402	(0.008)	0.402	(0.008)	0.379	(0.007)	0.368	(0.007)	0.367	(0.007)
σ_2	0.496	(0.010)	0.400	(0.008)	0.356	(0.007)	0.327	(0.006)	0.393	(0.008)
ρ	0.970	(0.001)	0.967	(0.001)	0.945	(0.001)	0.928	(0.002)	0.893	(0.002)
$-\log L$	15564		15026		17704		19031		22514	
Hit test	839.7	(0.000)	689.2	(0.000)	712.0	(0.000)	669.5	(0.000)	627.3	(0.000)
	$\lambda^{(-)}$	$\lambda^{(+)}$	$\lambda^{(-)}$	$\lambda^{(+)}$	$\lambda^{(-)}$	$\lambda^{(+)}$	$\lambda^{(-)}$	$\lambda^{(+)}$	$\lambda^{(-)}$	$\lambda^{(+)}$
sample	0.751	0.820	0.867	0.805	0.797	0.743	0.727	0.681	0.689	0.627
estim.	0.809	0.802	0.803	0.796	0.755	0.747	0.724	0.715	0.666	0.655
$t(\lambda^{(-)} - \lambda^{(+)})$	0.962		0.723		0.912		2.109		0.139	
	Past losers		Mom. 3-4		Mom. 5-6		Mom. 7-8		Past winners	
γ_1	-0.045	(0.010)	-0.056	(0.012)	-0.053	(0.013)	-0.065	(0.019)	-0.065	(0.018)
γ_2	-0.016	(0.010)	-0.026	(0.012)	-0.033	(0.012)	-0.066	(0.020)	-0.130	(0.022)
ν	2.338	(0.046)	2.610	(0.053)	2.868	(0.061)	2.923	(0.065)	2.842	(0.062)
σ_1	0.314	(0.006)	0.337	(0.007)	0.366	(0.007)	0.365	(0.007)	0.353	(0.007)
σ_2	0.466	(0.010)	0.344	(0.007)	0.343	(0.007)	0.365	(0.007)	0.513	(0.010)
ρ	0.876	(0.002)	0.937	(0.001)	0.960	(0.001)	0.959	(0.001)	0.926	(0.002)
$-\log L$	25773		19008		15783		16196		22089	
Hit test	801.0	(0.000)	775.0	(0.000)	728.1	(0.000)	572.4	(0.000)	490.4	(0.000)
	$\lambda^{(-)}$	$\lambda^{(+)}$	$\lambda^{(-)}$	$\lambda^{(+)}$	$\lambda^{(-)}$	$\lambda^{(+)}$	$\lambda^{(-)}$	$\lambda^{(+)}$	$\lambda^{(-)}$	$\lambda^{(+)}$
sample	0.576	0.583	0.731	0.716	0.793	0.754	0.754	0.785	0.638	0.638
estim.	0.670	0.664	0.753	0.747	0.795	0.790	0.792	0.784	0.728	0.712
$t(\lambda^{(-)} - \lambda^{(+)})$	8.547		7.644		5.400		7.467		18.391	

Figure 3: Actual and estimated number of realizations in some regions

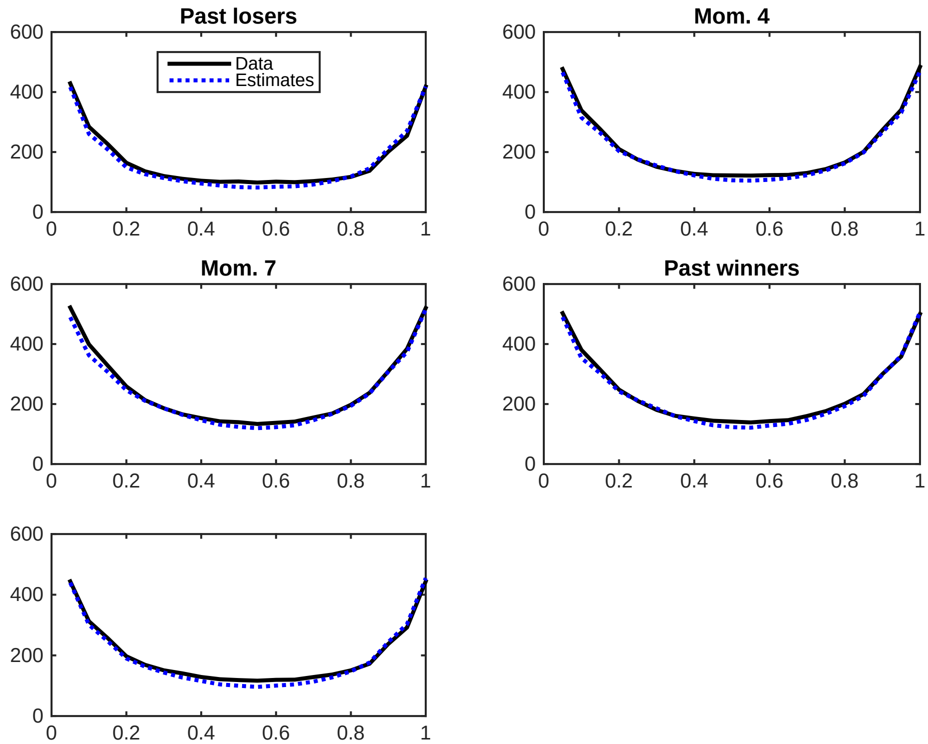
Panel A: Size portfolios - Unfiltered returns



Panel B: B/M portfolios - Unfiltered returns



Panel C: Momentum portfolios - Unfiltered returns

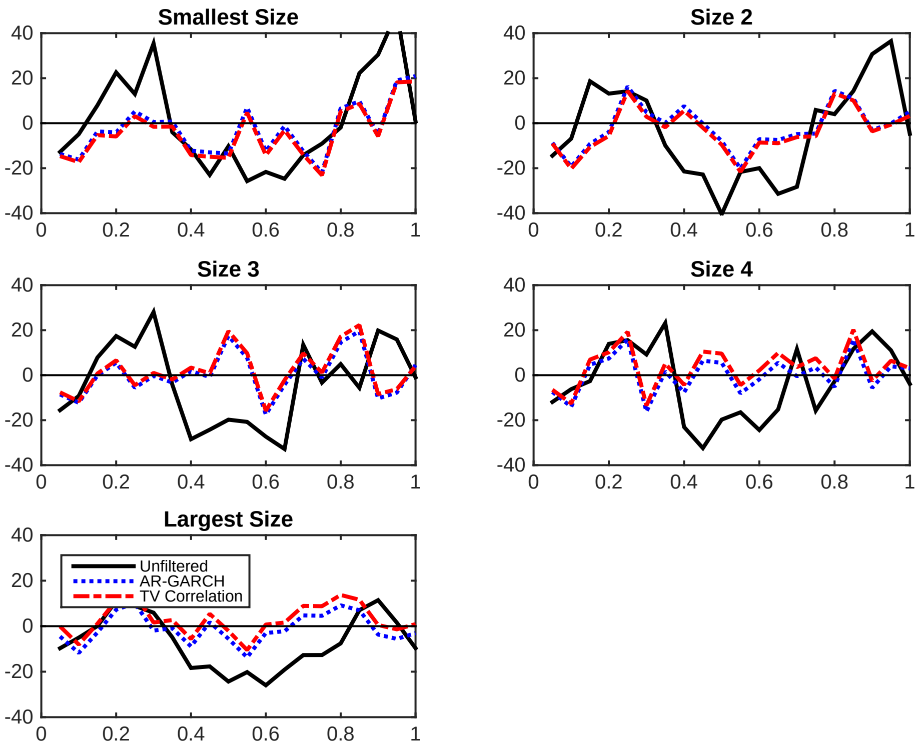


economic interest, i.e., the 20 regions defined by $[u_i < F_{(1)}(r_t^{(1)}) < u_{i+1}, u_i <_{(2)} (r_t^{(2)}) < u_{i+1}]$, with $u \in [0; 0.05; \cdots; 0.95; 1]$. Then, we compute the number of actual observations in each of these regions in the sample, N_i, where i corresponds to a given region. We also compute the number of observations found in these regions for a given model, $\hat{N}_i^{(m)}$ for model m. Finally, we compute the relative coverage of the model for this region, $(\hat{N}_i^{(m)} - N_i)/N_i$.

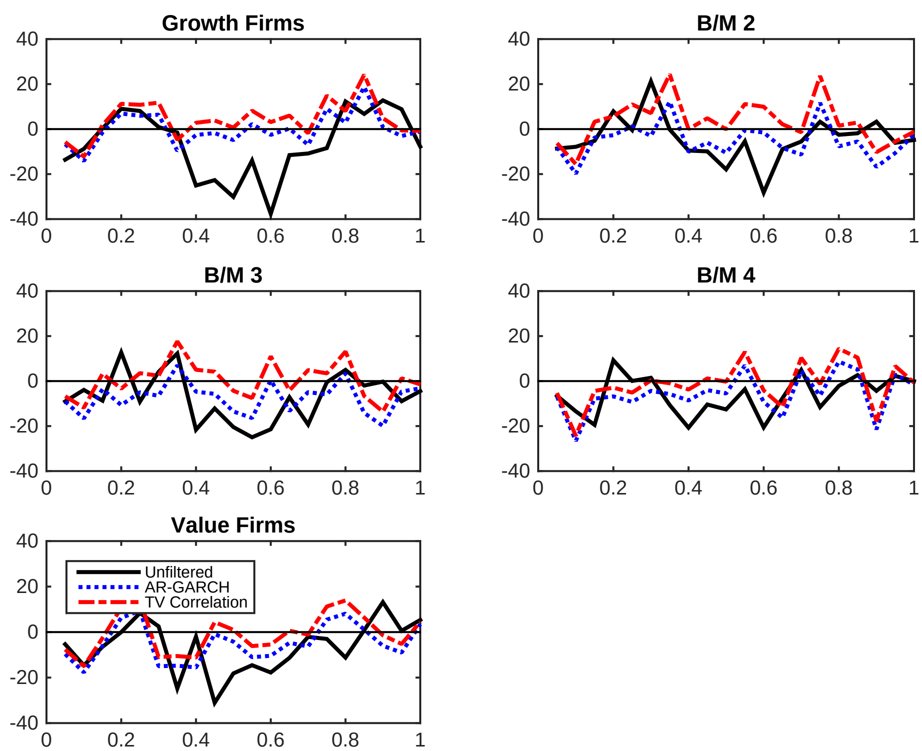
Figure 3 displays the relative coverage for each of the regions for size portfolios, book-to-market portfolios, and momentum portfolios, respectively. We observe that the relative coverage of models with filtered returns is almost always close to the zero line than the relative coverage of models with unfiltered returns. The model with AR-GARCH filtered and the model with time-varying correlations are in general much closer to each other. We notice, however, that the model with time-varying correlations provides a much better fit for book-to-market portfolios and intermediate momentum portfolios.

Figure 3: Relative coverage in the regions of interest

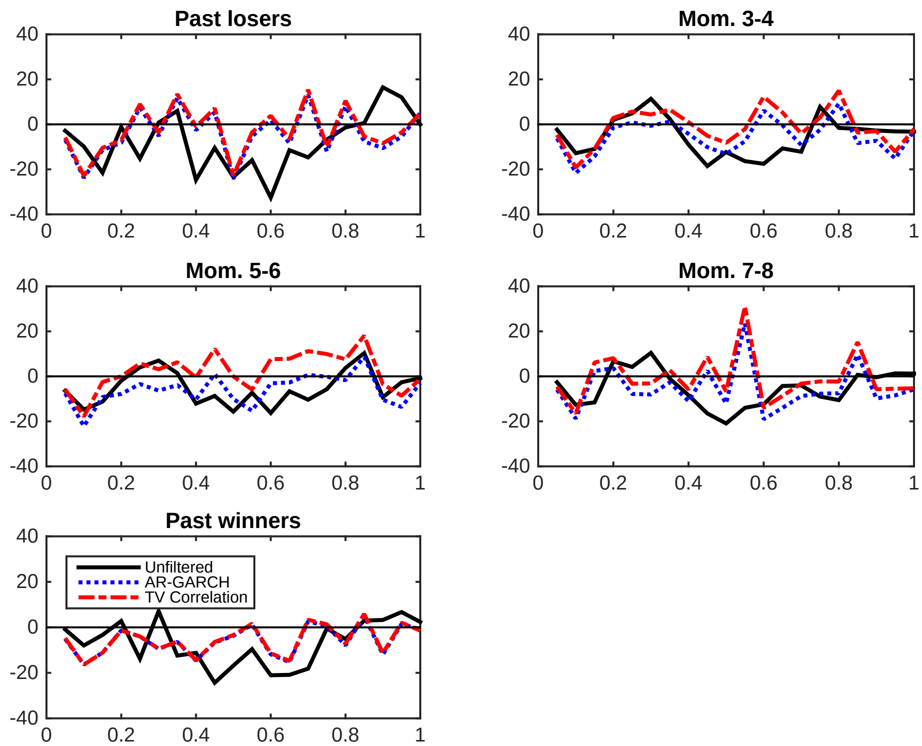
Panel A: Size portfolios



Panel B: B/M portfolios



Panel C: Momentum portfolios



References

Banachewicz, K., and A. van der Vaart, 2008, Tail dependence of skewed grouped t distributions, *Statistics and Probability Letters* 78, 2388–2399.

Christoffersen, P.F., 1998, Evaluating interval forecasts, *International Economic Review* 39, 841–862.

Gudendorf, G., 2008, Tail dependence for the two-dimensional kshirsagar t distribution, Master's thesis University of Lausanne.

Kotz, S., and S. Nadarahjah, 2004, *Multivariate* t *Distributions and their Applications* (Cambridge University Press).

Kshirsagar, A.M., 1961, Some extensions of the multivariate generalization t distribution and the multivariate generalization of the distribution of the regression coefficient, *Proceedings of the Cambridge Philosophical Society* 57, 80–85.

McNeil, A.J., R. Frey, and P. Embrechts, 2005, *Quantitative Risk Management: Concepts, Techniques, and Tools* (Princeton University Press).

Mencia, J., and E. Sentana, 2009, Multivariate location-scale mixtures of normals and mean-variance-skewness portfolio allocation, *Journal of Econometrics* 153, 105–121.

Patton, A., 2006, Modelling asymmetric exchange rate dependence, *International Economic Review* 47, 527–556.