The Dynamics of Squared Returns Under Contemporaneous Aggregation of GARCH Models (Technical Appendix)

Eric Jondeau*

Contents

1 The Minimum Distance Estimator (MDE) 2

2 Performance of the estimators 3

2.1 Calibration based on U.S. equities 3

2.2 Simulation: baseline case 5

2.3 Simulation: robustness check 6

3 Aggregate return and FIGARCH process 8

*Swiss Finance Institute and University of Lausanne, Faculty of Business and Economics, CH 1015 Lausanne, Switzerland. E-mail: eric.jondeau@unil.ch. Phone: (41) 21 692 33 49. Fax: (41) 21 692 33 45.

1 The Minimum Distance Estimator (MDE)

An alternative estimator, which does not rely on QMLE, has been proposed by Baillie and Chung (2001) in the context of a GARCH(1,1) process. This minimum distance estimator (MDE) is motivated by the idea of replicating some properties of the data. A typical example of such properties is the autocorrelogram of the aggregate squared returns. The objective of the estimator is then to minimize the distance between the theoretical autocorrelations and their empirical counterparts. It is described more precisely in the following definition.

Definition 1 *The Minimum-Distance Aggregation-Corrected Estimator MD-ACE(K_Λ, K_Φ,K_ρ), denoted by $\theta_{AC}^{MD} = (\Lambda_1, \cdots, \Lambda_{K_\Lambda}, \Phi_1, \cdots, \Phi_{K_\Phi})'$, is defined as:*

$$\theta_{AC}^{MD} \in \arg\min_{\theta} \quad \left(\hat{\rho} - \rho\left(\theta\right)\right)' W \left(\hat{\rho} - \rho\left(\theta\right)\right), \tag{1}$$

where $\rho\left(\theta\right) = \left(\rho_1(\theta), \cdots, \rho_{K_\rho}(\theta)\right)'$ and $\hat{\rho} = \left(\hat{\rho}_1, \cdots, \hat{\rho}_{K_\rho}\right)'$ denote the first K_ρ theoretical autocorrelations of an ARMA(K_Λ, K_Φ) process and their empirical counterparts, respectively, and W is a weighting matrix. The constant term is estimated by $\Omega_p = (1 - \sum_{k=1}^{K_\Lambda} \Lambda_k) E\left[X_{p,t}\right]$.

A usual choice for the weighting matrix W is a consistent estimate of the inverse of the covariance matrix of $\hat{\rho}$. The theoretical autocorrelations $\rho\left(\theta\right)$ are obtained using the approach described in Brockwell and Davis (1991, section 3.3). The asymptotic distribution of the MDE is given by:

$$\sqrt{T}(\hat{\theta}_{AC}^{MD} - \theta_0) \sim N(0, V_0^{MD}),$$

where $V_0^{MD} = (D_0' C_0^{-1} D_0)^{-1}$ is the asymptotic covariance matrix of $\hat{\theta}_{AC}^{MD}$, C_0 is the asymptotic covariance matrix of $\hat{\rho}$, and $D_0 = \partial \rho(\theta) / \partial \theta$ is evaluated at the true parameter value θ_0.

As the innovation $v_{p,t}$ is not an i.i.d. process, we follow the robust approach proposed by Baillie and Chung (2001). The asymptotic covariance matrix of $\hat{\rho}$, C_0, is estimated using the Newey and West (1987) procedure, $\hat{C}_{NW} = \hat{\gamma}_0^{-2} \hat{V}_v$, where $\hat{\gamma}_0 = (1/T) \sum_{t=1}^{T} v_{p,t}^2$

and

$$\hat{V}_v = \hat{\Gamma}_0 + \sum_{j=1}^{q} \left(1 - \frac{j}{1+q}\right) (\hat{\Gamma}_j + \hat{\Gamma}'_j),$$

with $\hat{\Gamma}_j = (1/T) \sum_{t=j+1}^{T} Z_t^* Z_{t-j}^{*\prime}$ and

$$Z_t^* = \begin{pmatrix} v_{p,t}v_{p,t-1} - \rho_1(\hat{\theta})v_{p,t}^2 \\ \vdots \\ v_{p,t}v_{p,t-K_\rho} - \rho_{K_\rho}(\hat{\theta})v_{p,t}^2 \end{pmatrix}.$$

2 Performance of the estimators

This section aims at evaluating the finite-sample properties of the estimator. This evaluation is based on Monte-Carlo simulations, which reproduce the properties of a large sample of U.S. equities. We find that the ACE provides unbiased estimates of the parameters driving the aggregate squared returns.

2.1 Calibration based on U.S. equities

The calibration of the individual parameters for the Monte-Carlo simulations is based on a sample of 70 U.S. companies between January 1988 and December 2013 for a total of $T = 6,783$ daily observations.[1] This subsection aims at describing the main properties of the parameter estimates, which will be used for the simulations.

Table 1 reports some summary statistics on the individual variance parameters of each of the 70 individual stocks and the covariance parameters of each of the 2,415 pairs of stocks, using the flexible GARCH approach of Ledoit et al. (2003). The cross-section mean estimates of the variance parameters α_i, β_i, and γ_i are 0.054, 0.938, and 0.991, respectively (Panel A), the mean estimates of the covariance parameters α_{ij}, β_{ij}, and γ_{ij} are 0.036, 0.944, and 0.980, respectively (Panel B).[2] For the calibration of the individual parameters, we adjust a Beta distribution to each set of parameters (α_i, α_{ij}, β_i, β_{ij}, γ_i, γ_{ij}).

[1] The sample is composed of all the companies belonging to the S&P 100 at the end of 2013, for which prices were available over the 1988-2013 period. The estimations starts in 1988 because the October 1987 crash was found to affect the estimation of the model.

[2] These numbers are consistent with the restrictions $\alpha_{ij}^2 < \alpha_i\alpha_j$ and $\beta_{ij}^2 < \beta_i\beta_j$, required to ensure the positive semi-definiteness of A and B (Assumption 1). This obviously results in the same observation for the persistence parameter, i.e., $\gamma_{ij}^2 < \gamma_i\gamma_j$.

The table reports the estimates of parameters p and q for each distribution. As there are some significant differences between the characteristics of the variance parameters and the covariance parameters, we mostly focus in the sequel on the covariance parameters, which have the dominant role in the aggregate variance (97.2% of the terms involved). **Figure 1** displays the histogram of the parameters $\{\vartheta_{ij}\}$ and the estimated Beta distribution $f_\vartheta(\vartheta)$, for $\vartheta = \alpha$, β, and γ. The fit of the actual parameters is very good for all the sets of parameters. We notice that the range of values is in fact rather narrow. All the estimates of α_{ij} range between 0.01 and 0.06 and all the estimates of γ_{ij} are above 0.94.

For simulation purpose, another important property is the dependence between the individual parameters. Clearly, we cannot simulate α and β independently from each other because it could imply values of γ larger than 1. To address this issue, we measure the correlation between the individual parameters estimated on U.S. equities (Panel C). The table reveals that γ_{ij} is positively and strongly correlated with β_{ij} but weakly correlated with α_{ij} (0.820 and -0.136, respectively). Therefore, in the Monte-Carlo experiments, we simulate parameters α_i and γ_i from independent Beta distributions with the parameters p and q reported in Panel A. Parameters α_{ij} and γ_{ij} are also drawn from independent Beta distributions with the parameters p and q reported in Panel B. Then, we define $\beta_i = \gamma_i - \alpha_i$ and $\beta_{ij} = \gamma_{ij} - \alpha_{ij}$.

Similarly, simulating ω and γ independently from each other could generate extremely erratic values for $h = \omega/(1-\gamma)$, when γ is close to 1. From Panel C, we notice that the correlation is highly negative between γ_{ij} and ω_{ij} but weakly positive between γ_{ij} and h_{ij} (-0.659 and 0.225, respectively). Therefore, we draw the unconditional variances h_i from a symmetric Beta distribution with $p_{h_v} = q_{h_v} = 3$ in the range $\left[\underline{h}_v, \overline{h}_v\right]$ and the unconditional covariances h_{ij} from a Beta distribution with $p_{h_c} = q_{h_c} = 3$ in the range $[\underline{h}_c, \overline{h}_c]$.[3] We then define the constant terms as $\omega_i = (1-\gamma_i)h_i$ and $\omega_{ij} = (1-\gamma_{ij})h_{ij}$.

[3] More precisely, if $\tilde{h}_i$ is drawn from a standard Beta(p_h,q_h) distribution, the unconditional variance is defined as from $h_i = \underline{h} + (\overline{h} - \underline{h})\tilde{h}_i$, where $\underline{h}$ and $\overline{h}$ denote the minimum and maximum estimates of the unconditional variances, respectively. The choice of $p_h = q_h = 3$ ensures that the resulting h_i are rather dispersed in the interval $\left[\underline{h}, \overline{h}\right]$.

2.2 Simulation: baseline case

For each simulation, samples of variance parameters $(\alpha_i, \gamma_i, h_i)$ and covariance parameters $(\alpha_{ij}, \gamma_{ij}, h_{ij})$ for $i, j = 1, \cdots, N$, are drawn from their respective distribution, as described in the previous subsection. Then N time-series of individual innovations $\{z_{i,t}\}_{t=1,\cdots,T}$ are drawn from a normal $N(0,1)$ distribution and the unexpected returns $\{\varepsilon_{i,t}\}_{t=1,\cdots,T}$ are constructed for $i = 1, \cdots, N$. The portfolio unexpected return $\{\varepsilon_{p,t}\}_{t=1,\cdots,T}$ is obtained by aggregation with portfolio weights $w = (1/N, \cdots, 1/N)'$. Finally, the parameters driving the aggregate squared return $X_{p,t} = \varepsilon_{p,t}^2$ are estimated from aggregate data only.

We consider two alternative estimators in order to evaluate the magnitude of the bias induced by imposing parameter homogeneity when deriving the aggregate squared return dynamics. The first one is the QMLE of the strong GARCH(1,1) process, which implicitly assumes parameter homogeneity:

$$h_{p,t} = \Omega_p + \Psi_1 X_{p,t-1} + \Phi_1 h_{p,t-1}. \tag{2}$$

The second estimator is the Least-Square ACE of the weak GARCH(K_Λ,K_Φ) process, which allows parameter heterogeneity:[4]

$$X_{p,t} = \Omega_p + \sum_{k=1}^{K_\Lambda} \Lambda_k X_{p,t-k} + v_{p,t} - \sum_{k=1}^{K_\Phi} \Phi_k v_{p,t-k}. \tag{3}$$

In the baseline case, the number of observations is $T = 6,000$ and the number of assets is $N = 20$ or 40. Each experiment is based on 1,000 replications. It should be noticed that these simulation experiments are not designed to exactly match all the features observed on U.S. equity returns, but rather to mimic some of their main properties.[5]

Table 2 reports summary statistics of parameter estimates for the QMLE and ACE procedures. We begin with the case $N = 20$, which is a realistic number of asset classes in a strategic allocation approach. For the ACE, we report the estimates of $\{\Omega_p, \Psi_1, \Phi_1, \Lambda_1\}$, for comparability with the QMLE, as well as the estimates of the Beta parameters p and

[4]The Least-Square ACE estimator is based on $K_\Lambda = 10$, 20, and 40 lags and $K_\Phi = 5$ lags, so that the first five terms Φ_i, $i = 1, \cdots, 5$, are freely estimated.

[5]For instance, actual data may be generated by asymmetric GARCH processes and/or fat-tailed innovations. These features are not introduced in the experiment.

q. As expected, the QMLE provides biased estimates of the variance parameters. The most striking result is the severe downward bias in the γ-type parameter ($\Lambda_1 = \Psi_1 + \Phi_1$). The median estimate is 0.835, while the expected value is 0.97. This bias is not due to the estimation of the α-type parameter (Ψ_1), as its median estimate is equal to 0.041, which is rather close to the expected value 0.033 with a narrow confidence interval. On the opposite, the median estimate of the β-type parameter (Φ_1) is far from the expected value (0.794 instead of 0.937) with a large uncertainty across simulations. Increasing the number of assets does not help estimating the persistence parameter, as the value of Λ_1 is still severely underestimated even with $N = 40$ (with a median estimate of 0.777). This result indicates that the QMLE is not able to generate the high persistence found in the simulated aggregate squared returns.

With regard to the ACE, the table reveals that for $N = 20$ the persistence parameter Λ_1 is correctly and precisely estimated to be 0.977 for both values of K_Λ (20 and 40). This result suggests that a moderate number of additional lags is sufficient to correct for the aggregation bias. The parameter Ψ_1 is also very well estimated, with median estimates of 0.034 and 0.039, respectively. Increasing the number of assets in the portfolio does not alter the parameter estimates significantly. This result is important, because it suggests that the ACE is able to reproduce rather closely the properties of the aggregate process, even for a relatively small number of assets.

2.3 Simulation: robustness check

To evaluate the robustness of the results presented above, we performed additional simulation experiments based on alternative assumptions regarding the range of the unconditional correlations, the distribution of the innovation process, and the choice of the portfolio weight vector. All simulation results, based on $T = 6,000$ and $N = 40$ assets, are reported in **Table 3**.[6]

[6] Other experiments essentially left the patterns already described unaltered. In particular, there is no sizable effect on the parameter estimates when the number of lags in the ACE (K_Φ and K_Λ) is increased or when the range of the unconditional variances is widened. The results, not reported in order to save space, are available upon request.

The first experiment relies on the effect of increasing the correlation between the assets. As outlined by Zaffaroni (2006), dynamic conditional heteroskedasticity of the aggregate process requires a sufficiently strong cross-correlation. While the baseline case was calibrated with a moderate positive correlation using the mean value found on U.S. stocks (0.167), this experiment considers the case of highly correlated assets ($\rho_{ij} \in [0.75; 0.9]$). As Panel A reveals, the median estimate of the persistence parameter Λ_1 obtained from the QMLE is slightly larger than in the baseline case (0.804), and therefore far below the expected value. The estimate of Ψ_1 remains close to the expected value. The ACE turns out to be very robust to changes in the range of correlations across assets. The estimate of Λ_1 only slightly decreases towards its expected value.

In the second experiment, the innovation process has a non-normal distribution. Although $z_{i,t}$ has been assumed to be normally distributed so far, it is well known that the empirical distribution of asset returns is often asymmetric and/or fat-tailed. The interaction between the variance dynamics and the distribution properties of returns has been highlighted by Engle (1982) and more recently by He and Teräsvirta (1999). To illustrate the consequences of innovations drawn from distributions with fat tails, Panel B reports the results for a t distribution with 5 degrees of freedom. As expected, the magnitude of the bias in the QMLE is increased. The median estimates of the parameter Λ_1 produced by the QMLE is increased from 0.777 for normal innovations to 0.871 for $t(5)$ innovations. Introducing asymmetry into the innovation distribution through a skewed t distribution does not further affect these parameter estimates with any significance. Again, the properties of the ACE are not altered by the change in the conditional distribution regardless of the number of lags K_Λ. This result is consistent with the fact that the ACE, which is based on Least-Square estimation, does not rely on any particular distributional assumption (provided the innovation's fourth moment is finite).

The last experiment evaluates the effect of the portfolio weights on the performance of the estimators. While the previous simulations were based on equal weights, we consider now a portfolio with short sales allowed: weights are randomly drawn between -0.2 and 0.2, with the sum of the weights equal to 1. Again, the table reveals that the QMLE severely underestimates the persistence parameter, although to a lesser extent (Panel

C). On the opposite, the ACE produces parameter estimates that are very close to the expected values. These results suggest that positivity restrictions on portfolio weights are not required to obtain consistent estimators of the parameters driving aggregate squared returns. The ACE easily accommodates portfolios with different weights or even with short sales.

3 Aggregate return and FIGARCH process

The literature has put forward that squared returns often display some long memory features, in particular through a slowly decreasing autocorrelation function. This result was conjectured by Ding and Granger (1996), in the light of the empirical evidence reported by Ding, Engle, and Granger (1993). Kazakevicius et al. (2004) and Zaffaroni (2006) have shown that in fact long memory does not hold in the n-component specification analyzed by Ding and Granger. The autocorrelation function is indeed slowly decaying, but the aggregate process is still covariance stationary. This result holds provided one assumes the individual processes to be fourth-order stationary. This is admittedly a strong assumption, although it is relatively standard in the literature.

To investigate this issue more in detail, we estimate a FIGARCH(1,d,1) process on the aggregate data, following the approach developed by Baillie et al. (1996). With our notations, we have:

$$(1-L)^d X_{p,t} = \tilde{\Omega}_p + \tilde{\Lambda}_1 (1-L)^d X_{p,t-1} + \upsilon_{p,t} - \tilde{\Phi}_1 \upsilon_{p,t-1}, \quad (4)$$

where $0 < d < 1$ is the parameter of fractional differenciation. When $d > 0$, the process has long memory, in the sense that the sequence of absolute autocorrelations does not converge to a finite limit.

Parameter estimates are reported in Table 4. The long-memory parameter d turns out to be significant with a p-value of 1.3%. However, its value (0.27) is relatively low compared to usual estimates found on other markets (such as commodities or currencies).

This suggests that there are some long memory features, although they are likely to be limited.

Finally, we compare the autocorrelation function obtained with the data to the one obtained with the QMLE, the ACE, and the FIGARCH process. For the FIGARCH process, we simulate a long sample (10 million draws with the parameters reported in the table). The autocorrelation function is displays in Figure 1. As the figure shows, the autocorrelation function of the ACE decreases much more slowly than the strong GARCH process, but faster than the FIGARCH process. The ACE is also closer to the data.

References

[1] Baillie, R.T., Bollerslev, T., Mikkelsen, H.O., 1996. Fractionally integrated generalized autoregressive conditional heteroskedasticity, Journal of Econometrics 74, 3–30.

[2] Baillie, R.T., Chung, H., 2001. Estimation of GARCH models from the autocorrelations of the squares of a process, Journal of Time Series Analysis, 22, 631–650.

[3] Bartlett, M.S., 1946. On the theoretical specification and sampling properties of autocorrelated time-series, Journal of Royal Statistical Society, B8, 27–41.

[4] Brockwell, P.J., Davis, R.A., 1991. Time Series: Theory and Methods, second edition, Springer-Verlag, New York.

[5] Ding, Z., Engle, R.F., 1994. Large scale conditional covariance matrix modeling, estimation and testing, University of California, San Diego Working Paper.

[6] Ding, Z., Granger, C.W.J., 1996. Modeling volatility persistence of speculative returns: A new approach, Journal of Econometrics 73, 185–215.

[7] Engle, R.F., 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation, Econometrica 50, 987–1007.

[8] He, C., Teräsvirta, T., 1999. Properties of moments of a family of GARCH processes, Journal of Econometrics 92, 173–192.

[9] Kazakevičius, V., Leipus, R., Viano, M.-C., 2004. Stability of random coefficient ARCH models and aggregation schemes, Journal of Econometrics, 120, 139–158.

[10] Ledoit, O., Santa-Clara, P., Wolf, M., 2003. Flexible multivariate GARCH modeling with an application to international stock markets, Review of Economics and Statistics 85, 735–747.

[11] Newey, W.K., West, K.D., 1987. A simple, positive semidefinite, heteroskedasticity and autocorrelation consistent covariance matrix, Econometrica 55, 703–708.

[12] Zaffaroni, P., 2006. Contemporaneous aggregation of GARCH processes, Journal of Time Series Analysis 28, 521–544.

Table 1: Summary statistics for the parameter estimates of conditional variance and covariance processes

	α_i	β_i	γ_i	$\omega_i(\times 100)$	$h_i(\times 100)$
Panel A: Conditional variances					
Mean	0.054	0.938	0.991	0.357	0.520
Std dev.	0.020	0.023	0.009	0.288	0.403
Skewness	1.230	-1.066	-3.042	2.372	3.295
Kurtosis	5.462	4.103	14.413	8.837	16.678
p	9.445	149.456	149.045	–	–
std err.	(0.261)	(4.294)	(4.711)		
q	184.909	8.899	1.140	–	–
std err.	(5.236)	(0.249)	(0.029)		
$\tilde{V}^{(w)1/2}$	0.015	0.018	0.007	–	–
$\tilde{S}^{(w)}$	0.598	-0.607	-1.834	–	–
$\tilde{K}^{(w)}$	3.503	3.512	7.970	–	–
	α_{ij}	β_{ij}	γ_{ij}	$\omega_{ij}(\times 100)$	$h_{ij}(\times 100)$
Panel B: Conditional covariances					
Mean	0.036	0.944	0.980	0.128	0.067
Std dev.	0.008	0.014	0.010	0.071	0.033
Skewness	0.415	-0.842	-1.531	1.184	2.073
Kurtosis	3.218	4.110	6.196	5.630	13.849
p	20.624	289.620	227.805	–	–
std err.	(0.602)	(8.187)	(6.743)		
q	548.987	17.351	4.724	–	–
std err.	(16.218)	(0.484)	(0.133)		
$\tilde{V}^{(w)1/2}$	0.008	0.013	0.009	–	–
$\tilde{S}^{(w)}$	0.415	-0.436	-0.886	–	–
$\tilde{K}^{(w)}$	3.247	3.265	4.147	–	–
	α_{ij}	β_{ij}	γ_{ij}	ω_{ij}	h_{ij}
Panel C: Correlation matrix					
α_{ij}	1				
β_{ij}	-0.679	1			
γ_{ij}	-0.136	0.820	1		
ω_{ij}	0.161	-0.581	-0.659	1	
h_{ij}	0.032	0.148	0.225	0.503	1

Note: The parameters of the conditional variance and covariance processes are estimated for U.S. equity returns over the 1988-2013 sample. Parameters are (α_i, β_i, γ_i, ω_i, h_i) for conditional variances (Panel A) and (α_{ij}, β_{ij}, γ_{ij}, ω_{ij}, h_{ij}) for conditional covariances (Panel B). Summary statistics are the mean, standard deviation, skewness, and kurtosis of the empirical distribution, the ML estimates of the parameters p and q of the corresponding Beta distribution (with the standard error in parentheses), and the standard deviation, skewness, and kurtosis implied by the estimated Beta distribution. Panel C provides the cross-correlations between the parameter estimates (α_{ij}, β_{ij}, γ_{ij}, ω_{ij}, h_{ij}) of the conditional covariance processes.

Table 2: Simulation experiments: Estimates of the aggregate GARCH parameters

	Expected value	QMLE	ACE(10, 5)	ACE(20, 5)	ACE(40, 5)
Panel A: $N = 20, T = 6,000$					
Ω_p	0.008	0.025	0.005	0.006	0.006
		(0.040)	(0.008)	(0.002)	(0.013)
Ψ_1	0.035	0.036	0.034	0.032	0.033
		(0.009)	(0.041)	(0.009)	(0.022)
Φ_1	0.939	0.823	0.935	0.941	0.935
		(0.218)	(0.067)	(0.012)	(0.084)
Λ_1	0.973	0.858	0.970	0.970	0.970
		(0.221)	(0.035)	(0.011)	(0.073)
Panel B: $N = 40, T = 6,000$					
Ω_p	0.008	0.031	0.005	0.003	0.005
		(0.034)	(0.008)	(0.006)	(0.009)
Ψ_1	0.035	0.030	0.030	0.030	0.033
		(0.011)	(0.034)	(0.039)	(0.045)
Φ_1	0.939	0.718	0.931	0.931	0.927
		(0.262)	(0.081)	(0.072)	(0.089)
Λ_1	0.973	0.748	0.961	0.961	0.960
		(0.267)	(0.063)	(0.047)	(0.065)

Note: The table provides the median of the GARCH parameter estimates for the QMLE and the ACE. The ACE is based on $K_\Lambda = 10$, 20 and 40 lags and $K_\Phi = 5$ lags. The median of the absolute deviations from the median is reported in parentheses.

Table 3: Simulation experiments: Estimates of the aggregate GARCH parameters – Robustness analysis

	Expected value	QMLE	ACE(10, 5)	ACE(20, 5)	ACE(40, 5)
Panel A: High correlations ($\rho_{ij} \in [0.75; 0.9]$)					
Ω_p	0.008	0.029	0.008	0.008	0.008
		(0.032)	(0.009)	(0.009)	(0.007)
Ψ_1	0.035	0.035	0.035	0.031	0.032
		(0.010)	(0.026)	(0.022)	(0.024)
Φ_1	0.939	0.768	0.909	0.911	0.909
		(0.244)	(0.074)	(0.074)	(0.062)
Λ_1	0.973	0.804	0.944	0.943	0.941
		(0.249)	(0.065)	(0.067)	(0.053)
Panel B: t distribution ($\nu = 5$)					
Ω_p	0.008	0.036	0.007	0.007	0.007
		(0.036)	(0.012)	(0.009)	(0.008)
Ψ_1	0.035	0.043	0.032	0.031	0.030
		(0.010)	(0.011)	(0.012)	(0.010)
Φ_1	0.939	0.828	0.941	0.942	0.942
		(0.165)	(0.080)	(0.034)	(0.035)
Λ_1	0.973	0.871	0.973	0.973	0.973
		(0.168)	(0.049)	(0.030)	(0.032)
Panel C: Alternative weight vector with short sales					
Ω_p	0.008	0.023	0.005	0.005	0.005
		(0.030)	(0.007)	(0.007)	(0.005)
Ψ_1	0.035	0.041	0.036	0.034	0.035
		(0.010)	(0.035)	(0.037)	(0.037)
Φ_1	0.939	0.782	0.923	0.925	0.924
		(0.238)	(0.072)	(0.074)	(0.063)
Λ_1	0.973	0.823	0.959	0.959	0.959
		(0.242)	(0.051)	(0.053)	(0.042)

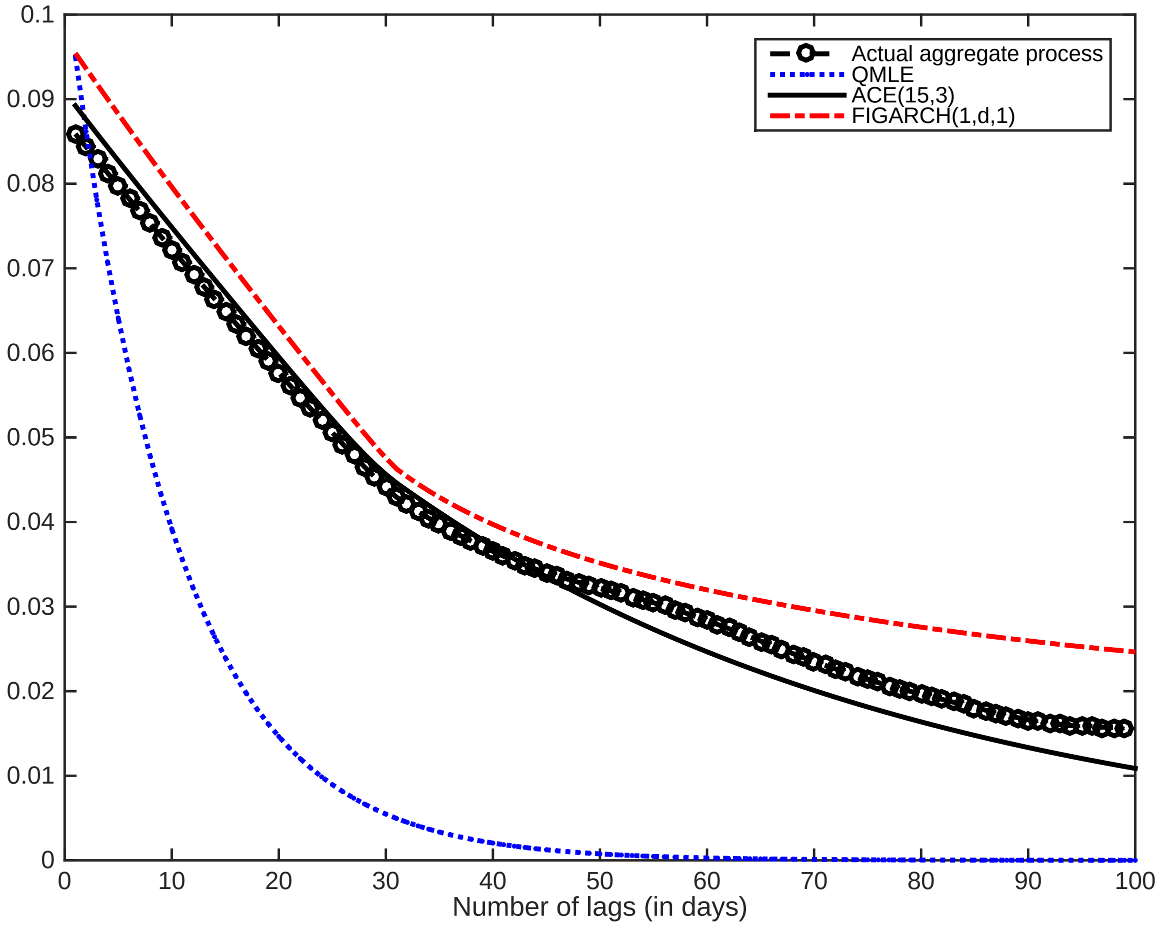
Note: The table provides the median of the GARCH parameter estimates for the QMLE and the ACE under various changes in the baseline experiment. The ACE is based on $K_\Lambda = 10$, 20 and 40 lags and $K_\Phi = 5$ lags. The median of the absolute deviations from the median is reported in parentheses.

Table 4: Estimation of a FIGARCH process on aggregate return

	Parameter estimate	Robust standard error
$\tilde{\Omega}_p$ ($\times 100$)	0.0011	0.0006
$\tilde{\Lambda}_1$	0.2391	0.3940
$\tilde{\Phi}_1$	0.4512	0.4691
d	0.2700	0.1206

Note: The table reports the estimation of a FIGARCH process (Equation (4)) on aggregate returns.

Figure 1: Empirical and estimated ACF of aggregate squared returns



Note: This figure displays the empirical ACF of aggregate squared returns is compared to the estimated ACF obtained from the QMLE, ACE$(15, 3)$, and a FIGARCH(1,d,1) process for the U.S. equity portfolio. The empirical ACF has been smoothed to emphasize its trend.