Lemma 8.1 There exists $K_1 > 0$ such that

$$V(x_2, c) - V(x_1, c) \leq K_1 (x_2 - x_1)$$

for all $0 \leq x_1 \leq x_2$ and $c \leq \min \{\tau, p\}$.

**Proof.** Take $\varepsilon > 0$ and $C \in \Pi_{x_2,c,\tau}$ such that

$$J(x_2; C) \geq V(x_2, c) - \varepsilon.$$  \hspace{1cm} (36)

Then the associated control process is given by

$$X_t^C = x_2 + \int_0^t (p - C_s)ds - \sum_{i=1}^{N_t} U_i,$$

where

$$X_t^C = x_1 + \int_0^t (p - C_s)ds - \sum_{i=1}^{N_t} U_i.$$

Let $\tau$ be the ruin time of the process $X_t^C$. Assume first that $\bar{\tau} \leq p$ and define $\bar{C} \in \Pi_{x_1,c,\tau}$ as $\bar{C}_t = C_t$,

where

$$X_t^{\bar{C}} = x_1 + \int_0^t (p - C_s)ds - \sum_{i=1}^{N_t} U_i.$$

For the ruin time $\bar{\tau} \leq \tau$ of the process $X_t^{\bar{C}}$, it holds $X_t^C - X_t^{\bar{C}} = x_2 - x_1$ for $t \leq \bar{\tau}$. Since $\bar{\tau} \leq p$, ruin can occur only at the arrival of a claim. Hence, using (36) we have

$$V(x_2, c) - V(x_1, c) \leq J(x_2; C) - J(x_1; \bar{C}) + \varepsilon$$

$$\leq E \left[ \int_0^\tau C_s e^{-qs}ds \right] + \varepsilon$$

$$\leq E \left[ \sum_{j=1}^\infty I \{ \bar{\tau} = \tau_j \text{ and } \tau > \tau_j \} \left( \int_{\tau_j}^\tau C_s e^{-qs}ds \right) \right] + \varepsilon$$

With the definitions

$$U_{j-1} := \sum_{i=1}^{j-1} U_i$$

and $A_j^C := \int_0^t (p - C_s)ds$,

we have

$$\{ \bar{\tau} = \tau_j \text{ and } \tau > \tau_j \} = \left\{ x_2 + A_j^C - U_{j-1} \geq U_j > x_1 + A_j^C - U_{j-1} \right\},$$

and by the i.i.d. assumptions $\tau_j, U_j$ and $U_{j-1}$ are mutually independent. This implies

$$E \left[ \sum_{j=1}^\infty e^{-q\tau_j} I \{ \bar{\tau} = \tau_j, \tau > \tau_j \} \right]$$

$$\leq K(x_2 - x_1)^\beta \sum_{j=1}^\infty \left[ \int_0^\infty e^{-qt} \left( \frac{2^{j-1}(j-1)!}{(j-1)!} \right) e^{-\beta t} dt \right]$$

$$\leq K \frac{\beta}{q} (x_2 - x_1),$$

because $F(A_t + x_2 - U_{j-1}) - F(x_1 + A_t - U_{j-1}) \leq K(x_2 - x_1)$. From (37) and (39) we get the result with $K_1 = K\beta\tau/q^2$.

Consider now $c \leq p < \tau$. The main difference in this case is that ruin can occur not only at the arrival of a claim but also if dividends are paid with current surplus zero at a rate greater than $p$.

Let us prove first the result for $c = p$. Consider $C \in \Pi_{x_2,p,\tau}$ as in (36) and

$$T = \min \left\{ t : \int_0^t (C_s - p) ds = x_2 - x_1 \right\}.$$  \hspace{1cm} (40)
We put $T = \infty$ in the event

$$\int_0^\tau (C_s - p) \, ds < x_2 - x_1.$$  

Define $\overline{C} \in \Pi_{x_1, p, \tau}$ as follows: $\overline{C}_t = p$ for $t \leq T$ and then $\overline{C}_t = C_t$ and $\overline{\tau} \leq \tau$ as the ruin time of the controlled process $X_{\overline{T}}^C$. Note that if $T \leq \tau$ we have $X_{\overline{T}}^C = X_{\overline{T}}^C$ because

$$X_{\overline{T}}^C - X_{\overline{T}}^C = x_2 - x_1 + \int_0^{\overline{T}} (p - C_s) \, ds = 0$$

and so $X_{\overline{T}}^C = X_{\overline{T}}^C$ for $T \leq t \leq \tau = \tau$. In the event that $T > \tau$, we have $0 < X_{\overline{T}}^C - X_{\overline{T}}^C \leq x_2 - x_1$ for all $t \leq \tau$, also $\tau$ coincides with the arrival of a claim since $\overline{C}_s = p$ for $s \leq \tau$. Therefore, from (40) and using the proof of (39) we can write

$$V(x_2, p) - V(x_1, p) \leq J(x_2; C) - J(x_1; \overline{C}) + \varepsilon$$

$$= \mathbb{E}[I_{T \leq \tau} \left( \int_0^T (C_s - p) e^{-q_s} \, ds \right)] + \mathbb{E}[I_{T > \tau} \int_0^\tau (C_s - \overline{C}_s) e^{-q_s} \, ds] + \mathbb{E}[I_{T > \tau} \int_0^\tau C_s e^{-q_s} \, ds] + \varepsilon$$

$$\leq 2(x_2 - x_1) + \mathbb{E}[I_{T \leq \tau} \sum_{j=1}^\infty I\left\{ \tau = \tau_j, \tau_j > \tau \right\} \left( \int_{\tau_j}^\tau C_s e^{-q_s} \, ds \right)] + \varepsilon$$

and so we get the result with $K_1 = 2 + \bar{\tau} K \beta / q^2$.

Let us consider now the case $c < p < \tau$, $C \in \Pi_{x_2, \varepsilon, \overline{\tau}}$ as in (36) and define

$$T_1 = \min\{t : C_t \geq p\}.$$

if $C_t \leq p$ for all $t \leq \tau$ then $T_1 = \infty$.

Since $V(\cdot, p)$ is non-decreasing and continuous, we can find (as in Lemma 1.2 of [7]) an increasing sequence $(y_j)$ with $y_1 = 0$ such that if $y \in [y_j, y_{j+1})$ then $0 \leq V(y, p) - V(y_j, p) \leq \varepsilon / 2$; consider admissible strategies $\overline{C}^i \in \Pi_{y_j, p, \tau}$ such that $V(y_j, p) - J(y_j, \overline{C}^i) \leq \varepsilon / 2$. Let us define the dividend payment strategy $\overline{C} \in \Pi_{x_2, \varepsilon, \overline{\tau}}$ as follows: $\overline{C}_t = C_t$ for $t < T_1$ and $\overline{C}_t = \overline{C}^i_{t - T_1}$ for $t \geq T_1$ in the case that $X_{\overline{T}}^C \in [y_j, y_{j+1})$; note that, with this definition, the strategy $\overline{C}$ turns out to be Borel measurable and so it is admissible. With arguments similar to the ones used before, we obtain

$$V(x_2, p) - V(x_1, p) \leq \left( 2 + 2\bar{\tau} K \beta / q^2 \right) (x_2 - x_1).$$

**Lemma 8.2** There exists $K_2 > 0$ such that

$$0 \leq V(x, c_1) - V(x, c_2) \leq K_2 (c_2 - c_1)$$

for all $x \geq 0$ and $0 \leq c_1 \leq c_2 \leq \min\{\tau, p\}$.

**Proof.** Take $\varepsilon > 0$ and $C \in \Pi_{x, c_2, \tau}$ such that

$$J(x; C) \geq V(x, c_1) - \varepsilon$$

and define the stopping time

$$\tilde{T} = \min\{t : C_t \geq c_2\}.$$

Recall that $\tau$ is the ruin time of the process $X_t^C$. Consider first the case $\bar{\tau} \leq p$ and define $\overline{C} \in \Pi_{x_2, \varepsilon, \overline{\tau}}$ as $\overline{C}_t = c_2 I_{t < \bar{\tau}} + C_t I_{t \geq \bar{\tau}}$; denote by $X_{\overline{T}}^C$ the associated controlled surplus process and by $\overline{\tau} \leq \tau$ the corresponding ruin time. Since $\bar{\tau} \leq p$, both $X_t^C$ and $X_{\overline{T}}^C$ are non-decreasing between claim arrivals, and ruin can only occur at the arrival of a claim. We also have that $\overline{C}_s - C_s \leq c_2 - c_1$. We can write

$$V(x, c_1) - V(x, c_2) \leq J(x; C) + \varepsilon - J(x; \overline{C})$$

$$= \mathbb{E}\left[ \int_0^{\tilde{T}} (C_s - \overline{C}_s) e^{-q_s} \, ds \right] + \mathbb{E}\left[ \int_0^{\overline{T}} C_s e^{-q_s} \, ds \right] + \varepsilon$$

$$\leq \mathbb{E}\left[ \sum_{j=1}^{\infty} I\left\{ \tau = \tau_j, \tau_j > \tau \right\} \left( \int_{\tau_j}^{\tau_j} C_s e^{-q_s} \, ds \right) \right] + \varepsilon.$$
Then, 

\[ E \left[ \left( e^{-q\tau} - e^{-q\tau} \right) I_{\{\tau \geq \tau_j, \tau > \tau_j\}} \right] \leq E \left[ e^{-q\tau} I_{\{\tau > \tau_j\}} \right]. \]

Using the definitions given in (38), we have

\[
\begin{aligned}
\{\tau_j = \tau, \tau > \tau_j\} &= \left\{ X_{\tau_j}^C = x + A_{\tau_j}^C - \mathcal{U}_{\tau_j-1} \geq 0 \text{ and } X_{\tau_j}^C = x + A_{\tau_j}^C - \mathcal{U}_{\tau_j-1} < 0 \right\} \\
&= \left\{ x + A_{\tau_j}^C - \mathcal{U}_{\tau_j-1} \geq U_j > x + A_{\tau_j}^C - \mathcal{U}_{\tau_j-1} \right\} \\
&\subseteq \left\{ x + A_{\tau_j}^C + (c_2 - c_1)\tau_j - \mathcal{U}_{\tau_j-1} \geq U_j > x + A_{\tau_j}^C - \mathcal{U}_{\tau_j-1} \right\}.
\end{aligned}
\]

Note that by the i.i.d. assumptions of the compound Poisson process we have that \( \tau_j, U_j \) and \( \mathcal{U}_{\tau_j-1} \) are mutually independent. Hence, 

\[
E \left[ \sum_{j=1}^{\infty} e^{-q\tau_j} I_{\{\tau_j = \tau, \tau > \tau_j\}} \right] \leq K(c_2 - c_1)\beta \sum_{j=1}^{\infty} \left[ e^{-q\tau_j} \left( \frac{\beta t^{j-1}}{j!} \right) t e^{-\beta t} dt \right]
\]

because \( F(x + A_{\tau_j}^C + (c_2 - c_1)\tau_j - u) - F(x + A_{\tau_j}^C - u) \leq (c_2 - c_1)t. \) From (44) and (45) we get the result with \( K_2 = K\beta / q^3. \)

Let us consider now the case \( \bar{c} > p. \) Take \( C \in \Pi_{x,c} \) as in (42) and \( \hat{T} \) as in (43). For

\[ T_1 := \min \{ t : C_t \geq p \}, \]

since \( c_2 \leq p, \) we have that \( T_1 \geq \hat{T}. \) Consider the increasing sequence \( (y_i), \) and the admissible strategies \( \hat{C}_i \in \Pi_{y_i,p} \) introduced in the proof of Lemma 8.1, and define the dividend payment strategy \( \hat{C} \in \Pi_{x,c2} \) as follows: take rate \( c_2 \) for \( t \leq \hat{T}, \) \( C_t \) for \( \hat{T} < t < T_1 \) and for \( t \geq T_1 \) take \( C_t = C_{t-1}, \) in the case that \( X_{t-1}^C \in [y_{i+1}, y_{i+1}]; \) as before, the strategy \( \hat{C} \) turns out to be Borel measurable and so it is admissible. With arguments similar to the ones used before, we obtain,

\[ V(x, c_1) - V(x, c_2) \leq \left( \frac{2}{q} + 2\bar{c} K \frac{\beta}{q^3} \right) (c_2 - c_1). \]

Proof of Proposition 3.3

Proposition 3.3 follows from the following two lemmas:

**Lemma 8.3** Assume that \( \bar{c} > p, \) then there exist constants \( K_2 > 0 \) and \( K_3 > 0 \) such that

\[ V(x, c) - V(x_1, c) \leq \left[ K_2 + \frac{K_3}{c - p} \right] (x_2 - x_1) \]

for all \( 0 \leq x_1 \leq x_2 \) and \( p < c \leq \bar{c}. \)

**Proof.** Take \( \varepsilon > 0 \) and \( C \in \Pi_{x,c}, \) such that

\[ J(x_2, C) \geq V(x_2, c) - \varepsilon. \]

Define \( \hat{C} \in \Pi_{x,c}, \) as \( \hat{C}_1 = C_t, \) and let us call \( \tau \) the ruin time of the process \( X_t^C, \) then \( X_t^C - X_t = x_2 - x_1 \) for \( t \leq \tau. \) Hence, using (46) and (39) we have,

\[
\begin{aligned}
V(x_2, c) - V(x_1, c) &= E \left[ \int_0^\tau C_s e^{-q\tau} ds \right] + \varepsilon \\
&\leq E \left[ \sum_{j=1}^{\infty} I_{\{\tau_j = \tau, \tau > \tau_j\}} \int_0^\tau C_s e^{-q\tau} ds \right] + E \left[ \sum_{j=1}^{\infty} I_{\{\tau_j = \tau, \tau > \tau_j\}} \int_0^\tau e^{-q\tau} C_s ds \right] + \varepsilon \\
&\leq \frac{E}{q} E \left[ \sum_{j=1}^{\infty} e^{-q\tau} I_{\{\tau_j = \tau, \tau > \tau_j\}} \right] + \frac{E}{q} E \left[ \sum_{j=1}^{\infty} I_{\{\tau_j = \tau, \tau > \tau_j\}} \right] e^{-q\tau} \left( e^{-q\tau} - e^{-q\tau} \right) + \varepsilon \\
&\leq \frac{2}{q} K_2 (x_2 - x_1) + \varepsilon.
\end{aligned}
\]
Lemma 8.4 Assume that \( \tau \in (\tau_j-1, \tau_j) \) (and so \( \tilde{\tau} < \tau \)). Then

\[
0 = X^C_{\tilde{\tau}} = x_1 + \int_0^{\tilde{\tau}} (p - C_s) \, ds - \sum_{k=1}^{j-1} U_k \quad \text{and} \quad 0 \leq X^C_{\tau} - x_2 + \int_0^{\tau} (p - C_s) \, ds - \sum_{k=1}^{j-1} U_k.
\]

Hence, we get

\[
0 \leq X^C_{\tau} - X^C_{\tilde{\tau}} \leq x_2 - x_1 + \int_{\tilde{\tau}}^{\tau} (p - C_s) \, ds \leq x_2 - x_1 + (p - c)(\tau - \tilde{\tau})
\]

and this implies

\[
\tau - \tilde{\tau} \leq \frac{x_2 - x_1}{c - p}.
\]

We also have

\[
E[\sum_{j=1}^{\infty} e^{-q\tau_j-1}] = 1 + \int_0^\infty e^{-q\beta \sum_{k=1}^{\infty} (\beta^{k-1}, k-1)!} e^{-\beta s} \, ds \leq 1 + \beta/q.
\]

So, from (47), (48), (49) and (50), we get the result with \( K_2 = \varepsilon K \beta/q^2 \) and \( K_3 = \varepsilon (1 + \beta/q) \).

**Lemma 8.4** Assume that \( \bar{\tau} > p \), then there exist constants \( K_2 > 0 \) and \( K_3 > 0 \) such that

\[
V(x, c_1) - V(x, c_2) \leq \left[ K_2 + \frac{K_3 x}{(c_1 - p)^2} \right] (c_2 - c_1)
\]

for all \( x \geq 0 \) and \( p < c_1 \leq c_2 \leq \bar{\tau} \).

**Proof.** If \( x = 0 \), \( V(x, c) = 0 \) for all \( c > p \). Consider now \( x > 0 \) and \( p < c_1 < c_2 \leq \bar{\tau} \). Take \( \varepsilon > 0 \) and \( C \in \Pi_{x, c_1, \bar{\tau}} \) such that \( J(x; C) \geq V(x, c_1) - \varepsilon \); we define the admissible strategy

\[
\hat{T} = \min\{t : C_t \geq c_2\}.
\]

\( \bar{C} \in \Pi_{x, c_2, \bar{\tau}} \) as \( \bar{C}_t = c_2 I_{\{t \leq \hat{T}\}} + C_t I_{\{t > \hat{T}\}} \), and the ruin times \( \tau \) and \( \bar{\tau} \) of the processes \( X^C_t \) and \( X^\bar{C}_t \) respectively. In this case both \( \tau \) and \( \bar{\tau} \) are finite with \( \tau \geq \bar{\tau} \). Note that

\[
\tau \leq \frac{x}{c_1 - p}.
\]

Let us define as \( T_0 = \min\{t : x + \int_0^t (p - C_s) \, ds = 0\} \) as the ruin time of the controlled process \( X^\bar{C}_t \).

In the event of no claims, we have \( \bar{\tau} \leq T_0 \). Since \( \bar{\tau} \geq \bar{C}_t \geq c_2 > p \), \( T_0 \) is finite and satisfies

\[
\frac{x}{c_2 - p} \leq T_0 \leq \frac{x}{c_2 - p}.
\]

So we have

\[
0 \leq \int_0^t (\bar{C}_s - C_s) \, ds \leq \begin{cases} (c_2 - c_1)t & \text{if } t \leq \hat{T} \\ (c_2 - c_1)\hat{T} & \text{if } t > \hat{T} \end{cases},
\]

and then

\[
X^C_t \leq X^C_{\tau} \leq X^C_{\bar{T}} \leq (c_2 - c_1)\bar{\tau} \leq (c_2 - c_1)T_0 \leq (c_2 - c_1) \frac{x}{c_2 - p}.
\]
We can write, using (45),
\[
V(x, c_1) - V(x, c_2) 
\leq \frac{\varepsilon}{p} \sum_{j=1}^{\infty} \mathbb{E} \left[ I_{\{r_j < \tau, \tau > r_j\}} e^{-q r_j} \right] + \frac{\varepsilon}{p} \sum_{j=1}^{\infty} \mathbb{E} \left[ I_{\{r_j < \tau, \tau > r_j\}} \right] \left( e^{-q \tau} - e^{-q r_j} \right) + \varepsilon
\]
\[
\leq \frac{\varepsilon}{q} (c_2 - c_1) + \frac{\varepsilon}{p} \sum_{j=1}^{\infty} \mathbb{E} \left[ I_{\{r_j < \tau, \tau > r_j\}} \right] \left( e^{-q \tau} - e^{-q r_j} \right) + \varepsilon.
\]  
(54)

In the case that \( \tau \in (\tau_j-1, \tau_j) \) and \( \tau > \tau \), we have that
\[
X_{\tau}^C + \int_{\tau}^{\tau} (p - c_1) ds \geq X_{\tau}^C \geq 0.
\]

Then we get, from (53),
\[
0 \leq \tau - \tau \leq \frac{X_{\tau}^C}{c_1 - p} \leq \frac{x}{(c_1 - p)(c_2 - p)} (c_2 - c_1).
\]  
(55)

Hence, by virtue of (48), (50) and (55),
\[
\frac{\varepsilon}{q} \sum_{j=1}^{\infty} \mathbb{E} \left[ I_{\{r_j < \tau, \tau > r_j\}} \right] \left( e^{-q \tau} - e^{-q r_j} \right)
\leq \frac{\varepsilon}{q} \sum_{j=1}^{\infty} \mathbb{E} \left[ I_{\{r_j < \tau, \tau > r_j\}} \right] q (\tau - \tau) e^{-q r_j-1}
\leq \sum_{j=1}^{\infty} \mathbb{E} \left[ I_{\{r_j < \tau, \tau > r_j\}} \right] e^{-q r_j-1} \leq \frac{\varepsilon}{(c_1 - p)(c_2 - p)} (c_2 - c_1)
\]
\[
= \frac{\varepsilon}{(c_1 - p)(c_2 - p)} \left( 1 + \frac{2}{q} \right) (c_2 - c_1).
\]
Therefore, from (54) the result is established with \( K_2 = \varepsilon q K_2 \) and \( K_3 = \varepsilon (1 + \beta / q) \).

**Proof of Proposition 3.4**

The proof of Proposition 3.4 is quite technical. In addition to some technical lemmas below, we will use the exponential inequality
\[
e^{-\frac{\alpha}{\eta}} \leq e^{-\frac{\alpha}{(\eta \gamma)^{1/\gamma}}}
\]  
(56)
for \( \alpha > 0, \gamma > 0 \) and \( \eta > 0 \), as well as the following elementary remark about convolutions of independent distribution functions.

**Remark 8.1** The distribution function \( F_j \) of the random variable \( U_j = U_1 + ... + U_j \) is Lipschitz with the same Lipschitz constant as \( F \). To see this, consider \( U_j = U_1 + U_2 \). Then
\[
P(a \leq U_1 + U_2 \leq a + h) = \int_{a-h}^{a+h} \int_{a-u}^{a+h-u} dF(v) dF(u) \leq K h \int_{0}^{a+h} dF(u) \leq Kh.
\]

With a recursive argument the proof extends to all \( U_j \) for \( j \geq 1 \).

Let us call \( J^c_p \) the value function of the strategy in \( \Pi_{c,p} \) that pays dividends at a constant rate \( c \) until ruin. We first compare \( J^p \) with \( J^c_p \) for \( c > p \).

**Lemma 8.5** If \( c > p \), there exists a positive constant \( K \), such that,
\[
\frac{c - p}{q} \leq J^p - J^c_p \leq K \left[ 1 + \frac{1}{x} + \frac{e^{-\frac{1}{x}}}{{x(1-\alpha)})^{1/(1-\alpha)} + \frac{x}{(c - p)^{\alpha}}} \right] (c - p),
\]
for any \( 0 < \alpha < 1 \) and \( x > 0 \).
Proof. Let us call $C \in \Pi_{x,p}$ the constant strategy $C_t = p$ and $\overline{C} \in \Pi_{x,c,p}$ the constant strategy $\overline{C}_t = c > p$ for all $t$. Define again $\tau$ as the ruin time of the process $X^{C}_t$ and $\overline{\tau}$ the one of the process $X^{\overline{C}}_t$. We have that $\tau$ coincides with the arrival of a claim and $\overline{\tau} \leq \tau$, so we get the first inequality since

$$J^p_x \leq \int_0^\infty (c-p) e^{-qs} ds + \int_0^\infty p e^{-qs} ds \leq \frac{c-p}{q} + J^p_x.$$  

We can write, using (45),

$$J^p_x - J^c_x \leq \frac{p}{q} \mathbb{E} \left[ e^{-q\overline{\tau}} - e^{-q\tau} \right] + \frac{p}{q} \sum_{j=1}^\infty \mathbb{E} \left[ I_{(\tau_1, \tau_j]} e^{q \tau_j} \right] + \frac{p}{q} \sum_{j=1}^\infty \mathbb{E} \left[ I_{(\overline{\tau}, \tau_j]} e^{q \tau_j} \right] \leq \frac{p \mathbb{K}}{q} \overline{x} (c-p) + \frac{p}{q} \sum_{j=1}^\infty \mathbb{E} \left[ I_{(\tau_1, \tau_j]} e^{q \tau_j} \right].$$

(57)

Note that if $\tau \in (\tau_j-1, \tau_j)$, then $\tau > \tau$.

In the event that $\tau \in (0, \tau_1)$ we have $\tau = x/(c-p)$. From (56), we get

$$\mathbb{E} \left[ e^{q \tau} I_{(\tau_1, \tau_2)} \right] \leq e^{q \tau} \mathbb{E} \left[ I_{(0, \tau_1)} \right] \leq \frac{e^{-1}}{qx} (c-p) \mathbb{E} \left[ I_{(0, \tau_1)} \right].$$  

(58)

In the event that $\tau \in (\tau_1, \tau_2)$, we have $X^{\overline{C}}_{\tau_1} = x - (c-p)\tau_1 - U_1 > 0$. We consider two cases: $X^{\overline{C}}_{\tau_1} \leq x(c-p)^\alpha$ and $X^{\overline{C}}_{\tau_1} > x(c-p)^\alpha$. In the first case, using the Lipschitz condition on $F$, we obtain

$$\mathbb{E} \left[ e^{q \tau} I_{(\tau_1, \tau_2)} \mathbb{1}_{\{X^{\overline{C}}_{\tau_1} \leq x(c-p)^\alpha \}} \right] \leq e^{q \tau} \mathbb{E} \left[ I_{(\tau_1, \tau_2)} \right] \leq \frac{e^{-1} \mathbb{K}}{x \alpha} \mathbb{E} \left[ I_{(\tau_1, \tau_2)} \right].$$

In the second case, we have $(\tau - \tau_1) = X^{\overline{C}}_{\tau_1} / (c-p) \geq x/(c-p)^{1-\alpha}$, and (56) yields

$$\mathbb{E} \left[ e^{q \tau} I_{(\tau_1, \tau_2)} \mathbb{1}_{\{X^{\overline{C}}_{\tau_1} > x(c-p)^\alpha \}} \right] = \mathbb{E} \left[ e^{q \tau} I_{(\tau_1, \tau_2)} \right] \leq e^{q \tau} \mathbb{E} \left[ I_{(\tau_1, \tau_2)} \right].$$

Hence,

$$\mathbb{E} \left[ e^{q \tau} I_{(\tau_1, \tau_2)} \right] \leq \mathbb{E} \left[ e^{q \tau_1} \right] \left( K x (c-p)^\alpha + \frac{e^{-1}}{x \alpha} \right).$$

(59)

In a similar way, and using Remark 8.1, we obtain,

$$\mathbb{E} \left[ e^{q \tau} I_{(\tau_{j-1}, \tau_j)} \right] \leq \mathbb{E} \left[ e^{q \tau_j} \right] \left( K x (c-p)^\alpha + \frac{e^{-1}}{x \alpha} \right)$$

for any $j \geq 3$ and so from (50), (57) and (58) we get the second inequality.\[\Box\]

In the next lemma, we give an alternative version of the Lipschitz condition for $x > 0$ and $c > p$. Here, for $x_2 > x_1 \geq \delta > 0$, the growth of the Lipschitz bound as $c \to p^+$, goes to infinity but slower than the bound obtained in Lemma 8.3.

**Lemma 8.6** For any $0 < \alpha < 1$ there exists a positive constant $\overline{K}$ such that

$$V(x_2, c) - V(x_1, c) \leq \overline{K} \left[ 1 + \frac{1}{x_1} + \frac{e^{1-x_1}}{x_1 (q x_1 (1-\alpha))^{1/(1-\alpha)}} + \frac{x_1}{(c-p)^{1-\alpha}} \right] (x_2 - x_1),$$

where $p < c \leq \overline{c}$ and $0 < x_1 < x_2$. 
Proof. Take $\varepsilon > 0$ and $C \in \Pi_{x, c, \pi}$ such that
\[ J(x_2; C) \geq V(x_2, c) - \varepsilon \] (60)
and call $\tau$ the ruin time of the process $X_t^C$. Define $\tilde{C} \in \Pi_{x_1, c, \pi}$ as $\tilde{C}_t = C_t$ and call $\tilde{\tau}$ the ruin time of the process $X_t^{\tilde{C}}$; it holds that $\tilde{\tau} \leq \tau$ and $X_t^{\tilde{C}} = x_2 - x_1$ for $t \leq \tilde{\tau}$. In the event that $\tilde{\tau} \in (\tau_j, \tau_j')$ (and so $\tilde{\tau} < \tau$), $X_t^{\tilde{C}} = 0$ and so $X_t^C = X_t^{\tilde{C}} + (x_2 - x_1) = x_2 - x_1$. Hence, since $C_s \geq \tilde{C}_s$ for $s \geq \tilde{\tau}$,
\[ \tau - \tilde{\tau} \leq \frac{1}{C_s - p} \int_{\tilde{\tau}}^{\tau} (C_s - p) \, ds \leq \frac{x_2 - x_1}{C_s - p}. \] (61)

From (47) and (61), we get
\[ V(x_2, c) - V(x_1, c) \leq \frac{\varepsilon}{q x_1}. \] (62)

In the event that $\tilde{\tau} \in (0, \tau_1)$,
\[ (C_s - p) \tilde{\tau} \geq \int_0^{\tilde{\tau}} (C_s - p) \, ds = x_1, \]
so $\tilde{\tau} \geq x_1 / (C_s - p)$. By (56), we get
\[ \mathbb{E} \left[ \frac{e^{-q \tilde{\tau}}}{C_s - p} I_{\tau \in (0, \tau_1)} \right] \leq \frac{e^{-1}}{q x_1}. \] (63)

In the event that $\tilde{\tau} \in (\tau_1, \tau_2)$, we consider two cases: $X_{\tau_{1, 2}} > x_1 (C_s - p)^{\alpha}$ and $0 < X_{\tau_{1, 2}} \leq x_1 (C_s - p)^{\alpha}$. Analogously to the proof of Lemma 8.5, we use the Lipschitz condition on $F$ in the first case and (56) in the second case to obtain
\[ \mathbb{E} \left[ I_{\tau \in (\tau_1, \tau_2)} \frac{e^{-q \tilde{\tau}}}{C_s - p} \right] = \mathbb{E} \left[ \frac{e^{-q \tilde{\tau}}}{C_s - p} I_{\tilde{\tau} \in (0, \tau_1)} \right] I_{0 < X_{\tau_{1, 2}} \leq x_1 (C_s - p)^{\alpha}} + \mathbb{E} \left[ \frac{e^{-q \tilde{\tau}}}{C_s - p} I_{\tilde{\tau} \in (\tau_1, \tau_2)} \right] I_{X_{\tau_{1, 2}} > x_1 (C_s - p)^{\alpha}} \leq \left( \frac{K x_1}{(c - p)^{1 - \alpha}} + \frac{e^{-1}}{q x_1 (1 - \alpha)} \right) \mathbb{E} \left[ e^{-q \tau_1} \right]. \]

In a similar way, and using Remark 8.1, we obtain
\[ \mathbb{E} \left[ I_{\tau \in (\tau_{j - 1}, \tau_j)} \frac{e^{-q \tilde{\tau}}}{C_s - p} \right] \leq \left( \frac{K x_1}{(c - p)^{1 - \alpha}} + \frac{e^{-1}}{q x_1 (1 - \alpha)} \right) \mathbb{E} \left[ e^{-q \tau_{j - 1}} \right] \]
for any $j \geq 3$ and so from (50), (63) and (62), we get the result.

Proof of Proposition 3.4. Consider $x > 0$, we need to prove that $\lim_{\pi \to p^+} V(x, c) = V(x, p)$. Let us call, as before, $J_p$ the value function of the strategy in $\Pi_{y, c, \pi}$ that pays dividends at a constant rate $c$ until ruin. Then, by Remark 2.1, $V(0, p) = J_y$. Also, we get $0 \leq J_y - J_0 = J_y - V(0, p)$, and from Proposition 3.2 there exists a $K_1 > 0$ such that $V(y, p) - V(0, p) \leq K_1 y$. Hence,
\[ V(y, p) - J_y \leq V(y, p) - V(0, p) + V(0, p) - J_0 \leq K_1 y. \]
So, given $\varepsilon > 0$ small enough and taking $\delta \leq \varepsilon / K_1$, we have
\[ V(y, p) - J_y \leq \varepsilon \] (64)
for all initial surplus levels $0 \leq y \leq \delta$. We assume $\delta < \min \{1/4, x\}$, so $\delta^{3/2} < \delta/2$. Consider $C \in \Pi_{x, p, \pi}$ such that $J(x; C) \geq V(x, p) - \varepsilon$ and define $T_1 := \min \{t \geq 0 : X_t^C \leq \delta\}$ and $T_2$ such that
\[
\int_{T_2}^{\infty} e^{-q\tau} ds = \frac{2}{q} e^{-qT_2} \leq \varepsilon.
\]

Take \(c \in (p, \bar{c})\) such that
\[
c - p \leq \min\{\delta^{3/2}/T_2, (\varepsilon/T_2)^{3/2}, \varepsilon, \delta^{3/2}\}.
\]

Let us define \(\hat{T} := \min(t : C_t \geq c)\). Since \(V(\cdot, c)\) is non-decreasing and continuous, we can find (as in Lemma 8.1) an increasing sequence \((y_i)\) with \(y_1 = 0\) such that if \(y \in [y_i, y_{i+1})\), then \(0 \leq V(y, c) - V(y, c) \leq \varepsilon/2\). Consider admissible strategies \(\overline{C}\) of \(V(y, c) - V(y, c) \leq \varepsilon/2\).

Let us now define the admissible strategy \(\overline{C}\) as follows: \(\overline{C}_t = c\) for \(t < \hat{T}\); in the event that \(T_1 \leq \hat{T}\) (and so \(X^C_\hat{T} \leq \delta\)), the strategy for \(t \geq \hat{T}\) consists of paying dividends at constant rate \(c\) until ruin; and in the event that \(T_1 > \hat{T}\) (and so \(X^C_\hat{T} > \delta\)), we define \(\overline{C}_t = C^c_{t-T_1}\) for \(t \geq \hat{T}\) in the case that \(X^C_\hat{T} \in [y_i, y_{i+1})\). Note that with this definition the strategy \(\overline{C}\) turns out to be admissible and \(C_s - \overline{C}_s \leq 0\) for \(s \leq \hat{T}\).

Let us call \(\tau\) and \(\overline{\tau}\) the ruin times of the processes \(X^C_t\) and \(X^C_{\overline{T}}\), respectively. In order to prove the result, we consider different cases depending on the value of \(\hat{T}\):

\[
\begin{align*}
V(x, p) - V(x, c) & \leq J(x; C) - J(x; \overline{C}) + \varepsilon \\
& = \mathbb{E}\left[I_{\{\hat{\tau} \geq \tau\}}(J(x; C) - J(x; \overline{C}))\right] + \mathbb{E}\left[I_{\{\hat{\tau} < \tau\}}(J(x; C) - J(x; \overline{C}))\right] + \varepsilon \\
& + \mathbb{E}\left[I_{\{\hat{\tau} < \tau, \hat{\tau} \leq T_2 \wedge T_1\}}(J(x; C) - J(x; \overline{C}))\right]
\end{align*}
\]

In the event \(\hat{T} \geq \tau\), using \(\tau \geq \tau\) and Lemma 8.5, we can show that
\[
\mathbb{E}\left[I_{\{\hat{\tau} \geq \tau\}}(J(x; C) - J(x; \overline{C}))\right] \leq K \left[1 + \frac{1}{q} + \frac{\varepsilon}{(q(1-\alpha))^{1/(1-\alpha)}} + \frac{\varepsilon}{(c-p)^{1-\alpha}} + \frac{1}{q}\right] (c - p).
\]

In the event that \(\hat{T} < \tau\) and \(\hat{T} > T_2\), by the definition of \(T_2\),
\[
\mathbb{E}\left[I_{\{\hat{\tau} < \tau, \hat{\tau} > T_2\}}(J(x; C) - J(x; \overline{C}))\right] \leq \mathbb{E}\left[\int_{T_2}^{\infty} e^{-q\tau} ds\right] \leq \varepsilon.
\]

In the event that \(\hat{T} < \tau, \hat{T} \leq T_2 \wedge T_1\), it holds that \(X^C_{\hat{T}} \geq \delta\) and
\[
0 \leq X^C_{\hat{T}} - X^C_{\tau} \leq (c - p)\hat{T} < (c - p)T_2 \leq \min\{\varepsilon, \delta^{3/2}\} < \delta/2.
\]

Therefore, since \(V(\cdot, c)\) is non-decreasing and \(X^C_{\tau} \in [\delta/2, x]\), we obtain from Lemma 8.6
\[
\mathbb{E}\left[I_{\{\hat{\tau} < \tau, \hat{\tau} \leq T_2 \wedge T_1\}}(J(x; C) - J(x; \overline{C}))\right] \leq K \left[1 + \frac{2}{q} + \frac{\varepsilon}{(q(1-\alpha)^{1/(1-\alpha)})}\right] \delta^{3/2} + \bar{K}(c - p)^{\alpha}T_2 + \varepsilon.
\]

In the event \(\hat{T} < \tau\) and \(\hat{T} \geq T_2\), the strategy is \(\overline{C}\) and \(\tau = t\) for all \(t\). If \(T_1\) does not coincide with the arrival of a claim, then \(X^C_{T_1} = \delta\) (and so \(X^C_{\overline{T}} \geq \delta/2\)). Then we can write, using (64), Proposition 3.2 and Lemma 8.5,
\[
\begin{align*}
\mathbb{E}\left[I_{\{\hat{\tau} < \tau, \hat{\tau} \notin [T_1, T_2]\}}(J(x; C) - J(x; \overline{C}))\right] & \leq \mathbb{E}\left[I_{\{\hat{\tau} < \tau, \hat{\tau} \notin [T_1, T_2]\}}(J(x; C) - J(x; \overline{C}))e^{-qT_1}(V(\delta, p) - J_{\delta/2})\right] \\
& \leq \mathbb{E}\left[(V(\delta, p) - J_{\delta/2}^p) + (J_{\delta/2}^p - J_{\delta/2}^p) + (J_{\delta/2}^p - J_{\delta/2}^p)\right] \\
& \leq \varepsilon + K_1 \frac{e}{2\kappa_1} + \bar{K}(1 + \frac{2}{q} + \frac{\varepsilon}{(q(1-\alpha)^{1/(1-\alpha)})}) \delta^{3/2} + \bar{K}(c - p)^{\alpha}.
\end{align*}
\]
Finally, in the event that $\hat{T} < \tau$, $\hat{T} \geq T_1$ and $T_1$ coincides with the $j$-th claim arrival, then $X_{\tau_1}^C = X_{\tau_j}^C \in (0, \delta)$ and $X_{\tau_j}^C \geq \delta$. Hence,

$$0 < X_{\tau_1}^C = X_{\tau_j}^C = X_{\tau_j}^C - U_j < \delta.$$ 

Therefore, $X_{\tau_j}^C > U_j > X_{\tau_j}^C - \delta \geq 0$. Since $F(X_{\tau_j}^C) - F(X_{\tau_j}^C - \delta) \leq K\delta$ and, by the compound Poisson assumptions, we obtain

$$E \left[ I \{ \hat{T} < \tau, \hat{T} \in [T_1, T_2) \} I \{ T_1 = \tau_j \} e^{-qT_j} \right] \leq K\delta E \left[ e^{-qT_j} \right].$$

So, by (50) and Proposition 3.3,

$$\sum_{j=1}^{\infty} E \left[ I \{ \hat{\rho} < \tau, \hat{\rho} \in [T_1, T_2) \} I \{ T_1 = \tau_j \} (J(x; C) - J(x; \overline{C})) \right] \leq K\delta V(\delta, c) \sum_{j=1}^{\infty} E \left[ e^{-qT_j} \right] \leq \frac{Kc\beta}{q} \delta.$$ (71)

Using (65)–(71) with $\alpha = 1/5$, and so $1/(1 - \alpha) = 5/4 < 3/2$, we get the result. ■