Can Optimism Solve the Entrepreneurial Earnings Puzzle?
Online Appendix

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Online Appendix A: Proofs

Proof of Proposition 1: The first step to determine the competitive equilibrium is to find out the labor market equilibrium condition. The labor demand from realistic entrepreneurs is

$$\begin{aligned} L_R^D &= N(1-\lambda)\int_{\hat{\theta}_R}^{\infty} l(\theta,w,r)g(\theta)d\theta \\ &= N(1-\lambda)\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}\int_{\hat{\theta}_R}^{\infty}\theta^{\frac{1}{1-\eta}}\rho\theta_m^{\rho}\theta^{-\rho-1}d\theta \\ &= N(1-\lambda)\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}\rho\theta_m^{\rho}\left[\frac{\theta^{\frac{1}{1-\eta}-\rho}}{\frac{1}{1-\eta}-\rho}\right]_{\hat{\theta}_R}^{\infty} \\ &= N(1-\lambda)\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}\rho\theta_m^{\rho}\frac{\hat{\theta}_R^{\frac{1}{1-\eta}-\rho}}{\rho-\frac{1}{1-\eta}}. \qquad (1)\end{aligned}$$

The labor demand from optimistic entrepreneurs is

$$\begin{aligned} L_O^D &= N\lambda\int_{\hat{\theta}_O}^{\infty} l(\gamma\theta,w,r)g(\theta)d\theta = N\lambda\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}\gamma^{\frac{1}{1-\eta}}\int_{\hat{\theta}_O}^{\infty}\theta^{\frac{1}{1-\eta}}\rho\theta_m^{\rho}\theta^{-\rho-1}d\theta \\ &= N\lambda\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}\gamma^{\frac{1}{1-\eta}}\rho\theta_m^{\rho}\frac{\hat{\theta}_O^{\frac{1}{1-\eta}-\rho}}{\rho-\frac{1}{1-\eta}}. \qquad (2)\end{aligned}$$

From (1) and (2), labor demand is equal to

$$L^D = L_R^D + L_O^D = N\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}\frac{\rho\theta_m^{\rho}}{\rho-\frac{1}{1-\eta}}\left[(1-\lambda)\hat{\theta}_R^{\frac{1}{1-\eta}-\rho}+\lambda\gamma^{\frac{1}{1-\eta}}\hat{\theta}_O^{\frac{1}{1-\eta}-\rho}\right].$$

Since each worker provides a unit of labor, labor supply is

$$\begin{aligned} L^S &= N\left[(1-\lambda)L_R^S + \lambda L_O^S\right] = N\left[(1-\lambda)\int_{\theta_m}^{\hat{\theta}_R} g(\theta)d\theta + \lambda\int_{\theta_m}^{\hat{\theta}_O} g(\theta)d\theta\right] \\ &= N\left[(1-\lambda)G(\hat{\theta}_R) + \lambda G(\hat{\theta}_O)\right] = N\left[(1-\lambda)(1-\theta_m^\rho\hat{\theta}_R^{-\rho}) + \lambda(1-\theta_m^\rho\hat{\theta}_O^{-\rho})\right] \\ &= N\left[1-\theta_m^\rho\left[(1-\lambda)\hat{\theta}_R^{-\rho} + \lambda\hat{\theta}_O^{-\rho}\right]\right]. \end{aligned}$$

In equilibrium, labor demand must equal labor supply:

$$\begin{aligned} &\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}\frac{\rho\theta_m^\rho}{\rho-\frac{1}{1-\eta}}\left[(1-\lambda)\hat{\theta}_R^{\frac{1}{1-\eta}-\rho} + \lambda\gamma^{\frac{1}{1-\eta}}\hat{\theta}_O^{\frac{1}{1-\eta}-\rho}\right] \\ &\quad = 1-\theta_m^\rho\left[(1-\lambda)\hat{\theta}_R^{-\rho} + \lambda\hat{\theta}_O^{-\rho}\right]. \end{aligned} \tag{3}$$

The second step to determine the competitive equilibrium is to find out the capital market equilibrium condition. The capital demand from realistic entrepreneurs is

$$\begin{aligned} K_R^D &= N(1-\lambda)\int_{\hat{\theta}_R}^\infty k(\theta,w,r)g(\theta)d\theta \\ &= N(1-\lambda)\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{1-\alpha}{1-\eta}}\int_{\hat{\theta}_R}^\infty \theta^{\frac{1}{1-\eta}}\rho\theta_m^\rho\theta^{-\rho-1}d\theta \\ &= N(1-\lambda)\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{1-\alpha}{1-\eta}}\rho\theta_m^\rho\frac{\hat{\theta}_R^{\frac{1}{1-\eta}-\rho}}{\rho-\frac{1}{1-\eta}}. \end{aligned} \tag{4}$$

The capital demand from optimistic entrepreneurs is

$$\begin{aligned} K_O^D &= N\lambda\int_{\hat{\theta}_O}^\infty k(\gamma\theta,w,r)g(\theta)d\theta \\ &= N\lambda\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{1-\alpha}{1-\eta}}\gamma^{\frac{1}{1-\eta}}\int_{\hat{\theta}_O}^\infty \theta^{\frac{1}{1-\eta}}\rho\theta_m^\rho\theta^{-\rho-1}d\theta \\ &= N\lambda\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{1-\alpha}{1-\eta}}\gamma^{\frac{1}{1-\eta}}\rho\theta_m^\rho\frac{\hat{\theta}_O^{\frac{1}{1-\eta}-\rho}}{\rho-\frac{1}{1-\eta}}. \end{aligned} \tag{5}$$

From (4) and (5), capital demand is equal to

$$\begin{aligned} K^D &= K_R^D + K_O^D \\ &= N\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{1-\alpha}{1-\eta}}\frac{\rho\theta_m^\rho}{\rho-\frac{1}{1-\eta}}\left[(1-\lambda)\hat{\theta}_R^{\frac{1}{1-\eta}-\rho} + \lambda\gamma^{\frac{1}{1-\eta}}\hat{\theta}_O^{\frac{1}{1-\eta}-\rho}\right]. \end{aligned}$$

In equilibrium, capital demand must equal the exogenous capital supply:

$$\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{1-\alpha}{1-\eta}}\frac{\rho\theta_m^\rho}{\rho-\frac{1}{1-\eta}}\left[(1-\lambda)\hat{\theta}_R^{\frac{1}{1-\eta}-\rho}+\lambda\gamma^{\frac{1}{1-\eta}}\hat{\theta}_O^{\frac{1}{1-\eta}-\rho}\right]=K/N. \tag{6}$$

The third step to determine the competitive equilibrium is to find out $\hat{\theta}_R$ and $\hat{\theta}_O$. A realist with ability $\hat{\theta}_R$ is indifferent between being an entrepreneur and a worker when

$$\hat{\theta}_R\left[l(\hat{\theta}_R,w,r)\right]^\alpha\left[k(\hat{\theta}_R,w,r)\right]^\beta-wl(\hat{\theta}_R,w,r)+r\left[K/N-k(\hat{\theta}_R,w,r)\right]=w+rK/N,$$

or

$$\alpha^\alpha\beta^\beta(1-\eta)^{1-\eta}\hat{\theta}_R=w^{1-\beta}r^\beta. \tag{7}$$

An optimist with perception of ability $\theta^*=\gamma\hat{\theta}_O$ and ability $\hat{\theta}_O$ is indifferent between being an entrepreneur and a worker when

$$\gamma\hat{\theta}_O\left[l(\gamma\hat{\theta}_O,w,r)\right]^\alpha\left[k(\gamma\hat{\theta}_O,w,r)\right]^\beta-wl(\gamma\hat{\theta}_O,w,r)+r\left[K/N-k(\gamma\hat{\theta}_O,w,r)\right]=w+rK/N,$$

or

$$\alpha^\alpha\beta^\beta(1-\eta)^{1-\eta}\gamma\hat{\theta}_O=w^{1-\beta}r^\beta. \tag{8}$$

It follows from (7) and (8) that

$$\alpha^\alpha\beta^\beta(1-\eta)^{1-\eta}\gamma\hat{\theta}_O=\alpha^\alpha\beta^\beta(1-\eta)^{1-\eta}\hat{\theta}_R,$$

or

$$\hat{\theta}_O=\frac{1}{\gamma}\hat{\theta}_R. \tag{9}$$

Substituting (7) and (9) into (3) we obtain

$$\alpha\rho\theta_m^\rho(1-\lambda+\lambda\gamma^\rho)\hat{\theta}_R^{\frac{1}{1-\eta}-\rho}=(1-\eta)\hat{\theta}_R^{\frac{1}{1-\eta}}\left(\rho-\frac{1}{1-\eta}\right)\left[1-\theta_m^\rho(1-\lambda+\lambda\gamma^\rho)\hat{\theta}_R^{-\rho}\right],$$

or

$$\alpha\rho\theta_m^\rho(1-\lambda+\lambda\gamma^\rho)=(1-\eta)\left(\rho-\frac{1}{1-\eta}\right)\left[\hat{\theta}_R^\rho-\theta_m^\rho(1-\lambda+\lambda\gamma^\rho)\right],$$

or

$$\begin{aligned}\hat{\theta}_R^{\rho} &= \frac{\alpha\rho\theta_m^{\rho}(1-\lambda+\lambda\gamma^{\rho})}{(1-\eta)\left(\rho-\frac{1}{1-\eta}\right)} + \theta_m^{\rho}(1-\lambda+\lambda\gamma^{\rho}) \\ &= \theta_m^{\rho}(1-\lambda+\lambda\gamma^{\rho})\left[\frac{\alpha\rho}{(1-\eta)\left(\rho-\frac{1}{1-\eta}\right)}+1\right] \\ &= \theta_m^{\rho}(1-\lambda+\lambda\gamma^{\rho})\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}.\end{aligned}$$

Hence, the ability of the marginal realistic entrepreneur is

$$\hat{\theta}_R = \theta_m(1-\lambda+\lambda\gamma^{\rho})^{\frac{1}{\rho}}\left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}}. \tag{10}$$

Note that (10), $\rho > 1/(1-\eta)$, and $\eta = \alpha+\beta \in (0,1)$, imply $\hat{\theta}_R > \theta_m$. From (9) and (10) the ability of the marginal optimistic entrepreneur is

$$\hat{\theta}_O = \frac{1}{\gamma}\theta_m(1-\lambda+\lambda\gamma^{\rho})^{\frac{1}{\rho}}\left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}}.$$

From (3) and (6) we have

$$\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{1-\alpha}{1-\eta}}\frac{1-\theta_m^{\rho}\left[(1-\lambda)\hat{\theta}_R^{-\rho}+\lambda\hat{\theta}_O^{-\rho}\right]}{\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}} = \frac{K}{N},$$

or

$$1-\theta_m^{\rho}\left[(1-\lambda)\hat{\theta}_R^{-\rho}+\lambda\hat{\theta}_O^{-\rho}\right] = \frac{K}{N}\left(\frac{\alpha}{w}\right)^{\frac{1-\beta-\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta-1+\alpha}{1-\eta}},$$

or

$$\frac{\alpha}{w}\frac{r}{\beta}\frac{K}{N} = 1-\theta_m^{\rho}\left[(1-\lambda)\hat{\theta}_R^{-\rho}+\lambda\hat{\theta}_O^{-\rho}\right],$$

or

$$\frac{\alpha}{w}\frac{r}{\beta}\frac{K}{N} = 1-\left[\frac{\rho(1-\eta)-1}{\rho(1-\beta)-1}\right],$$

or

$$r = w\frac{N}{K}\frac{\beta\rho}{\rho(1-\beta)-1} \tag{11}$$

Substituting (11) into (7) we obtain

$$\alpha^\alpha \beta^\beta (1-\eta)^{1-\eta} \theta_m (1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}} \left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}} = w^{1-\beta} \left[w \frac{N}{K} \frac{\beta\rho}{\rho(1-\beta)-1}\right]^\beta .$$

Solving this equality with respect to w we obtain the equilibrium wage:

$$w^* = \alpha^\alpha \beta^\beta (1-\eta)^{1-\eta} \theta_m (1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}} \left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}} \left[\frac{N}{K} \frac{\beta\rho}{\rho(1-\beta)-1}\right]^{-\beta} . \quad (12)$$

The equilibrium rental cost of capital is equal to

$$\begin{aligned} r^* &= w^* \frac{N}{K} \frac{\beta\rho}{\rho(1-\beta)-1} \\ &= \frac{\alpha^\alpha \beta^\beta (1-\eta)^{1-\eta} \theta_m (1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}} \left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}}}{\left[\frac{N}{K} \frac{\beta\rho}{\rho(1-\beta)-1}\right]^\beta} \frac{N}{K} \frac{\beta\rho}{\rho(1-\beta)-1} \\ &= \alpha^\alpha \beta^\beta (1-\eta)^{1-\eta} \theta_m (1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}} \left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}} \left[\frac{N}{K} \frac{\beta\rho}{\rho(1-\beta)-1}\right]^{1-\beta} \quad (13) \end{aligned}$$

The equilibrium labor force is equal to

$$\begin{aligned} L^* &= N \left[1 - \theta_m^\rho \left[(1-\lambda)\hat{\theta}_R^{-\rho} + \lambda \hat{\theta}_O^{-\rho}\right]\right]. \\ &= N \left[1 - \theta_m^\rho (1-\lambda+\lambda\gamma^\rho) \hat{\theta}_R^{-\rho}\right] \\ &= N \left[1 - \frac{\rho(1-\eta)-1}{\rho(1-\beta)-1}\right] \\ &= N \frac{\alpha\rho}{\rho(1-\beta)-1}. \end{aligned}$$

The equilibrium output level is

$$\begin{aligned} Y^* &= (1-\lambda) N \int_{\hat{\theta}_R}^{\infty} \theta [l(\theta, w^*, r^*)]^\alpha [k(\theta, w^*, r^*)]^\beta g(\theta) d\theta \\ &\quad + \lambda N \int_{\hat{\theta}_O}^{\infty} \theta [l(\gamma\theta, w^*, r^*)]^\alpha [k(\gamma\theta, w^*, r^*)]^\beta g(\theta) d\theta. \end{aligned}$$

This can be simplified to

$$
\begin{aligned}
Y^* &= N\left(\frac{\alpha}{w^*}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r^*}\right)^{\frac{\beta}{1-\eta}}\left[(1-\lambda)\int_{\hat{\theta}_R}^{\infty}\theta^{\frac{1}{1-\eta}}g(\theta)d\theta+\lambda\gamma^{\frac{\eta}{1-\eta}}\int_{\hat{\theta}_O}^{\infty}\theta^{\frac{1}{1-\eta}}g(\theta)d\theta\right] \\
&= N\left(\frac{\alpha}{w^*}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r^*}\right)^{\frac{\beta}{1-\eta}}\left[(1-\lambda)\rho\theta_m^{\rho}\frac{\hat{\theta}_R^{\frac{1}{1-\eta}-\rho}}{\rho-\frac{1}{1-\eta}}+\lambda\gamma^{\frac{\eta}{1-\eta}}\rho\theta_m^{\rho}\frac{\hat{\theta}_O^{\frac{1}{1-\eta}-\rho}}{\rho-\frac{1}{1-\eta}}\right] \\
&= N\left(\frac{\alpha}{w^*}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r^*}\right)^{\frac{\beta}{1-\eta}}\frac{\rho\theta_m^{\rho}}{\rho-\frac{1}{1-\eta}}(1-\lambda+\lambda\gamma^{\rho-1})\hat{\theta}_R^{\frac{1}{1-\eta}-\rho} \\
&= N\left(\frac{\alpha}{w^*}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r^*}\right)^{\frac{\beta}{1-\eta}}\frac{\rho\theta_m^{\frac{1}{1-\eta}}}{\rho-\frac{1}{1-\eta}}\frac{1-\lambda+\lambda\gamma^{\rho-1}}{(1-\lambda+\lambda\gamma^{\rho})^{1-\frac{1}{\rho(1-\eta)}}}\left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho(1-\eta)}-1}.
\end{aligned}
$$

Substituting w^* and r^* by (12) and (13), respectively, and simplifying terms we obtain

$$
\begin{aligned}
Y^* &= N\alpha^{\alpha}\beta^{\beta}\theta_m\frac{1}{(1-\eta)^{\eta}}\frac{\rho}{\rho-\frac{1}{1-\eta}}(1-\lambda+\lambda\gamma^{\rho})^{\frac{1-\rho}{\rho}}(1-\lambda+\lambda\gamma^{\rho-1})\times \\
&\quad \left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1-\rho}{\rho}}\left[\frac{N}{K}\frac{\beta\rho}{\rho(1-\beta)-1}\right]^{-\beta} \\
&= N^{1-\beta}K^{\beta}\alpha^{\alpha}\theta_m(1-\eta)^{1-\eta}\frac{1-\lambda+\lambda\gamma^{\rho-1}}{(1-\lambda+\lambda\gamma^{\rho})^{1-\frac{1}{\rho}}}\left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}}\left[\frac{\rho}{\rho(1-\beta)-1}\right]^{1-\beta}
\end{aligned}
$$

For the equilibrium to be well defined we must have that

$$\hat{\theta}_O \geq \theta_m,$$

or

$$\frac{1}{\gamma}\theta_m(1-\lambda+\lambda\gamma^{\rho})^{\frac{1}{\rho}}\left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}} \geq \theta_m$$

or

$$(1-\lambda+\lambda\gamma^{\rho})\left[\rho(1-\beta)-1\right] \geq \gamma^{\rho}\left[\rho(1-\eta)-1\right],$$

or

$$\frac{1-\lambda+\lambda\gamma^{\rho}}{\gamma^{\rho}} \geq \frac{\rho(1-\eta)-1}{\rho(1-\beta)-1}.$$

Q.E.D.

Proof of Proposition 2:

(i) We know from Proposition 1 that output in a competitive equilibrium with optimists is

$$Y^* = \theta_m \alpha^\alpha (1-\eta)^{1-\eta} N^{1-\beta} K^\beta \frac{1-\lambda+\lambda\gamma^{\rho-1}}{(1-\lambda+\lambda\gamma^\rho)^{1-\frac{1}{\rho}}} \left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}} \left[\frac{\rho}{\rho(1-\beta)-1}\right]^{1-\beta}.$$

Setting $\lambda = 0$ (or $\gamma = 1$) we obtain output in the competitive equilibrium without optimists:

$$Y_0^* = \theta_m \alpha^\alpha (1-\eta)^{1-\eta} N^{1-\beta} K^\beta \left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}} \left[\frac{\rho}{\rho(1-\beta)-1}\right]^{1-\beta}.$$

Hence, the existence of optimists (the case $\lambda > 0$ and $\gamma > 1$) lowers output provided that

$$\frac{1-\lambda+\lambda\gamma^{\rho-1}}{(1-\lambda+\lambda\gamma^\rho)^{1-\frac{1}{\rho}}} < 1,$$

when $\rho > 1$, $\gamma > 1$, and $\lambda \in (0,1)$. Define

$$\psi(\lambda) = \frac{1-\lambda+\lambda\gamma^{\rho-1}}{(1-\lambda+\lambda\gamma^\rho)^{1-\frac{1}{\rho}}}. \tag{14}$$

We prove this result by showing that (a) $\psi(0) = \psi(1) = 1$, (b) $\psi\prime(0) < 0$, (c) $\psi\prime(1) > 0$, and (d) there exists only one $\lambda \in (0,1)$ such that $\psi\prime(\lambda) = 0$. Results (a), (b), (c), and (d) imply that: $\psi(\lambda)$ is convex in $[0,1]$, $\psi(\lambda)$ attains a maxima of 1 at $\lambda = 0$ and at $\lambda = 1$, and $\psi(\lambda)$ attains a minimum at an $\lambda \in (0,1)$. Hence, $\psi(\lambda) < 1$ when $\rho > 1$, $\gamma > 1$, and $\lambda \in (0,1)$. Substituting $\lambda = 0$ in (14) we have $\psi(0) = 1$. Substituting $\lambda = 1$ in (14) we obtain $\psi(1) = 1$. Hence, $\psi(0) = \psi(1) = 1$. This proves (a). Next, we show that $\psi'(0) < 0$ when $\rho > 1$ and $\gamma > 1$. The first derivative of $\psi(\lambda)$ with respect to λ is:

$$\psi'(\lambda) = \frac{\gamma^{\rho-1} - \left(1-\frac{1}{\rho}\right)\gamma^\rho - \frac{1}{\rho} + \frac{\lambda}{\rho}(\gamma^{\rho-1}-1)(\gamma^\rho-1)}{(1-\lambda+\lambda\gamma^\rho)^{2-\frac{1}{\rho}}}. \tag{15}$$

From (15) we have

$$\psi'(0) = \gamma^{\rho-1} - \left(1 - \frac{1}{\rho}\right)\gamma^{\rho} - \frac{1}{\rho}.$$

Define

$$\varphi(\gamma) = \gamma^{\rho-1} - \left(1 - \frac{1}{\rho}\right)\gamma^{\rho} - \frac{1}{\rho}.$$

Setting $\gamma = 1$ in $\varphi(\gamma)$ we obtain $\varphi(1) = 0$. Taking the derivative of $\varphi(\gamma)$ with respect to γ we obtain

$$\varphi'(\gamma) = (\rho - 1)\gamma^{\rho-2} - \left(1 - \frac{1}{\rho}\right)\rho\gamma^{\rho-1} = -(\rho - 1)\gamma^{\rho-1}\left(1 - \frac{1}{\gamma}\right) < 0,$$

when $\rho > 1$ and $\gamma > 1$. If $\varphi(1) = 0$ and $\varphi'(\gamma) < 0$ when $\rho > 1$ and $\gamma > 1$, then $\varphi(\gamma) < 0$ when $\rho > 1$ and $\gamma > 1$. Since $\psi'(0) = \varphi(\gamma)$ it follows that $\psi'(0) < 0$ when $\rho > 1$ and $\gamma > 1$. This proves (b). Next, we show that $\psi'(1) > 0$ when $\rho > 1$ and $\gamma > 1$.

From (15) we have

$$\psi'(1) = \frac{\left[\gamma^{\rho-1} - \left(1 - \frac{1}{\rho}\right)\gamma^{\rho} - \frac{1}{\rho}\right] + \frac{1}{\rho}(\gamma^{\rho-1} - 1)(\gamma^{\rho} - 1)}{\gamma^{2\rho-1}} = \frac{\rho - 1 + \gamma^{\rho} - \rho\gamma}{\rho\gamma^{\rho}}.$$

Define

$$\omega(\gamma) = \rho - 1 + \gamma^{\rho} - \rho\gamma.$$

Setting $\gamma = 1$ in $\omega(\gamma)$ we obtain $\omega(1) = 0$. Taking the derivative of $\omega(\gamma)$ with respect to γ we obtain

$$\omega'(\gamma) = \rho(\gamma^{\rho-1} - 1) > 0,$$

when $\rho > 1$ and $\gamma > 1$. If $\omega(1) = 0$ and $\omega'(\gamma) > 0$ when $\rho > 1$ and $\gamma > 1$, then $\omega(\gamma) > 0$ when $\rho > 1$ and $\gamma > 1$. Since $sign\,(\psi'(1)) = sign\,(\omega(\gamma))$ it follows that $\psi'(1) > 0$ when $\rho > 1$ and $\gamma > 1$. This proves (c). Finally, we show that there exists only one $\lambda \in (0, 1)$ such that $\psi\prime(\lambda) = 0$. From (15), $\psi'(\lambda) = 0$ is equivalent to

$$\gamma^{\rho-1} - \left(1 - \frac{1}{\rho}\right)\gamma^{\rho} - \frac{1}{\rho} + \frac{\lambda}{\rho}(\gamma^{\rho-1} - 1)(\gamma^{\rho} - 1) = 0.$$

Hence, the unique λ which solves $\psi'(\lambda) = 0$ is equal to

$$\tilde{\lambda} = \frac{-\rho\gamma^{\rho-1} + (\rho - 1)\,\gamma^{\rho} + 1}{(\gamma^{\rho-1} - 1)(\gamma^{\rho} - 1)}. \tag{16}$$

We see from (16) that $\tilde{\lambda} > 0$ since $-\rho\gamma^{\rho-1} + (\rho - 1)\,\gamma^{\rho} + 1 = -\rho\psi'(0)$, and, as we have shown above, $\psi'(0) < 0$. We now show that $\tilde{\lambda} < 1$. This is the case as long as

$$(\gamma^{\rho-1} - 1)(\gamma^{\rho} - 1) > -\rho\gamma^{\rho-1} + (\rho - 1)\,\gamma^{\rho} + 1,$$

or

$$\frac{1}{\gamma}\gamma^{2\rho} - \frac{1}{\gamma}\gamma^{\rho} - \gamma^{\rho} + 1 > -\rho\gamma^{\rho-1} + \rho\gamma^{\rho} - \gamma^{\rho} + 1,$$

or

$$\frac{1}{\gamma}\gamma^{\rho}(\gamma^{\rho} - 1) > \rho\gamma^{\rho}(1 - \gamma^{-1}),$$

or

$$\gamma^{\rho} - \rho\gamma + \rho - 1 > 0,$$

which is true since $\gamma^{\rho} - \rho\gamma + \rho - 1 = \omega(\gamma)$, and, as we have shown above, $\omega(\gamma) > 0$. This proves (d).

(ii) The mean ability of the pool of entrepreneurs in a competitive equilibrium without optimists is

$$E(\theta|\theta \geq \hat{\theta}_0) = \frac{\int_{\hat{\theta}_0}^{+\infty} \theta g(\theta) d\theta}{\int_{\hat{\theta}_0}^{+\infty} g(\theta) d\theta} = \frac{\rho}{\rho - 1}\hat{\theta}_0. \tag{17}$$

In a competitive equilibrium with optimists, the mean ability of realistic entrepreneurs is

$$E(\theta|\theta \geq \hat{\theta}_R) = \frac{\int_{\hat{\theta}_R}^{+\infty} \theta g(\theta) d\theta}{\int_{\hat{\theta}_R}^{+\infty} g(\theta) d\theta} = \frac{\rho}{\rho - 1}\hat{\theta}_R,$$

and the mean ability of optimistic entrepreneurs is

$$E(\theta|\theta \geq \hat{\theta}_O) = \frac{\int_{\hat{\theta}_O}^{+\infty} \theta g(\theta) d\theta}{\int_{\hat{\theta}_O}^{+\infty} g(\theta) d\theta} = \frac{\rho}{\rho - 1}\hat{\theta}_O.$$

Hence, the mean ability of the pool of entrepreneurs in a competitive equilibrium with optimists is equal to

$$\begin{aligned} E(\theta|E^*) &= \frac{E_R^*}{E^*}E(\theta|\theta \geq \hat{\theta}_R) + \frac{E_O^*}{E^*}E(\theta|\theta \geq \hat{\theta}_O) \\ &= \frac{(1-\lambda)\theta_m^\rho \hat{\theta}_R^{-\rho}}{(1-\lambda)\theta_m^\rho \hat{\theta}_R^{-\rho} + \lambda\theta_m^\rho \hat{\theta}_O^{-\rho}} \frac{\rho}{\rho-1}\hat{\theta}_R + \frac{\lambda\theta_m^\rho \hat{\theta}_O^{-\rho}}{(1-\lambda)\theta_m^\rho \hat{\theta}_R^{-\rho} + \lambda\theta_m^\rho \hat{\theta}_O^{-\rho}} \frac{\rho}{\rho-1}\hat{\theta}_O \\ &= \frac{1-\lambda}{1-\lambda+\lambda\gamma^\rho}\frac{\rho}{\rho-1}\hat{\theta}_R + \frac{\lambda\gamma^\rho}{1-\lambda+\lambda\gamma^\rho}\frac{\rho}{\rho-1}\hat{\theta}_O. \end{aligned} \tag{18}$$

It follows from (17) and (18) that $E(\theta|E^*) < E(\theta|\theta \geq \hat{\theta}_0)$ as long as

$$\frac{1-\lambda}{1-\lambda+\lambda\gamma^\rho}\frac{\rho}{\rho-1}\hat{\theta}_R + \frac{\lambda\gamma^\rho}{1-\lambda+\lambda\gamma^\rho}\frac{\rho}{\rho-1}\hat{\theta}_O < \frac{\rho}{\rho-1}\hat{\theta}_0$$

or

$$\frac{1-\lambda}{1-\lambda+\lambda\gamma^\rho}(1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}} + \frac{\lambda\gamma^{\rho-1}}{1-\lambda+\lambda\gamma^\rho}(1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}} < 1$$

or

$$\frac{1-\lambda+\lambda\gamma^{\rho-1}}{(1-\lambda+\lambda\gamma^\rho)^{1-\frac{1}{\rho}}} < 1,$$

which we know to hold from part (i). This proves result (ii).

(iii) The result follows directly from Proposition 1 and the definition of mean returns to entrepreneurship:

$$\bar{\pi}^* = \frac{\Pi^*}{E^*} = \frac{1}{E^*}\left[Y^* - w^*L^* - r^*\left(1 - \frac{E^*}{N}\right)K\right],$$

where Π^* denotes aggregate profits in the economy. *Q.E.D.*

Proof of Proposition 3: In equilibrium, the mean returns to entrepreneurship of realists is:

$$
\begin{aligned}
\bar{\pi}_R &= E(\pi(\theta, l(\theta, w^*, r^*), k(\theta, w^*, r^*))|\theta \geq \hat{\theta}_R) \\
&= \frac{\int_{\hat{\theta}_R}^{+\infty} \pi(\theta, l(\theta, w^*, r^*), k(\theta, w^*, r^*))g(\theta)d\theta}{\int_{\hat{\theta}_R}^{+\infty} g(\theta)d\theta} \\
&= \frac{\left(\frac{\alpha}{w^*}\right)^{\frac{\alpha}{1-\eta}} \left(\frac{\beta}{r^*}\right)^{\frac{\beta}{1-\eta}} (1-\eta)\,\rho\theta_m^\rho \int_{\hat{\theta}_R}^{+\infty} \theta^{\frac{1}{1-\eta}}\theta^{-\rho-1}d\theta}{G(+\infty)-G(\hat{\theta}_R)} + r^*\frac{K}{N} \\
&= \frac{\left(\frac{\alpha}{w^*}\right)^{\frac{\alpha}{1-\eta}} \left(\frac{\beta}{r^*}\right)^{\frac{\beta}{1-\eta}} (1-\eta)\,\rho\theta_m^\rho \int_{\hat{\theta}_R}^{+\infty} \theta^{\frac{1}{1-\eta}}\theta^{-\rho-1}d\theta}{1-\left[1-\left(\frac{\theta_m}{\hat{\theta}_R}\right)^\rho\right]} + r^*\frac{K}{N} \\
&= \left(\frac{\alpha}{w^*}\right)^{\frac{\alpha}{1-\eta}} \left(\frac{\beta}{r^*}\right)^{\frac{\beta}{1-\eta}} (1-\eta)\,\rho\frac{\hat{\theta}_R^{\frac{1}{1-\eta}}}{\rho-\frac{1}{1-\eta}} + r^*\frac{K}{N} \\
&= \alpha^{\frac{\alpha}{1-\eta}}\beta^{\frac{\beta}{1-\eta}}(w^*)^{-\frac{\alpha}{1-\eta}}(r^*)^{-\frac{\beta}{1-\eta}}(1-\eta)\,\rho\frac{\hat{\theta}_R^{\frac{1}{1-\eta}}}{\rho-\frac{1}{1-\eta}} + r^*\frac{K}{N} \\
&= \alpha^{\frac{\alpha}{1-\eta}}\beta^{\frac{\beta}{1-\eta}}w^{-\frac{\eta}{1-\eta}}\hat{\theta}_R^{\frac{1}{1-\eta}}\left[\frac{N}{K}\frac{\beta\rho}{\rho(1-\beta)-1}\right]^{-\frac{\beta}{1-\eta}}(1-\eta)\,\frac{\rho}{\rho-\frac{1}{1-\eta}} + r^*\frac{K}{N} \\
&= \alpha^{\frac{\alpha}{1-\eta}}\beta^{\frac{\beta}{1-\eta}}\left[\theta_m\alpha^\alpha\beta^\beta(1-\eta)^{1-\eta}(1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}}\left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}}\left[\frac{N}{K}\frac{\beta\rho}{\rho(1-\beta)-1}\right]^{-\beta}\right]^{-\frac{\eta}{1-\eta}} \\
&\times\left[\theta_m(1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}}\left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}}\right]^{\frac{1}{1-\eta}}\left[\frac{N}{K}\frac{\beta\rho}{\rho(1-\beta)-1}\right]^{-\frac{\beta}{1-\eta}}\frac{\rho\,(1-\eta)}{\rho-\frac{1}{1-\eta}} + r^*\frac{K}{N} \\
&= \frac{\rho\theta_m\alpha^\alpha\beta^\beta(1-\eta)^{2-\eta}}{\rho(1-\eta)-1}(1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}}\left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}}\left[\frac{N}{K}\frac{\beta\rho}{\rho(1-\beta)-1}\right]^{-\beta} + r^*\frac{K}{N}
\end{aligned}
\tag{19}
$$

Substituting r^* into (19) we have

$$
\begin{aligned}
\bar{\pi}_R &= \rho\theta_m\alpha^\alpha\beta^\beta(1-\eta)^{1-\eta}(1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}}\left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}}\left[\frac{N}{K}\frac{\beta\rho}{\rho(1-\beta)-1}\right]^{-\beta} \\
&\times\left[\frac{1-\eta}{\rho(1-\eta)-1}+\frac{\beta}{\rho(1-\beta)-1}\right].
\end{aligned}
$$

In equilibrium, the mean returns to entrepreneurship of optimists is:

$$
\begin{aligned}
\bar{\pi}_O^* &= E(\pi(\theta, l(\gamma\theta, w^*, r^*), k(\gamma\theta, w^*, r^*))|\theta \geq \hat{\theta}_O) \\
&= \frac{\int_{\hat{\theta}_O}^{+\infty} \pi(\theta, l(\gamma\theta, w^*, r^*), k(\gamma\theta, w^*, r^*))g(\theta)d\theta}{\int_{\hat{\theta}_O}^{+\infty} g(\theta)d\theta} \\
&= \frac{\left(\frac{\alpha}{w^*}\right)^{\frac{\alpha}{1-\eta}} \left(\frac{\beta}{r^*}\right)^{\frac{\beta}{1-\eta}} \left(\gamma^{-1}-\eta\right)\gamma^{\frac{1}{1-\eta}} \rho\theta_m^\rho \int_{\hat{\theta}_O}^{+\infty} \theta^{\frac{1}{1-\eta}} \theta^{-\rho-1} d\theta}{G(+\infty) - G(\hat{\theta}_O)} + r^* \frac{K}{N} \\
&= \frac{\left(\frac{\alpha}{w^*}\right)^{\frac{\alpha}{1-\eta}} \left(\frac{\beta}{r^*}\right)^{\frac{\beta}{1-\eta}} \left(\gamma^{-1}-\eta\right)\gamma^{\frac{1}{1-\eta}} \rho\theta_m^\rho \int_{\hat{\theta}_O}^{+\infty} \theta^{\frac{1}{1-\eta}} \theta^{-\rho-1} d\theta}{1 - \left[1 - \left(\frac{\theta_m}{\hat{\theta}_O}\right)^\rho\right]} + r^* \frac{K}{N} \\
&= \frac{\left(\frac{\alpha}{w^*}\right)^{\frac{\alpha}{1-\eta}} \left(\frac{\beta}{r^*}\right)^{\frac{\beta}{1-\eta}} \left(\gamma^{-1}-\eta\right)\gamma^{\frac{1}{1-\eta}} \rho\theta_m^\rho \frac{\hat{\theta}_O^{\frac{1}{1-\eta}-\rho}}{\rho - \frac{1}{1-\eta}}}{\left(\frac{\theta_m}{\hat{\theta}_O}\right)^\rho} + r^* \frac{K}{N} \\
&= \alpha^{\frac{\alpha}{1-\eta}} \beta^{\frac{\beta}{1-\eta}} (w^*)^{-\frac{\alpha}{1-\eta}} (r^*)^{-\frac{\beta}{1-\eta}} \left(\gamma^{-1}-\eta\right)\gamma^{\frac{1}{1-\eta}} \rho \frac{\hat{\theta}_O^{\frac{1}{1-\eta}}}{\rho - \frac{1}{1-\eta}} + r^* \frac{K}{N} \\
&= \alpha^{\frac{\alpha}{1-\eta}} \beta^{\frac{\beta}{1-\eta}} (w^*)^{-\frac{\eta}{1-\eta}} \hat{\theta}_O^{\frac{1}{1-\eta}} \left[\frac{N}{K}\frac{\beta\rho}{\rho(1-\beta)-1}\right]^{-\frac{\beta}{1-\eta}} \left(\gamma^{-1}-\eta\right)\gamma^{\frac{1}{1-\eta}} \frac{\rho}{\rho - \frac{1}{1-\eta}} + r^* \frac{K}{N} \\
&= \alpha^\alpha \beta^\beta (1-\eta)^{-\eta} \theta_m (1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}} \left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}} \left[\frac{N}{K}\frac{\beta\rho}{\rho(1-\beta)-1}\right]^{\frac{\beta\eta}{1-\eta}} \times \\
&\quad \gamma^{-\frac{1}{1-\eta}} \left(\frac{N}{K}\frac{\beta\rho}{\rho(1-\beta)-1}\right)^{-\frac{\beta}{1-\eta}} \frac{\rho\left(\gamma^{-1}-\eta\right)\gamma^{\frac{1}{1-\eta}}}{\rho - \frac{1}{1-\eta}} + r^* \frac{K}{N} \\
&= \frac{\rho\theta_m \alpha^\alpha \beta^\beta (1-\eta)^{1-\eta}}{\rho(1-\eta)-1} \frac{(1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}}}{\left(\gamma^{-1}-\eta\right)^{-1}} \left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}} \left[\frac{N}{K}\frac{\beta\rho}{\rho(1-\beta)-1}\right]^{-\beta} + r^* \frac{K}{N}
\end{aligned}
\tag{20}
$$

Substituting r^* into (20) we have

$$
\begin{aligned}
\bar{\pi}_O^* \quad = \quad & \frac{\rho\theta_m \alpha^\alpha \beta^\beta (1-\eta)^{1-\eta}}{\rho(1-\eta)-1} \frac{(1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}}}{\left(\gamma^{-1}-\eta\right)^{-1}} \left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}} \left[\frac{N}{K}\frac{\beta\rho}{\rho(1-\beta)-1}\right]^{-\beta} \\
& + \theta_m \alpha^\alpha \beta^\beta (1-\eta)^{1-\eta} (1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}} \left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}} \left[\frac{N}{K}\frac{\beta\rho}{\rho(1-\beta)-1}\right]^{1-\beta} \frac{K}{N},
\end{aligned}
$$

or

$$\bar{\pi}_O^* = \rho\theta_m \alpha^\alpha \beta^\beta (1-\eta)^{1-\eta}(1-\lambda+\lambda\gamma^\rho)^{\frac{1}{\rho}} \left[\frac{\rho(1-\beta)-1}{\rho(1-\eta)-1}\right]^{\frac{1}{\rho}} \left[\frac{N}{K}\frac{\beta\rho}{\rho(1-\beta)-1}\right]^{-\beta}$$
$$\times \left[\frac{\gamma^{-1}-\eta}{\rho(1-\eta)-1} + \frac{\beta}{\rho(1-\beta)-1}\right].$$

Comparing (19) to (20) we see that the mean returns to entrepreneurship of realists is greater than the mean returns to entrepreneurship of optimists as long as

$$(1-\eta)^{2-\eta} > (1-\eta)^{1-\eta}\left(\gamma^{-1}-\eta\right),$$

or

$$1-\eta > \frac{1}{\gamma} - \eta,$$

which is always the case when $\gamma > 1$. *Q.E.D.*

Online Appendix B: Pareto Firm Size Distribution

In this appendix we show that if ability is distributed according to a Pareto distribution so is firm size. We start by doing it in the model without optimists. After that we show that the result also holds in the model with optimists.

Setting $\gamma = 1$ in the first-order conditions of an entrepreneur's problem we have

$$k = \frac{\beta}{\alpha}\frac{w}{r}l.$$

The scale of a firm is given by

$$\beta\gamma\theta l^\alpha k^{\beta-1} = r,$$

or

$$\beta\gamma\theta l^\alpha \left(\frac{\beta}{\alpha}\frac{w}{r}l\right)^{\beta-1} = r,$$

or

$$\theta = \frac{1}{\gamma}\left(\frac{w}{\alpha}\right)^{1-\beta}\left(\frac{r}{\beta}\right)^{\beta} l^{1-\eta}. \tag{21}$$

Let $S(l)$ denote the probability that a randomly select firm has fewer than l employees. Then under (21) $S(l)$ will be the probability that θ is less than $\left(\frac{w}{\alpha}\right)^{1-\beta}\left(\frac{r}{\beta}\right)^{\beta} l^{1-\eta}$ conditional on $\theta \geq \hat{\theta}_0^*$, or

$$\begin{aligned} S(l) &= \Pr\left[\theta \leq \left(\frac{w}{\alpha}\right)^{1-\beta}\left(\frac{r}{\beta}\right)^{\beta} l^{1-\eta} | \theta \geq \hat{\theta}_0^*\right] \\ &= \frac{G\left[\left(\frac{w}{\alpha}\right)^{1-\beta}\left(\frac{r}{\beta}\right)^{\beta} l^{1-\eta}\right] - G(\hat{\theta}_0^*)}{1 - G(\hat{\theta}_0^*)} \end{aligned}$$

for $l \geq$ and 0 otherwise. If ability is distributed according to a Pareto cumulative distribution function $G(\theta) = 1 - \theta_m^{\rho}\theta^{-\rho}$ we have

$$\begin{aligned} S(l) &= \frac{G\left[\left(\frac{w}{\alpha}\right)^{1-\beta}\left(\frac{r}{\beta}\right)^{\beta} l^{1-\eta}\right] - G(\hat{\theta}_0^*)}{1 - G(\hat{\theta}_0^*)} \\ &= \frac{1 - \theta_m^{\rho}\left[\left(\frac{w}{\alpha}\right)^{1-\beta}\left(\frac{r}{\beta}\right)^{\beta} l^{1-\eta}\right]^{-\rho} - \left[1 - \theta_m^{\rho}(\hat{\theta}_0^*)^{-\rho}\right]}{1 - \left[1 - \theta_m^{\rho}(\hat{\theta}_0^*)^{-\rho}\right]} \\ &= 1 - \frac{\left[\left(\frac{w}{\alpha}\right)^{1-\beta}\left(\frac{r}{\beta}\right)^{\beta} l^{1-\eta}\right]^{-\rho}}{(\hat{\theta}_0^*)^{-\rho}} \\ &= 1 - (\hat{\theta}_0^*)^{\rho}\left[\left(\frac{w}{\alpha}\right)^{1-\beta}\left(\frac{r}{\beta}\right)^{\beta}\right]^{-\rho} l^{-\rho(1-\eta)} \\ &= 1 - (\hat{\theta}_0^*)^{\rho}\left(\frac{w^{1-\beta}r^{\beta}}{\alpha^{1-\beta}\beta^{\beta}}\right)^{-\rho} l^{-\rho(1-\eta)}. \end{aligned}$$

Using the equilibrium condition

$$w^{1-\beta}r^{\beta} = \alpha^{\alpha}\beta^{\beta}(1-\eta)^{1-\eta}\hat{\theta}_0^*,$$

we have

$$S(l) = 1-(\hat{\theta}_0^*)^{\rho}\left[\frac{(1-\eta)^{1-\eta}\hat{\theta}_0^*}{\alpha^{1-\eta}}\right]^{-\rho} l^{-\rho(1-\eta)}$$
$$= 1-\left(\frac{\alpha}{1-\eta}\right)^{\rho(1-\eta)} l^{-\rho(1-\eta)}.$$

We now show that optimism does not have an impact on the firm size distribution. Note that if $\theta_r < x$:

$$\begin{aligned}
\Pr(X<x) &= \Pr(X_o<x)\lambda+\Pr(X_r<x)(1-\lambda) \\
&= \left[1-\left(\frac{\hat{\theta}_O}{x}\right)^{\alpha}\right]\lambda+\left[1-\left(\frac{\hat{\theta}_R}{x}\right)^{\alpha}\right](1-\lambda) \\
&= -\lambda\left(\frac{\hat{\theta}_O}{x}\right)^{\alpha}+1-(1-\lambda)\left(\frac{\hat{\theta}_R}{x}\right)^{\alpha} \\
&= 1-\lambda\left(\frac{\hat{\theta}_O}{x}\right)^{\alpha}-(1-\lambda)\left(\frac{\hat{\theta}_R}{x}\right)^{\alpha} \\
&= 1-\left[\lambda\left(\hat{\theta}_O\right)^{\alpha}-(1-\lambda)\left(\hat{\theta}_R\right)^{\alpha}\right]\left(\frac{1}{x}\right)^{\alpha} \\
&= 1-\left[\frac{\left(\lambda\hat{\theta}_O^{\alpha}-(1-\lambda)\hat{\theta}_R^{\alpha}\right)^{\frac{1}{\alpha}}}{x}\right]^{\alpha}.
\end{aligned}$$

If $\hat{\theta}_O < x < \hat{\theta}_R$:

$$\Pr(X<x)=\Pr(X_o<x)=1-\left(\frac{\hat{\theta}_O}{x}\right)^{\alpha}$$

Note that the marginal optimist entrepreneur perceived ability is $\gamma\hat{\theta}_O=\hat{\theta}_R$. Therefore, the firm size associated to this entrepreneur is the same as that associated to the marginal realist entrepreneur. The firm size distribution can be derived as follows:

$$l > \left[\gamma\hat{\theta}_O\left(\frac{w}{\alpha}\right)^{\beta-1}\left(\frac{r}{\beta}\right)^{-\beta}\right]^{\frac{1}{1-\eta}}$$

or, using the equilibrium condition, $w^{1-\beta} r^{\beta} = \alpha^{\alpha} \beta^{\beta} (1-\eta)^{1-\eta} \gamma \hat{\theta}_O$,

$$l > \left[\gamma \hat{\theta}_O \left(\frac{1}{\alpha} \right)^{\beta - 1} \left(\frac{1}{\beta} \right)^{-\beta} \alpha^{-\alpha} \beta^{-\beta} (1-\eta)^{\eta - 1} \gamma^{-1} \hat{\theta}_O^{-1} \right]^{\frac{1}{1-\eta}}$$

or

$$l > \left[\left(\frac{\alpha}{1-\eta} \right)^{1-\eta} \right]^{\frac{1}{1-\eta}}$$

or

$$l > \frac{\alpha}{1-\eta}.$$

Then

$$\begin{aligned}
\Pr(S < l) &= \Pr\left(\theta \leq \frac{1}{\gamma} \left(\frac{w}{\alpha} \right)^{1-\beta} \left(\frac{r}{\beta} \right)^{\beta} l^{1-\eta} \middle| \theta \geq \hat{\theta}_O \right) \lambda \\
&\quad + \Pr\left(\theta \leq \left(\frac{w}{\alpha} \right)^{1-\beta} \left(\frac{r}{\beta} \right)^{\beta} l^{1-\eta} \middle| \theta \geq \hat{\theta}_R \right) (1-\lambda) \\
&= \frac{1 - \theta_m^{\rho} \left(\frac{1}{\gamma} \left(\frac{w}{\alpha} \right)^{1-\beta} \left(\frac{r}{\beta} \right)^{\beta} l^{1-\eta} \right)^{-\rho} - \left(1 - \theta_m^{\rho} \hat{\theta}_O^{-\rho} \right)}{1 - \left(1 - \theta_m^{\rho} \hat{\theta}_O^{-\rho} \right)} \lambda \\
&\quad + \frac{1 - \theta_m^{\rho} \left(\left(\frac{w}{\alpha} \right)^{1-\beta} \left(\frac{r}{\beta} \right)^{\beta} l^{1-\eta} \right)^{-\rho} - \left(1 - \theta_m^{\rho} \hat{\theta}_R^{-\rho} \right)}{1 - \left(1 - \theta_m^{\rho} \hat{\theta}_R^{-\rho} \right)} (1-\lambda) \\
&= \frac{1 - \theta_m^{\rho} \left(\frac{1}{\gamma} \left(\frac{w}{\alpha} \right)^{1-\beta} \left(\frac{r}{\beta} \right)^{\beta} l^{1-\eta} \right)^{-\rho} - \left(1 - \theta_m^{\rho} \hat{\theta}_O^{-\rho} \right)}{1 - \left(1 - \theta_m^{\rho} \hat{\theta}_O^{-\rho} \right)} \\
&\quad + \frac{1 - \theta_m^{\rho} \left(\left(\frac{w}{\alpha} \right)^{1-\beta} \left(\frac{r}{\beta} \right)^{\beta} l^{1-\eta} \right)^{-\rho} - \left(1 - \theta_m^{\rho} \gamma^{-\rho} \hat{\theta}_O^{-\rho} \right)}{1 - \left(1 - \theta_m^{\rho} \gamma^{-\rho} \hat{\theta}_O^{-\rho} \right)} (1-\lambda)
\end{aligned}$$

or λ

$$\begin{aligned}
\Pr\left(S<l\right) &= 1-\left[\lambda\frac{\left(\frac{1}{\gamma}\left(\frac{w}{\alpha}\right)^{1-\beta}\left(\frac{r}{\beta}\right)^{\beta}l^{1-\eta}\right)^{-\rho}}{\hat{\theta}_O^{-\rho}}\right]-\left[\left(1-\lambda\right)\frac{\left(\left(\frac{w}{\alpha}\right)^{1-\beta}\left(\frac{r}{\beta}\right)^{\beta}l^{1-\eta}\right)^{-\rho}}{\gamma^{-\rho}\hat{\theta}_O^{-\rho}}\right] \\
&= 1-\left\{\left[\lambda\frac{\left(\frac{1}{\gamma}\right)^{-\rho}}{\hat{\theta}_O^{-\rho}}\right]+\left[\left(1-\lambda\right)\frac{1}{\gamma^{-\rho}\hat{\theta}_O^{-\rho}}\right]\right\}\left(\left(\frac{w}{\alpha}\right)^{1-\beta}\left(\frac{r}{\beta}\right)^{\beta}\right)^{-\rho}l^{-\rho(1-\eta)} \\
&= 1-\hat{\theta}_O^{\rho}\left(\frac{\left(\frac{w}{\alpha}\right)^{1-\beta}\left(\frac{r}{\beta}\right)^{\beta}}{\gamma}\right)^{-\rho}l^{-\rho(1-\eta)} \\
&= 1-\hat{\theta}_O^{\rho}\left(\frac{\left(\frac{1}{\alpha}\right)^{1-\beta}\left(\frac{1}{\beta}\right)^{\beta}\alpha^{\alpha}\beta^{\beta}\left(1-\eta\right)^{1-\eta}\gamma\hat{\theta}_O}{\gamma}\right)^{-\rho}l^{-\rho(1-\eta)} \\
&= 1-\left(\left(\frac{1}{\alpha}\right)^{1-\beta}\left(\frac{1}{\beta}\right)^{\beta}\alpha^{\alpha}\beta^{\beta}\left(1-\eta\right)^{1-\eta}\right)^{-\rho}l^{-\rho(1-\eta)} \\
&= 1-\left(\frac{\alpha}{1-\eta}\right)^{\rho(1-\eta)}l^{-\rho(1-\eta)}.
\end{aligned}$$

Therefore the firm size distribution is the same that would apply in the absence of optimism. The intuition is that the marginal optimist behaves, in terms of labor demand, as the marginal realist, under the assumption that the ability distribution is the same for optimists and realists, the labor market is the same, etc.

Online Appendix C: Non-Pecuniary Benefits

This appendix derives and calibrates the competitive equilibrium with non-pecuniary benefits. We consider additive and multiplicative non-pecuniary benefits since both types have been studied in the literature. Assume fraction $\mu\in(0,1)$ of the population derives a non-pecuniary benefit from entrepreneurship and fraction $1-\mu$ does not. The utility of an entrepreneur who employs l workers and rents k units of capital is

$$u(\theta, w, r, B) = \pi(\theta, w, r) + B = \theta l^{\alpha} k^{\beta} - wl + r(K/N - k) + B, \tag{22}$$

where $B \geq 0$. The parameter B measures the intensity of additive non-pecuniary benefits. Entrepreneurs for whom $B = 0$ are called R-entrepreneurs and those with $B > 0$ are called B-entrepreneurs. An entrepreneur solves

$$\max_{l,k} \left[\theta l^{\alpha} k^{\beta} - wl + r(K/N - k) + B\right].$$

The first-order conditions are $\alpha\theta l^{\alpha-1}k^{\beta} = w$, and $\beta\theta l^{\alpha}k^{\beta-1} = r$. Solving for l and k we obtain the input demands:

$$l(\theta, w, r) = \theta^{\frac{1}{1-\eta}} \left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}} \left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}, \tag{23}$$

and

$$k(\theta, w, r) = \theta^{\frac{1}{1-\eta}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}} \left(\frac{\beta}{r}\right)^{\frac{1-\alpha}{1-\eta}}. \tag{24}$$

The input demands determine the size of the firm given the ability of the entrepreneur, the wage, the rental cost of capital. Substituting (23) and (24) into (22) we obtain the reduced form utility of an entrepreneur:

$$u(\theta, w, r, B) = \theta^{\frac{1}{1-\eta}} (1-\eta) \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}} \left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} + r\frac{K}{N} + B. \tag{25}$$

The returns to paid employment are given by

$$w + r\frac{K}{N}. \tag{26}$$

The ability of the marginal R-entrepreneur, $\hat{\theta}_R$, is obtained by setting $B = 0$ in (25) and equating this to (26). Hence, an individual with ability $\hat{\theta}_R$ is indifferent between being an entrepreneur and a worker when

$$\hat{\theta}_R^{\frac{1}{1-\eta}} (1-\eta) \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}} \left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} = w. \tag{27}$$

The ability of the marginal B-entrepreneur, $\hat{\theta}_B$, is obtained by equating (25) to (26). Hence, an individual with ability $\hat{\theta}_B$ is indifferent between being an entrepreneur and a worker when

$$\hat{\theta}_B^{\frac{1}{1-\eta}}(1-\eta)\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} = w - B. \tag{28}$$

Since the reduced form utility of an entrepreneur is an increasing and convex function of θ it follows from (27) and (28) that there exist a unique ability cut-off between R-entrepreneurs and R-workers–$\hat{\theta}_R$ is unique–and an unique ability cut-off between B-entrepreneurs and B-workers–$\hat{\theta}_B$ is unique. Moreover, from (27) and (28) we have

$$\hat{\theta}_B = \hat{\theta}_R\left(\frac{w-B}{w}\right)^{1-\eta} < \hat{\theta}_R. \tag{29}$$

The labor market equilibrium condition is

$$\begin{aligned}\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} \frac{\rho\theta_m^\rho}{\rho-\frac{1}{1-\eta}}\left[(1-\mu)\hat{\theta}_R^{\frac{1}{1-\eta}-\rho}+\mu\hat{\theta}_B^{\frac{1}{1-\eta}-\rho}\right] \\ = 1-\theta_m^\rho\left[(1-\mu)\hat{\theta}_R^{-\rho}+\mu\hat{\theta}_B^{-\rho}\right],\end{aligned} \tag{30}$$

and the capital market equilibrium condition is

$$\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{1-\alpha}{1-\eta}} \frac{\rho\theta_m^\rho}{\rho-\frac{1}{1-\eta}}\left[(1-\mu)\hat{\theta}_R^{\frac{1}{1-\eta}-\rho}+\mu\hat{\theta}_B^{\frac{1}{1-\eta}-\rho}\right] = K/N. \tag{31}$$

Equations (27), (28), (30), and (31) form a system of four equations and four unknowns ($\hat{\theta}_R$, $\hat{\theta}_B$, w, r) which defines a unique competitive equilibrium. Solving (27) and (28) for the unique cut-offs $\hat{\theta}_R$ and $\hat{\theta}_B$ and substituting these into (30) and (31) we obtain the unique equilibrium vector of input prices (w^*, r^*). Finally, from $(\hat{\theta}_R, \hat{\theta}_B, w^*, r^*)$ we obtain the equilibrium labor force

$$L^* = N\left[1-\theta_m^\rho\left[(1-\mu)\hat{\theta}_R^{-\rho}+\mu\hat{\theta}_B^{-\rho}\right]\right],$$

and output level

$$\begin{aligned}Y^* &= (1-\mu)N\int_{\hat{\theta}_R}^{\infty}\theta[l(\theta,w^*,r^*)]^\alpha[k(\theta,w^*,r^*)]^\beta g(\theta)d\theta \\ &\quad +\mu N\int_{\hat{\theta}_B}^{\infty}\theta[l(\theta,w^*,r^*)]^\alpha[k(\theta,w^*,r^*)]^\beta g(\theta)d\theta.\end{aligned}$$

For the equilibrium to be well defined we must have $\hat{\theta}_B \geq \theta_m$. Note that this condition together with (29) imposes an upper bound on B given by

$$B \leq w \left[1 - \left(\frac{\theta_m}{\hat{\theta}_R} \right)^{\frac{1}{1-\eta}} \right].$$

It follows from this inequality that B has to be smaller than the equilibrium wage. We calibrate the model using the technology parameters in Table 1. The behavioral parameters μ and B are calibrated as follows. According to Hurst and Pugsley (2011, pp.75): "Over 50 percent of these new business owners cite non-pecuniary benefits—for example, "wanting flexibility over schedule" or "to be one's own boss"—as a primary reason for starting the business." Hence, we calibrate the fraction of individuals with non-pecuniary benefits at 50 percent, i.e., $\mu = 0.5$.

It is very hard to infer the nature and magnitude of non-pecuniary benefits directly from empirical data. Bartling et al. (2014) estimate experimentally the non-pecuniary benefit of decision rights (i.e., the non-pecuniary benefit of autonomy and control over decision making) to 16.7 percent of the payoff associated to a decision. Owens et al. (2014) find that the average participant in an experiment is willing to sacrifice 8 percent to 15 percent of expected asset earnings to retain control. Jones and Pratap (2020), using a panel of owner-operated New York dairy farms, estimate that for a market wage w equal to $\$15,000$, the non-pecuniary benefit from farming offsets a $\$6,300$ (42 percent) drop in consumption. Based on these three studies we consider two alternative values to calibrate the parameter for additive non-pecuniary benefits. First, we set $B = 0.5$ which corresponds to non-pecuniary benefits of approximately 20 percent of the equilibrium wage w^*. We consider $B = 0.5$ as a lower bound for additive non-pecuniary benefits. Next, we set $B = 1$ which corresponds to non-pecuniary benefits of approximately 40 percent of the equilibrium wage w^*. We consider $B = 1$ as an upper bound for additive non-pecuniary benefits.

Table 3 reports the results with additive non-pecuniary benefits. The third column in Table 3 shows that when $B = 0.5$ the mean returns to entrepreneurship are 26.2 times greater than the wage. The fourth column in Table 3 shows that when $B = 1$ the mean returns to entrepreneurship are 22.4 times greater than the wage. Hence, the model with additive non-pecuniary benefits can explain between one tenth and one fourth of the size of the entrepreneurial earnings puzzle in the U.K.

Table 3. Calibration Results: Additive Non-Pecuniary Benefits (ANPB)

	Lucas' (1978) model	Model with ANPB	Model with ANPB
B	0	0.5	1
Output	6.0200	6.0190	6.0139
Wage	2.5422	2.5521	2.5669
Rental cost of capital	0.0797	0.0797	0.0797
Mean returns to paid employment	3.6457	3.6559	3.6698
Mean returns to entrepreneurship	74.7700	66.9920	57.3710

We now derive and calibrate the model assuming that the non-pecuniary benefits of entrepreneurship are multiplicative instead of additive. That is, we assume the utility of an entrepreneur who employs l workers and rents k units of capital is

$$u(\theta, w, r, B) = B\pi(\theta, w, r) = B[\theta l^{\alpha} k^{\beta} - wl - rK] + rK/N, \tag{32}$$

where $B \geq 1$. The parameter B measures the intensity of multiplicative non-pecuniary benefits. Under the multiplicative specification, non-pecuniary benefits change the utility of the entrepreneur in proportion to the profits of the firm. As before, fraction $\mu \in (0, 1)$ of the population derives a non-pecuniary benefit from entrepreneurship and fraction $1 - \mu$ does not. Entrepreneurs for whom $B = 1$ are called R-entrepreneurs and those with $B > 1$ are called B-entrepreneurs. The input demand functions are left unchanged. Hence, the reduced form utility of an entrepreneur is:

$$u(\theta, w, r, B) = B\left[\theta^{\frac{1}{1-\eta}}(1-\eta)\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}\right] + r\frac{K}{N}. \tag{33}$$

The returns to paid employment are given by

$$w + r\frac{K}{N}. \tag{34}$$

The ability of the marginal R-entrepreneur, $\hat{\theta}_R$, is obtained by setting $B = 1$ in (33) and equating this to (34). Hence, an individual with ability $\hat{\theta}_R$ is indifferent between being an entrepreneur and a worker when

$$\hat{\theta}_R^{\frac{1}{1-\eta}}(1-\eta)\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} = w. \tag{35}$$

The ability of the marginal B-entrepreneur, $\hat{\theta}_B$, is obtained by equating (33) to (34). Hence, an individual with ability $\hat{\theta}_B$ is indifferent between being an entrepreneur and a worker when

$$B\hat{\theta}_B^{\frac{1}{1-\eta}}(1-\eta)\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} = w. \tag{36}$$

Since the reduced form utility of an entrepreneur is an increasing and convex function of θ it follows from (35) and (36) that there exist a unique ability cut-off between R-entrepreneurs and R-workers–$\hat{\theta}_R$ is unique–and an unique ability cut-off between B-entrepreneurs and B-workers–$\hat{\theta}_B$ is unique. Moreover, from (35) and (36) we have

$$\hat{\theta}_B = \frac{\hat{\theta}_R}{B^{1-\eta}} < \hat{\theta}_R. \tag{37}$$

Equations (35), (36), (30), and (31) form a system of four equations and four unknowns ($\hat{\theta}_R$, $\hat{\theta}_B$, w, r) which defines a unique competitive equilibrium. Solving (35) and (36) for the unique cut-offs $\hat{\theta}_R$ and $\hat{\theta}_B$ and substituting these into (30) and (31) we obtain the unique equilibrium vector of input prices (w^*, r^*). Finally, from $(\hat{\theta}_R, \hat{\theta}_B, w^*, r^*)$ we obtain the equilibrium labor force L^* and output level Y^*. For the equilibrium to be well defined we must have $\hat{\theta}_B \geq \theta_m$. Note that this condition together with (37) imposes an upper bound on B given by

$$B \leq \left(\frac{\hat{\theta}_R}{\theta_m}\right)^{\frac{1}{1-\eta}}.$$

We calibrate the model using the technology parameters in Table 1. As before, we set $\mu = 0.5$. As far as we know, only one study provides estimates for multiplicative non-pecuniary benefits. Using Swedish data, Hårsman and Mattsson (2020) estimate multiplicative non-pecuniary benefits of B of 1.05 for electrical engineers. Based on this study we consider two values to calibrate B. First, we set $B = 1.05$, that is, we assume multiplicative non-pecuniary benefits are worth 5% of profits, which we consider as a lower bound. Next, we set $B = 1.2$, that is, we assume multiplicative non-pecuniary benefits are worth 20% of profits, which we consider as an upper bound. Table 4 reports the results with multiplicative non-pecuniary benefits.

Table 4. Calibration Results: Multiplicative Non-Pecuniary Benefits (MNPB)

	Lucas' (1978) model	Model with MNPB	Model with MNPB
B	1	1.05	1.2
Output	6.0200	6.0199	6.0193
Wage	2.5422	2.5445	2.5505
Rental cost of capital	0.0797	0.0797	0.0797
Mean returns to paid employment	3.6457	3.6484	3.6543
Mean returns to entrepreneurship	74.7700	73.0320	68.2720

The third column in Table 4 shows that when $B = 1.05$ the mean returns to entrepreneurship are 28.7 times greater than the wage. The fourth column in Table IV shows that when $B = 1.2$ the mean returns to entrepreneurship are 26.8 times greater than the wage. Hence, the model with multiplicative non-pecuniary benefits can explain between two hundredth and nine hundredth of the size of the entrepreneurial earnings puzzle in the U.K.

Online Appendix D: Introducing Firms without Employees and Heterogeneous Wages

In this appendix we extend the model by introducing firms without employees and heterogeneous wages. We maintain the assumption that the technology of firms with employees is a generalized Cobb-Douglas with decreasing returns to scale, i.e., $y = \theta l^{\alpha} k^{\beta}$, with $\alpha > 0$, $\beta > 0$, and $\alpha + \beta < 1$, and that entrepreneurial ability is distributed according to a Pareto cumulative distribution, i.e., $G(\theta) = 1 - (\theta_m/\theta)^{\rho}$ for $\theta \geq \theta_m > 0$. We assume firms without employees generate a benefit of $B > 0$. To introduce heterogeneous wages we follow Poschke (2013), and assume that the wage earned by a worker with ability θ is $w\theta$, where w is the wage rate per efficiency unit, which is determined endogenously in general equilibrium. Note that now l denotes not the number of workers but, rather, the total number of efficiency units of skill a firm employs.

Under these assumptions, the allocation of individuals into occupations is determined as follows. A realist with ability θ chooses to open a firm without employees if

$$B + r\frac{K}{N} \geq w\theta + r\frac{K}{N},$$

chooses to become a worker if

$$B + r\frac{K}{N} < w\theta + r\frac{K}{N} < \pi(\theta, w, r),$$

and chooses to open a firm with employees if

$$\pi(\theta, w, r) \geq w\theta + r\frac{K}{N},$$

where

$$\pi(\theta, w, r) = \theta^{\frac{1}{1-\eta}}(1-\eta)\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} + r\frac{K}{N}.$$

Hence, a realist with ability $\hat{\theta}_R^L$ is indifferent between opening a firm without employees and being a worker when

$$B = w\hat{\theta}_R^L, \tag{38}$$

and a realist with ability $\hat{\theta}_R^H$ is indifferent between being a worker and opening a firm with employees when

$$(\hat{\theta}_R^H)^{\frac{1}{1-\eta}}(1-\eta)\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} = w\hat{\theta}_R^H,$$

or

$$(\hat{\theta}_R^H)^{\frac{\eta}{1-\eta}}(1-\eta)\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} = w. \tag{39}$$

Similarly, an optimist with ability θ and perception of ability $\gamma\theta$ chooses to open a firm without employees if

$$B + r\frac{K}{N} \geq w\gamma\theta + r\frac{K}{N},$$

chooses to become a worker if

$$B + r\frac{K}{N} < w\gamma\theta + r\frac{K}{N} < \pi(\gamma\theta, w, r),$$

and chooses to open a firm with employees if

$$\pi(\gamma\theta, w, r) \geq w\gamma\theta + r\frac{K}{N},$$

where

$$\pi(\gamma\theta, w, r) = \gamma^{\frac{1}{1-\eta}}\theta^{\frac{1}{1-\eta}}(1-\eta)\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} + r\frac{K}{N}.$$

Hence, an optimist with ability $\hat{\theta}_O^L$ is indifferent between opening a firm without employees and being a worker when

$$B = w\gamma\hat{\theta}_O^L, \tag{40}$$

and an optimist with ability $\hat{\theta}_O^H$ is indifferent between being a worker and opening a firm with employees when

$$\gamma^{\frac{1}{1-\eta}}(\hat{\theta}_O^H)^{\frac{1}{1-\eta}}(1-\eta)\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} = w\gamma\hat{\theta}_O^H,$$

or

$$\gamma^{\frac{\eta}{1-\eta}}(\hat{\theta}_O^H)^{\frac{\eta}{1-\eta}}(1-\eta)\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} = w. \tag{41}$$

It follows from (38), (40), and $\gamma > 1$ that $\hat{\theta}_O^L = \hat{\theta}_R^L/\gamma < \hat{\theta}_R^L$. This result implies that optimists are *less likely* to open a firm without employees than realists. It follows from (39), (41), and $\gamma > 1$ that $\hat{\theta}_O^H/\gamma < \hat{\theta}_R^H$. This result implies that optimists are *more likely* to open a firm with employees than realists.

The demand for efficiency units of labor from realistic entrepreneurs is

$$L_R^D = N(1-\lambda)\int_{\hat{\theta}_R^H}^{\infty} l(\theta, w, r)g(\theta)d\theta = N(1-\lambda)\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}\rho\theta_m^\rho\frac{(\hat{\theta}_R^H)^{\frac{1}{1-\eta}-\rho}}{\rho-\frac{1}{1-\eta}}.$$

The demand for efficiency units of labor from optimistic entrepreneurs is

$$L_O^D = N\lambda\int_{\hat{\theta}_O^H}^{\infty} l(\gamma\theta, w, r)g(\theta)d\theta = N\lambda\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}\gamma^{\frac{1}{1-\eta}}\rho\theta_m^\rho\frac{(\hat{\theta}_O^H)^{\frac{1}{1-\eta}-\rho}}{\rho-\frac{1}{1-\eta}}.$$

The supply of efficiency units of labor is

$$\begin{aligned}
L^S &= N\left[(1-\lambda)\int_{\hat{\theta}_R^L}^{\hat{\theta}_R^H}\theta g(\theta)d\theta + \lambda\int_{\hat{\theta}_O^L}^{\hat{\theta}_O^H}\theta g(\theta)d\theta\right] \\
&= N\left[(1-\lambda)\int_{\hat{\theta}_R^L}^{\hat{\theta}_R^H}\theta\rho\theta_m^\rho\theta^{-\rho-1}d\theta + \lambda\int_{\hat{\theta}_O^L}^{\hat{\theta}_O^H}\theta\rho\theta_m^\rho\theta^{-\rho-1}d\theta\right] \\
&= N\rho\theta_m^\rho\left[(1-\lambda)\int_{\hat{\theta}_R^L}^{\hat{\theta}_R^H}\theta^{-\rho}d\theta + \lambda\int_{\hat{\theta}_O^L}^{\hat{\theta}_O^H}\theta^{-\rho}d\theta\right] \\
&= N\rho\theta_m^\rho\left[(1-\lambda)\left[\frac{\theta^{-\rho+1}}{-\rho+1}\right]_{\hat{\theta}_R^L}^{\hat{\theta}_R^H} + \lambda\left[\frac{\theta^{-\rho+1}}{-\rho+1}\right]_{\hat{\theta}_O^L}^{\hat{\theta}_O^H}\right] \\
&= N\frac{\rho\theta_m^\rho}{\rho-1}\left\{(1-\lambda)\left[\left(\hat{\theta}_R^L\right)^{-\rho+1} - \left(\hat{\theta}_R^H\right)^{-\rho+1}\right] + \lambda\left[\left(\hat{\theta}_O^L\right)^{-\rho+1} - \left(\hat{\theta}_O^H\right)^{-\rho+1}\right]\right\}.
\end{aligned}$$

The labor market clearing condition is

$$\begin{aligned}
&\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}\frac{\rho\theta_m^\rho}{\rho-\frac{1}{1-\eta}}\left[(1-\lambda)(\hat{\theta}_R^H)^{\frac{1}{1-\eta}-\rho} + \lambda\gamma^{\frac{1}{1-\eta}}(\hat{\theta}_O^H)^{\frac{1}{1-\eta}-\rho}\right] \\
&= \frac{\rho\theta_m^\rho}{\rho-1}\left\{(1-\lambda)\left[\left(\hat{\theta}_R^L\right)^{-\rho+1} - \left(\hat{\theta}_R^H\right)^{-\rho+1}\right] + \lambda\left[\left(\hat{\theta}_O^L\right)^{-\rho+1} - \left(\hat{\theta}_O^H\right)^{-\rho+1}\right]\right\}. \quad (42)
\end{aligned}$$

The capital demand from realistic entrepreneurs is

$$K_R^D = N(1-\lambda)\int_{\hat{\theta}_R^H}^{\infty} k(\theta,w,r)g(\theta)d\theta = N(1-\lambda)\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{1-\alpha}{1-\eta}}\rho\theta_m^\rho\frac{(\hat{\theta}_R^H)^{\frac{1}{1-\eta}-\rho}}{\rho-\frac{1}{1-\eta}}$$

The capital demand from optimistic entrepreneurs is

$$K_O^D = N\lambda\int_{\hat{\theta}_O^H}^{\infty} k(\gamma\theta,w,r)g(\theta)d\theta = N\lambda\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{1-\alpha}{1-\eta}}\gamma^{\frac{1}{1-\eta}}\rho\theta_m^\rho\frac{(\hat{\theta}_O^H)^{\frac{1}{1-\eta}-\rho}}{\rho-\frac{1}{1-\eta}}$$

The capital market clearing condition is

$$\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{1-\alpha}{1-\eta}}\frac{\rho\theta_m^\rho}{\rho-\frac{1}{1-\eta}}\left[(1-\lambda)(\hat{\theta}_R^H)^{\frac{1}{1-\eta}-\rho}+\lambda\gamma^{\frac{1}{1-\eta}}(\hat{\theta}_O^H)^{\frac{1}{1-\eta}-\rho}\right] = K/N. \tag{43}$$

Equations (38), (39), (40), (41), (42), and (43), form a system of six equations and six unknowns $(\hat{\theta}_R^L, \hat{\theta}_R^H, \hat{\theta}_O^L, \hat{\theta}_O^H, w, r)$ which defines a unique competitive equilibrium. Solving (38) for $\hat{\theta}_R^L$ we obtain

$$\hat{\theta}_R^L = \frac{B}{w}. \tag{44}$$

Solving (40) for $\hat{\theta}_O^L$ we have

$$\hat{\theta}_O^L = \frac{B}{\gamma w}. \tag{45}$$

Substituting (44) and (45) into the labor market clearing condition (42) gives us

$$\begin{aligned}&\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}\frac{\rho\theta_m^\rho}{\rho-\frac{1}{1-\eta}}\left[(1-\lambda)(\hat{\theta}_R^H)^{\frac{1}{1-\eta}-\rho}+\lambda\gamma^{\frac{1}{1-\eta}}(\hat{\theta}_O^H)^{\frac{1}{1-\eta}-\rho}\right]\\ &= \frac{\rho\theta_m^\rho}{\rho-1}\left\{(1-\lambda)\left[\left(\frac{B}{w}\right)^{-\rho+1}-\left(\hat{\theta}_R^H\right)^{-\rho+1}\right]+\lambda\left[\left(\frac{B}{\gamma w}\right)^{-\rho+1}-\left(\hat{\theta}_O^H\right)^{-\rho+1}\right]\right\}.\end{aligned} \tag{46}$$

Equations (39), (41), (43), and (46) form a system of four equations and four unknowns $(\hat{\theta}_R^H, \hat{\theta}_O^H, w, r)$. Solving (39) for $\hat{\theta}_R^H$ we have

$$\hat{\theta}_R^H = (1-\eta)^{\frac{\eta-1}{\eta}}\alpha^{-\frac{\alpha}{\eta}}\beta^{-\frac{\beta}{\eta}}w^{\frac{1-\beta}{\eta}}r^{\frac{\beta}{\eta}}. \tag{47}$$

Solving (41) for $\hat{\theta}_O^H$ we have

$$\hat{\theta}_O^H = \frac{\hat{\theta}_R^H}{\gamma} = (1-\eta)^{\frac{\eta-1}{\eta}} \alpha^{-\frac{\alpha}{\eta}} \beta^{-\frac{\beta}{\eta}} w^{\frac{1-\beta}{\eta}} r^{\frac{\beta}{\eta}} / \gamma. \tag{48}$$

Substituting (47) and (48) into (43) and (46) we obtain a system of two equations and two unknowns (w, r). Note that the solution to this system of six equations and six unknowns $(\hat{\theta}_R^L, \hat{\theta}_R^H, \hat{\theta}_O^L, \hat{\theta}_O^H, w, r)$ is well defined as long as $\hat{\theta}_O^L > \theta_m$, that is, the lowest ability worker has an ability higher than the minimum ability. When $\hat{\theta}_O^L \leq \theta_m$ the general equilibrium is the solution to a system of five equations and five unknowns $(\hat{\theta}_R^L, \hat{\theta}_R^H, \hat{\theta}_O^H, w, r)$. The five equations being (38), (39), (41), (43), and

$$\begin{aligned} &\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}} \left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} \frac{\rho\theta_m^\rho}{\rho - \frac{1}{1-\eta}} \left[(1-\lambda)(\hat{\theta}_R^H)^{\frac{1}{1-\eta}-\rho} + \lambda\gamma^{\frac{1}{1-\eta}}(\hat{\theta}_O^H)^{\frac{1}{1-\eta}-\rho}\right] \\ &= \frac{\rho\theta_m^\rho}{\rho-1}\left\{(1-\lambda)\left[\left(\hat{\theta}_R^L\right)^{-\rho+1} - \left(\hat{\theta}_R^H\right)^{-\rho+1}\right] + \lambda\left[\theta_m^{-\rho+1} - \left(\hat{\theta}_O^H\right)^{-\rho+1}\right]\right\}. \end{aligned} \tag{49}$$

The parameter B, representing benefits from opening a firm without employees plays a critical role determining how labor is allocated across the three occupations. If B is set too low, then the fraction of individuals who open firms without employees is too low. If B is set too high, then the fraction of individuals who open firms without employees is too high. In the British Household Panel Survey the share of employment in firms with 0 employees is roughly 6 percent. In the UK's Business Population Estimates of 2000 the share of private sector employment of firms with 0 employees is roughly 13 percent. Our calibrations with optimists show that values for B less than 2.5 imply that the fraction of individuals who opens firms without employed workers is less than 1 percent whereas values for B greater than 3 imply that the fraction of individuals who opens firms without employees is greater than 16 percent. Hence, we set $B = 2.75$. The results of the calibration are displayed in Table 5.

Table 5. Calibration Results: Firms without Employees and Heterogeneous Wages

	Model without optimists	Model with optimists	Percent change
Output	7.3808	6.9301	−6.11
Wage rate per efficiency unit	2.2420	2.5608	14.22
Rental cost of capital	0.0830	0.1239	49.18
Mean returns to			
Opening a firm without employees	3.9007	4.4677	14.52
Paid employment	5.4814	6.1446	12.10
Opening a firm with employees	74.8160	30.9610	−58.62
Fraction of individuals who			
Manage a firm without employees	0.4043	0.0963	−76.18
Work as paid employees	0.5909	0.8991	52.15
Manage a firm with employees	0.0048	0.0046	−3.97

The second column in Table 5 reports the competitive equilibrium of the model without optimists. This model generates mean returns to opening a firm with employees 13.6 times greater than the mean returns to paid employment. The third column in Table 5 reports the competitive equilibrium of the model with optimists. This model generates mean returns to opening a firm with employees 5 times greater than the mean returns to paid employment. Hence, the calibration shows that optimism can still explain more than half of the size of the entrepreneurial earnings puzzle in the U.K.

The fourth column in Table 5 reports the percent change in the variables common to both models. It shows that optimism leads to a 6.11 percent decline in output, a 14.22 percent increase in the wage rate per efficiency unit, a 49.18 percent increase in the rental rate of capital, a 14.52 increase in the mean returns to opening a firm without employees, a 12.1 percent increase in the mean returns to paid employment, and a 58.62 percent decline in the mean returns to opening a firm with employees. Note that in this model optimism leads to two misallocations of talent. First, it shifts individuals from opening a firm without employees to paid employment (there is a 76.18 percent decline in the fraction of individuals who open a firm without employees and a 52.15 percent increase in the fraction of workers). This happens because individuals overestimate their productivity as workers. Second, as in the baseline model, optimists crowd out realists from opening a firm with employees. This lowers the average ability of the pool of managers of firm with employees by raising the fraction of optimistic managers (who have, on average, lower ability and earn lower mean returns) and lowering the fraction of realistic managers (who have, on average, higher ability and earn higher mean returns). As in the baseline model, optimism also leads to an increase in input prices. These effects lower the mean returns to opening a firm with employees and raise the mean returns to paid employment.

Note that in this model the mean returns to paid employment are computed as $wE(\theta|\text{Work}) + rK/N$, where

$$\begin{aligned}
wE(\theta|\text{Work}) &= w\left\{\lambda E[\theta|\theta \in [\theta_m, \hat{\theta}_O^H]] + (1-\lambda)E[\theta|\theta \in [\hat{\theta}_R^L, \hat{\theta}_R^H]]\right\} \\
&= w\left[\lambda\frac{\int_{\theta_m}^{\hat{\theta}_O^H}\theta g(\theta)d(\theta)}{\int_{\theta_m}^{\hat{\theta}_O^H}g(\theta)d(\theta)} + (1-\lambda)\frac{\int_{\hat{\theta}_R^L}^{\hat{\theta}_R^H}\theta g(\theta)d(\theta)}{\int_{\hat{\theta}_R^L}^{\hat{\theta}_R^H}g(\theta)d(\theta)}\right] \\
&= \frac{\rho\theta_m^\rho}{\rho-1}w\left[\lambda\frac{(\theta_m)^{-\rho+1} - (\hat{\theta}_O^H)^{-\rho+1}}{1-\left(\frac{\theta_m}{\hat{\theta}_O^H}\right)^\rho} + (1-\lambda)\frac{(\hat{\theta}_R^L)^{-\rho+1} - (\hat{\theta}_R^H)^{-\rho+1}}{\left(\frac{\theta_m}{\hat{\theta}_R^L}\right)^\rho - \left(\frac{\theta_m}{\hat{\theta}_R^H}\right)^\rho}\right].
\end{aligned}$$

Online Appendix E: Introducing a Corporate Sector

In this appendix we extend the model by introducing a corporate sector in the economy. We solve the model with the corporate sector and calibrate it. We maintain the assumptions that the technology employed by the entrepreneurial sector is a generalized Cobb-Douglas with decreasing returns to scale, i.e., $y = \theta l^{\alpha} k^{\beta}$, with $\alpha > 0$, $\beta > 0$, and $\alpha + \beta < 1$, and that entrepreneurial ability is distributed according to a Pareto cumulative distribution, i.e., $G(\theta) = 1 - (\theta_m/\theta)^{\rho}$ for $\theta \geq \theta_m > 0$. Following Quadrini (2000), the technology employed by the corporate sector is a Cobb-Douglas with constant returns to scale:

$$Y_c = F(L_c, K_c) = AL_c^{\upsilon} K_c^{1-\upsilon},$$

where $\nu \in (0, 1)$, and $A > 0$ is the corporate sector's total factor productivity. The representative firm in the corporate sector solves

$$\max_{L_c, K_c} AL_c^{\upsilon} K_c^{1-\upsilon} - wL_c - rK_c.$$

The first-order conditions of the representative corporate sector firm are $\upsilon AL_c^{\upsilon-1} K_c^{1-\upsilon} = w$ and $(1-\upsilon)AL_c^{\upsilon} K_c^{-\upsilon} = r$. The competitive equilibrium of the two-sector model with optimists is the solution to the following system of six equations and six unknowns $(\hat{\theta}_R, \hat{\theta}_O, w, r, L_c, K_c)$:

$$\upsilon AL_c^{\upsilon-1} K_c^{1-\upsilon} = w,$$

and

$$(1-\upsilon)AL_c^{\upsilon} K_c^{-\upsilon} = r,$$

and

$$\hat{\theta}_R{}^{\frac{1}{1-\eta}} (1-\eta) \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}} \left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} = w,$$

and

$$\gamma^{\frac{1}{1-\eta}} \hat{\theta}_O^{\frac{1}{1-\eta}} (1-\eta) \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}} \left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}} = w.$$

and

$$\left(\frac{\alpha}{w}\right)^{\frac{1-\beta}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\eta}}\frac{\rho\theta_m^{\rho}}{\rho-\frac{1}{1-\eta}}\left[(1-\lambda)\hat{\theta}_R^{\frac{1}{1-\eta}-\rho}+\lambda\gamma^{\frac{1}{1-\eta}}\hat{\theta}_O^{\frac{1}{1-\eta}-\rho}\right]+L_c$$
$$=1-\theta_m^{\rho}\left[(1-\lambda)\hat{\theta}_R^{-\rho}+\lambda\hat{\theta}_O^{-\rho}\right],$$

and

$$\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\eta}}\left(\frac{\beta}{r}\right)^{\frac{1-\alpha}{1-\eta}}\frac{\rho\theta_m^{\rho}}{\rho-\frac{1}{1-\eta}}\left[(1-\lambda)\hat{\theta}_R^{\frac{1}{1-\eta}-\rho}+\lambda\gamma^{\frac{1}{1-\eta}}\hat{\theta}_O^{\frac{1}{1-\eta}-\rho}\right]+K_c=K/N.$$

The calibration uses the same parameters as in Table 1. The parameter υ, representing the labor income share in the corporate sector, is set to equalize the labor income share of the entrepreneurial sector at $\upsilon=\alpha/(\alpha+\beta)=0.69$. The parameter A, representing TFP in the corporate sector plays a critical role determining how labor and capital are allocated across the two sectors. If A is set too low, then the entrepreneurial sector will employ much more labor and capital than the corporate sector. If A is set too high, then the reverse happens. In the British Household Panel Survey the share of employment in firms with less than 200 employees is roughly 40 percent (for firms larger than 0 employees). In the UK's Business Population Estimates of 2000 the share of private sector employment in firms with less than 250 employees is roughly 60 percent (the usual definition of small and medium sized enterprises (SMEs) is any business with fewer than 250 employees). In addition, in UK's Business Population Estimates of 2000 the share of private sector business turnover of firms with less than 250 employees is roughly 52 percent. Our calibrations show that values for A less than 2 imply a corporate sector that is too small whereas values for A greater than 4 imply a corporate sector that is too large. Hence, our baseline calibration sets $A=3$. The results of the calibration are displayed in Table 6.

Table 6. Calibration Results: Introducing a Corporate Sector

	Model without optimists	Model with optimists	Percent change
Output $Y = Y_c + Y_e$	7.6919	7.2964	−5.14
Wage	4.6852	4.6963	0.24
Rental cost of capital	0.1510	0.1502	−0.53
Mean returns to paid employment	6.7754	6.7756	0
Mean returns to entrepreneurship	75.7570	31.3270	−58.65
Output of corporate sector Y_c	5.3675	3.8268	−28.70
L_c	0.7905	0.5623	−28.87
K_c	11.0220	7.9001	−28.32
Output of entrepreneurial sector Y_e	2.3244	3.4696	49.27
L_e	0.2025	0.4231	108.94
K_e	2.8240	5.9459	110.55

The second column in Table 6 reports the competitive equilibrium of the two-sector model without optimists. This model generates mean returns to entrepreneurship 16.2 times greater than the wage and 11.2 times greater than the mean returns to paid employment. These values are lower as the ones obtained in the one-sector model without optimists. The reason is that the existence of a corporate sector raises the equilibrium wage significantly above the value of the equilibrium wage in the one-sector model.

The third column in Table 6 reports the competitive equilibrium of the two-sector model with optimists. This model generates mean returns to entrepreneurship 6.7 times greater than the wage and 4.6 times greater than the mean returns to paid employment. These values are lower than the ones obtained in the one-sector model with optimists. Again, this is due to the effect of the corporate sector on the equilibrium wage. Hence, the calibration of the two-sector model shows that optimism can explain more than half of the size of the entrepreneurial earnings puzzle in the U.K.

The fourth column in Table 6 reports the percent change in the variables common to both models. It shows that optimism leads to a 5.14 percent decline in output, a 0.24 percent increase in the wage, a 0.53 percent decrease in the rental rate of capital, no change in the mean returns to paid employment, and a 58.65 percent decline in the mean returns to entrepreneurship. In this two-sector model, the large decline in the mean returns to entrepreneurship happens due to three channels. First, the misallocation of talent due to the fact that optimists crowd out realists from entrepreneurship. This lowers the average ability of the pool of entrepreneurs by raising the fraction of optimistic entrepreneurs (who have, on average, lower ability and earn lower mean returns) and lowering the fraction of realistic entrepreneurs (who have, on average, higher ability and earn higher mean returns). Second, the misallocation of inputs due to the fact that optimistic entrepreneurs hire an excessive amount of labor and capital in relation to their true ability. Third, the misallocation of inputs from the corporate sector to the entrepreneurial sector.

Online Appendix F: Sensitivity Analysis for Shape Parameter of Firm Size Distribution

In this appendix we perform a sensitivity analysis with respect to ξ the shape parameter of the firm size distribution in the UK. In our baseline calibration we take $\xi = 1.0357$. This value for ξ was obtained applying the method of moments to the BHPS firm size distribution. Previous estimates for ξ in the U.K. are 0.995 (Fujiwara et al. 2004) and 1.0089 (Görg et al. 2017). Hence, the value for the parameter should not be far from 1. Note that the model is not well defined when $\xi \leq 1$. Furthermore, most calibrations for the degree of decreasing returns to scale η fall inside a range which goes from 0.5 up to 0.85 (Atkenson and Kehoe 2005, Hsieh and Klenow 2009, Bachmann and Elstner 2015, Garicano et al. 2016). The fact that α, β, and ρ are the solution to

$$\begin{cases} \xi = \rho(1-\alpha-\beta) \\ \alpha/\beta = 0.69/0.31 \\ 0.9667 = \frac{\alpha\rho}{\rho(1-\beta)-1} \end{cases},$$

implies that η increases with ξ. Solving this system of three equations and three unknowns such that $\eta = 0.85$ we obtain $\xi = 1.155$. Hence, to determine how sensitive our results are to the Pareto shape parameter for the firm size distribution we consider values for ξ in a range which goes from 1 up to 1.155. Within this range we consider six values: 1.0089, 1.025, 1.050, 1.075, 1.100, and 1.125.

In all cases we set the capital-output ratio to 2.3, the intensity of optimistic beliefs to $\gamma = 1.6607$, and the fraction of optimistic entrepreneurs to 72.2 percent. Table 7 summarizes the sensitivity analysis' model parameters.

Table 7. Sensitivity Analysis for Shape Parameter ξ: Model Parameters

ξ	1.0089	1.025	1.050	1.075	1.100	1.125
Technology parameters						
α	0.1868	0.3495	0.4603	0.5147	0.5470	0.5684
β	0.0839	0.1570	0.2068	0.2312	0.2458	0.2554
$\eta = \alpha + \beta$	0.2707	0.5065	0.6671	0.7459	0.7928	0.8238
K	104.3150	22.1865	9.3974	6.4396	5.2262	4.5870
N	1	1	1	1	1	1
ρ	1.3833	2.0768	3.1536	4.2304	5.3073	6.3841
θ_m	1	1	1	1	1	1
Behavioral parameters						
γ	1.6607	1.6607	1.6607	1.6607	1.6607	1.6607
λ	0.5629	0.4753	0.3441	0.2330	0.1496	0.0925

Table 7 shows that as the shape parameter ξ increases from 1.0089 to 1.125 the degree of decreasing returns to scale parameter η increases from 0.2707 to 0.8238. Hence, this interval contains the usual range of the parameter η. We also see from the table that as ξ increases the the shape parameter of the ability distribution ρ increases and the fraction of optimists in the population λ decreases. This happens because we assume the fraction of optimistic entrepreneurs is fixed at 72.2 percent.

Table 8 reports the results of the sensitivity analysis for Lucas' (1978) model.

Table 8. Sensitivity Analysis for Shape Parameter ξ:
Calibration Results for Lucas' (1978) Model

ξ	1.0089	1.025	1.050	1.075	1.100	1.125
Output	45.354	9.6462	4.0858	2.7998	2.2723	1.9943
Wage	8.7615	3.487	1.9453	1.4906	1.2857	1.1726
Rental cost of capital	0.0365	0.0683	0.0899	0.1005	0.1069	0.1110
Returns to paid employment	12.567	5.0015	2.7902	2.1380	1.8441	1.6819
Returns to entrepreneurship	1000.5	144.46	41.703	22.016	14.702	11.063

Table 8 shows that as the shape parameter ξ increases from 1.0089 to 1.125 the output and the wage monotonically decrease. In contrast, the rental rate of capital monotonically increases. More interestingly, the mean returns to paid employment decrease at a slower rate than the decrease in the mean returns to entrepreneurship. This means that the Lucas' model predicts a smaller entrepreneurial earnings premium for higher values of the shape parameter of the firm size distribution. This makes sense since as the shape parameter increases more of the mass of the distribution comes from the lower ability entrepreneurs.

Table 9 reports the results of the sensitivity analysis for the model with optimists.

Table 9. Sensitivity Analysis for Shape Parameter ξ:
Calibration Results for Model with Optimists

ξ	1.0089	1.025	1.050	1.075	1.100	1.125
Output	44.841	9.3357	3.8233	2.5366	1.9994	1.7109
Wage	12.153	4.7348	2.5539	1.8947	1.5872	1.4114
Rental cost of capital	0.0506	0.0927	0.1180	0.1278	0.1319	0.1336
Returns to paid employment	17.432	6.7912	3.6632	2.7176	2.2765	2.0244
Returns to entrepreneurship	843.3	83.188	8.4729	−2.7188	−6.0493	−7.3898

Table 9 shows that as the shape parameter ξ increases from 1.0089 to 1.125 the model with optimists predicts an increasingly smaller entrepreneurial earnings premium. In fact, for the value $\xi = 1.075$ the mean returns to entrepreneurship are negative. Comparing the values in Tables 8 and 9 we see that the model with optimists is still able to account for a reasonable size of the entrepreneurial earnings puzzle even with the minimum value of $\xi = 1.0089$ (Görg et al. 2017). Furthermore, if the shape parameter is higher than the one in the baseline calibration, the model is able to account for an even higher size of the puzzle.