Overconfidence in Elimination Contests
Online Appendix

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1 Elimination Contest with Tullock CSF

This section shows that our main results obtained with Alcalde and Dahm's (2007) CSF also extend to Tullock's (1980) CSF. As in the paper, we consider a two-stage elimination contest where player 1 is overconfident and players 2, 3, and 4 are rational. Players 1 and 2 are paired in one semifinal and players 3 and 4 are paired in the other semifinal. Now, player i's probability of winning when paired with j is

$$p_{ij} = \begin{cases} \frac{q(e_i)}{q(e_i)+q(e_j)} & \text{if} \quad q(e_i)+q(e_j) > 0 \\ \frac{1}{2} & \text{if} \quad q(e_i)+q(e_j) = 0 \end{cases}$$

where the function $q(e_i)$ satisfies $q(0) \geqslant 0$, $q'(e_i) > 0$, and $q''(e_i) \leqslant 0$. Following Santos-Pinto and Sekeris (2023), an overconfident player i's perceived probability of winning

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when paired with j is

$$\widetilde{p}_{ij} = \begin{cases} \frac{\lambda_i q(e_i)}{\lambda_i q(e_i)+q(e_j)} & \text{if} \quad \lambda_i q(e_i) + q(e_j) > 0 \\ \frac{1}{2} & \text{if} \quad \lambda_i q(e_i) + q(e_j) = 0 \end{cases}$$

where $\lambda_i > 1$. We start by analyzing the final between the overconfident player 1 and the rational player 3. This leads to three results. First, the overconfident player exerts less effort at equilibrium. Second, the overconfident player's perceived equilibrium expected utility of the final increases in his bias. Third, the overconfident player's perceived equilibrium winning probability is always greater than 1/2.

1.1 Final

Consider a final between an overconfident player 1 and a rational player 3. Player 1 chooses the optimal effort level that maximizes his perceived expected utility:

$$\begin{aligned} \widetilde{E}^f(U_{13}) &= \widetilde{p}_{13}[u(w_1) - u(w_2)] - ce_1 + u(w_2) \\ &= \frac{\lambda_1 q(e_1)}{\lambda_1 q(e_1) + q(e_3)}[u(w_1) - u(w_2)] - ce_1 + u(w_2). \end{aligned}$$

Similarly, player 3 chooses the optimal level of effort that maximizes her expected utility:

$$\begin{aligned} E^f(U_{31}) &= p_{31}[u(w_1) - u(w_2)] - ce_3 + u(w_2) \\ &= \frac{q(e_3)}{q(e_1) + q(e_3)}[u(w_1) - u(w_2)] - ce_3 + u(w_2). \end{aligned}$$

The first-order conditions are

$$\frac{\lambda_1 q'(e_1^f)q(e_3^f)}{[\lambda_1 q(e_1^f) + q(e_3^f)]^2}[u(w_1) - u(w_2)] = c, \tag{1}$$

and

$$\frac{q'(e_3^f)q(e_1^f)}{[q(e_1^f) + q(e_3^f)]^2}[u(w_1) - u(w_2)] = c. \tag{2}$$

It is not possible to solve (1) and (2) explicitly for the equilibrium efforts in the final. Nevertheless, Santos-Pinto and Sekeris (2023) characterize the equilibrium of a one-shot Tullock contest between two overconfident players. Here, we show that their results also apply to a one-stage Tullock contest between an overconfident player 1 and a rational player 3. Namely, the overconfident player 1 exerts less effort than the rational player 3.

To prove this result we follow the approach in Santos-Pinto and Sekeris (2023). We know from Lemma 4 in Santos-Pinto and Sekeris (2023) that the best response of player 1, $R_1(e_3)$, is quasiconcave in e_3 and reaches a maximum at $q(e_3) = \lambda_1 q(e_1)$. Moreover, the slope of the best response of player 1, $R_1'(e_3)$, is positive for $\lambda_1 q(e_1) > q(e_3)$, zero for $\lambda_1 q(e_1) = q(e_3)$, and negative for $\lambda_1 q(e_1) < q(e_3)$. This implies that $R_1(e_3)$ increases in e_3 for $\lambda_1 q(e_1) > q(e_3)$, reaches the maximum at $\lambda_1 q(e_1) = q(e_3)$, and decreases in e_3 for $\lambda_1 q(e_1) < q(e_3)$. We now show, using the same approach as Proposition 2 in Santos-Pinto and Sekeris (2023), that the best response of the overconfident player 1 crosses the 45 degree line at a lower value of effort than the best response of the rational player 3. At the 45 degree line the best response of player 1 takes the value e_L given by

$$\frac{\lambda_1 q'(e_L)}{(1 + \lambda_1)^2 q(e_L)}[u(w_1) - u(w_2)] = c. \tag{3}$$

At 45 degree line the best response of player 3 takes the value e_H given by

$$\frac{q'(e_H)}{4q(e_H)}[u(w_1) - u(w_2)] = c. \tag{4}$$

Note that $\lambda_1 > 1$ implies

$$\frac{\lambda_1}{(1+\lambda_1)^2} < \frac{1}{4}, \tag{5}$$

Therefore (3), (4), and (5) imply

$$\frac{q'(e_H)}{q(e_H)} < \frac{q'(e_L)}{q(e_L)}.$$

Given that $q(.)$ is (weakly) concave, this inequality can only be satisfied provided $e_L < e_H$. This and the shape of the players' best responses imply the equilibrium lies above the 45 degree line. Hence, the overconfident player 1 exerts less effort than the rational player 3 in the final.

Letting (e_1^f, e_3^f) denote the players' equilibrium efforts in the final, the overconfident player's perceived equilibrium expected utility of the final is

$$\widetilde{E}^f(U_{13}) = \frac{\lambda_1 q(e_1^f)}{\lambda_1 q(e_1^f) + q(e_3^f)}[u(w_1) - u(w_2)] - ce_1^f + u(w_2).$$

The impact of player 1's confidence on his perceived equilibrium expected utility of the final is

$$\begin{aligned}\frac{\partial \widetilde{E}^f(U_{13})}{\partial \lambda_1} &= \left[\frac{q(e_1^f)q(e_3^f)}{[\lambda_1 q(e_1^f) + q(e_3^f)]^2} + \frac{\lambda_1 q'(e_1^f)q(e_3^f)}{[\lambda_1 q(e_1^f) + q(e_3^f)]^2}\frac{\partial e_1^f}{\partial \lambda_1} - \frac{\lambda_1 q(e_1^f)q'(e_3^f)}{[\lambda_1 q(e_1^f) + q(e_3^f)]^2}\frac{\partial e_3^f}{\partial \lambda_1}\right] \\ &\quad [u(w_1) - u(w_2)] - c\frac{\partial e_1^f}{\partial \lambda_1} \\ &= \frac{q(e_1^f)}{[\lambda_1 q(e_1^f) + q(e_3^f)]^2}\left[q(e_3^f) - \lambda_1 q'(e_3^f) R_3'(e_1^f)\frac{\partial e_1^f}{\partial \lambda_1}\right][u(w_1) - u(w_2)]\end{aligned}$$

where the second equality follows from the Envelope Theorem and $e_3^f = R_3(e_1^f)$. Observe that $R_3'(e_1^f)$ is positive and $\partial e_1^f/\partial \lambda_1$ is negative. Hence, the overconfident player's perceived equilibrium expected utility of the final increases in his bias. As player 1's overconfidence increases, his best response, $R_1(e_3)$, shifts inwards for low values of e_3. Since the best response of player 3, $R_3(e_1)$, is positively sloped for low values of e_3 and remains unaffected by this shift in player 1's overconfidence, both players' equilibrium efforts decrease in the bias of player 1.

Observe also that $R_1'(e_3^f)$ is positive. This and Lemma 4 in Santos-Pinto and Sekeris (2023) implies $\lambda_1 q(e_1^f) > q(e_3^f)$, which, in turn, implies

$$\widetilde{p}_{13}^f = \frac{\lambda_1 q(e_1^f)}{\lambda_1 q(e_1^f) + q(e_3^f)} > \frac{1}{2}.$$

Hence, the overconfident player's perceived equilibrium probability of winning the final is always greater than 1/2.

1.2 Semifinals

Let us now turn our attention to the semifinals of a two-stage elimination Tullock contest. We start with the semifinal between the overconfident player 1 and the rational player 2. Player 1 chooses the optimal effort level that maximizes his perceived expected utility:

$$\widetilde{E}^s(U_{12}) = \widetilde{p}_{12}\widetilde{v}_1 - ce_1 = \frac{\lambda_1 q(e_1)}{\lambda_1 q(e_1) + q(e_2)}\widetilde{v}_1 - ce_1,$$

where $\widetilde{v}_1 = \widetilde{E}^f(U_{13})$. Similarly, player 2 chooses the optimal level of effort that maximizes her expected utility:

$$E^s(U_{21}) = p_{21}v_2 - ce_2 = \frac{q(e_2)}{q(e_1) + q(e_2)}v_2 - ce_2.$$

where $v_2 = E^f(U_{23})$. Hence, in the semifinal between an overconfident player 1 and a rational player 2, the equilibrium efforts (e_1^s, e_2^s) satisfy

$$\frac{\lambda_1 q'(e_1^s)q(e_2^s)}{[\lambda_1 q(e_1^s) + q(e_2^s)]^2}\widetilde{v}_1 = c,$$

and

$$\frac{q'(e_2^s)q(e_1^s)}{[q(e_1^s) + q(e_2^s)]^2}v_2 = c,$$

Since the overconfident player's perceived continuation value $\widetilde{v}_1$ increases in his bias, overconfidence has an encouragement effect in the semifinal of a two-stage elimination Tullock contest. The complacency effect is given by

$$\frac{\partial mg\widetilde{p}_{12}^s}{\partial \lambda_1}\frac{\lambda_1}{mg\widetilde{p}_{12}^s} = -q(e_2^s)q'(e_1^s)\frac{\lambda_1 q(e_1^s) - q(e_2^s)}{(\lambda_1 q(e_1^s) + q(e_2^s))^3}\frac{\lambda_1}{\frac{\lambda_1 q(e_2^s)q'(e_1^s)}{(\lambda_1 q(e_1^s)+q(e_2^s))^2}} = -\frac{\lambda_1 q(e_1^s) - q(e_2^s)}{\lambda_1 q(e_1^s) + q(e_2^s)}. \quad (6)$$

It follows from (6) that there is a complacency effect when $\lambda_1 q(e_1^s) > q(e_2^s)$. Hence, the effect of overconfidence on the equilibrium efforts in the semifinal between an overconfident and a rational player depends on the sizes of the encouragement and complacency effects.

In the semifinal between rational players 3 and 4, the positive probability of meeting the overconfident player 1 in the final raises both players' continuation values. Hence, both players will exert higher effort in the semifinal than if all players were rational.

Overall, we see that the effects of overconfidence on the semifinals of a two-stage Tullock contest are similar to those on the semifinals of a two-stage Alcalde and Dahm contest.

2 Elimination Contest where Overconfidence is Unobsevable

This section shows that our results also hold when the overconfident player's rivals cannot observe his bias. As in the paper, we assume player 1 is overconfident and players 2, 3, and 4 are rational with $\lambda_1 > 1 = \lambda_2 = \lambda_3 = \lambda_4$. Players 1 and 2 are paired in one semifinal and players 3 and 4 are paired in the other semifinal. The overconfident player's bias is not observable by the rational players.

2.1 Final

Proposition A1 *In a final between an overconfident player and a rational player where the overconfident player's bias is not observable by the rational player, the equilibrium effort of the overconfident player is*

$$e_1^f = \frac{\alpha}{2c}\lambda_1^{-\frac{1}{\alpha+1}}[u(w_1) - u(w_2)],$$

and the equilibrium effort of the rational player is

$$e_3^f = \frac{\alpha}{2c}[u(w_1) - u(w_2)].$$

with $e_1^f < e_3^f = \overline{e}^f$. *The perceived equilibrium winning probabilities are*

$$\widetilde{p}_{13}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{1}{\alpha+1}},$$

$$\widetilde{p}_{31}^f = \frac{1}{2}$$

and the true equilibrium winning probabilities are

$$p_{13}^f = \frac{1}{2}\lambda_1^{-\frac{\alpha}{\alpha+1}}$$

$$p_{31}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{\alpha+1}}$$

with $\widetilde{p}_{13}^f > p_{31}^f > 1/2 = \widetilde{p}_{31}^f > p_{13}^f$. *The perceived equilibrium expected utilities are*

$$\widetilde{E}^f(U_{13}) = \left(1 + \frac{u(w_2)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)[u(w_1) - u(w_2)],$$

$$\widetilde{E}^f(U_{31}) = \left(1 + \frac{u(w_2)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)[u(w_1) - u(w_2)]$$

with $\widetilde{E}^f(U_{13}) > \widetilde{E}^f(U_{31}) = \overline{E}^f(U)$.

Proof of Proposition A1

Since the rational player 3 is unaware that player 1 is overconfident, she chooses the benchmark effort $e_3^f = \overline{e}^f$. The overconfident player 1 chooses a best response to $e_3^f = \overline{e}^f$. Assume the equilibrium satisfies $\lambda_1(e_1^f)^\alpha \geqslant (e_3^f)^\alpha$. In this case, the best response to $e_3^f = \overline{e}^f$ is the solution to

$$\frac{\alpha}{2\lambda_1}\frac{(\overline{e}^f)^\alpha}{(e_1^f)^{\alpha+1}}[u(w_1) - u(w_2)] = c.$$

Substituting $\overline{e}^f$ by $\frac{\alpha}{2c}[u(w_1) - u(w_2)]$ and solving for e_1^f we have

$$\widetilde{e}_1^f = \frac{\alpha}{2c}\lambda_1^{-\frac{1}{\alpha+1}}[u(w_1) - u(w_2)].$$

Note that this solution satisfies $\lambda_1(e_1^f)^\alpha \geqslant (e_3^f)^\alpha$ since

$$(\lambda_1)^{\frac{1}{\alpha}}\widetilde{e}_1^f = (\lambda_1)^{\frac{1}{\alpha}}\frac{\alpha}{2c}\lambda_1^{-\frac{1}{\alpha+1}}[u(w_1) - u(w_2)] = \frac{\alpha}{2c}\lambda_1^{\frac{1}{\alpha(\alpha+1)}}[u(w_1) - u(w_2)] > \frac{\alpha}{2c}[u(w_1) - u(w_2)] = \overline{e}^f.$$

Now, assume the equilibrium satisfies $\lambda_1(e_1^f)^\alpha \leqslant (e_3^f)^\alpha$. In this case, the best response to $e_3^f = \bar{e}^f$ is the solution to

$$\frac{\alpha\lambda_1}{2}\frac{(e_1^f)^{\alpha-1}}{(\bar{e}^f)^\alpha}[u(w_1) - u(w_2)] = c.$$

Substituting $\bar{e}^f$ by $\frac{\alpha}{2c}[u(w_1) - u(w_2)]$ and solving for e_1^f we have

$$\widetilde{e}_1^f = \frac{\alpha}{2c}\lambda_1^{\frac{1}{1-\alpha}}[u(w_1) - u(w_2)].$$

This is not a feasible solution since it fails to satisfy $\lambda_1(e_1^f)^\alpha \leqslant (e_3^f)^\alpha$. Hence, player 1' equilibrium effort is

$$\widetilde{e}_1^f = \frac{\alpha}{2c}\lambda_1^{-\frac{1}{\alpha+1}}[u(w_1) - u(w_2)].$$

Therefore, player 1's perceived winning probability is

$$\widetilde{p}_{13}^f = 1 - \frac{1}{2}\frac{(\bar{e}^f)^\alpha}{\lambda_1(\widetilde{e}_1^f)^\alpha} = 1 - \frac{1}{2}\frac{\left(\frac{\alpha}{2c}[u(w_1) - u(w_2)]\right)^\alpha}{\lambda_1\left(\frac{\alpha}{2c}\lambda_1^{-\frac{1}{\alpha+1}}[u(w_1) - u(w_2)]\right)^\alpha} = 1 - \frac{1}{2}\lambda_1^{-\frac{1}{\alpha+1}} > \frac{1}{2}.$$

Player 3's perceived winning probability is $\widetilde{p}_{31}^f = 1/2$ since she thinks, mistakenly, player 1 is rational. Player 1's true winning probability is

$$p_{13}^f = \frac{1}{2}\frac{(\widetilde{e}_1^f)^\alpha}{(\bar{e}^f)^\alpha} = \frac{1}{2}\frac{\left(\frac{\alpha}{2c}\lambda_1^{-\frac{1}{\alpha+1}}[u(w_1) - u(w_2)]\right)^\alpha}{\left(\frac{\alpha}{2c}[u(w_1) - u(w_2)]\right)^\alpha} = \frac{1}{2}\lambda_1^{-\frac{\alpha}{\alpha+1}} < \frac{1}{2}.$$

Player 3's true winning probability is

$$p_{31}^f = 1 - p_{13}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{\alpha+1}} > \frac{1}{2}.$$

The perceived expected utility of player 1 is

$$\widetilde{E}^f(U_{13}) = \widetilde{p}_{13}^f[u(w_1) - u(w_2)] - c\widetilde{e}_1^f + u(w_2) = \left(1 + \frac{u(w_2)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)[u(w_1) - u(w_2)].$$

The perceived expected utility of player 3 is

$$\widetilde{E}^f(U_{31}) = \widetilde{p}_{31}^f[u(w_1) - u(w_2)] - c\bar{e}^f + u(w_2) = \left(1 + \frac{u(w_2)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)[u(w_1) - u(w_2)].$$

2.2 Semifinals

Proposition A2 *Consider a semifinal between an overconfident player and a rational player of a two-stage elimination contest where player 1 is overconfident, players 2, 3 and 4 are rational, and the overconfident player's bias is not observable by the rational players.*
(i) If $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2}{\alpha}$ *and* $\lambda_1 < \underline{\lambda}$ *where* $\underline{\lambda}$ *solves* $\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\underline{\lambda}^{-\frac{1}{\alpha+1}}\right) = \underline{\lambda}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)$*, then the equilibrium efforts and winning probabilities satisfy* $\widetilde{e}_1^s > \widetilde{e}_2^s = \bar{e}^s$ *and* $\widetilde{p}_{12}^s > p_{12}^s > 1/2 = \widetilde{p}_{21}^s > p_{21}^s$.
(ii) If either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2}{\alpha}$ *or* $\lambda_1 \geqslant \underline{\lambda}$*, then the equilibrium efforts and winning probabilities satisfy* $\widetilde{e}_1^s \leqslant \widetilde{e}_2^s = \bar{e}^s$ *and* $\widetilde{p}_{12}^s > p_{21}^s \geqslant \frac{1}{2} = \widetilde{p}_{21}^s \geqslant p_{12}^s$.

Proof of Proposition A2

The perceived continuation value of the overconfident player 1 is

$$\begin{aligned}\widetilde{v}_1 &= p^f_{34}\widetilde{E}^f(U_{13}) + p^f_{43}\widetilde{E}^f(U_{14}) \\ &= \widetilde{E}^f(U_{13}) \\ &= \left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)[u(w_1)-u(w_2)] \\ &> \overline{v}\end{aligned}$$

The continuation value of the rational player 2 is

$$\begin{aligned}v_2 &= p^f_{34}E^f(U_{23}) + p^f_{43}E^f(U_{24}) \\ &= E^f(U_{23}) \\ &= \left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)[u(w_1)-u(w_2)] \\ &= \overline{v}\end{aligned}$$

Since the rational player 2 is unaware that player 1 is overconfident, she chooses the benchmark effort $e^s_2 = \overline{e}^s$. The overconfident player 1 chooses a best response to $e^s_2 = \overline{e}^s$. Assume the equilibrium (e^s_1, e^s_2) satisfies $\lambda_1(e^s_1)^\alpha \geqslant (e^s_2)^\alpha$. In this case, the best response to $e^s_2 = \overline{e}^s$ is the solution to

$$\frac{\alpha}{2\lambda_1}\frac{(\overline{e}^s)^\alpha}{(e^s_1)^{\alpha+1}}\widetilde{v}_1 = c.$$

Substituting $\overline{e}^s$ by $\frac{\alpha}{2c}\overline{v}$ and solving for e^s_1 we have

$$\widetilde{e}^s_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{1}{1+\alpha}}(\widetilde{v}_1)^{\frac{1}{1+\alpha}}(\overline{v})^{\frac{\alpha}{1+\alpha}}.$$

Note that this solution satisfies $\lambda_1(e^s_1)^\alpha \geqslant (e^s_2)^\alpha$ since

$$(\lambda_1)^{\frac{1}{\alpha}}\widetilde{e}^s_1 = (\lambda_1)^{\frac{1}{\alpha}}\frac{\alpha}{2c}\lambda_1^{-\frac{1}{1+\alpha}}(\widetilde{v}_1)^{\frac{1}{1+\alpha}}(\overline{v})^{\frac{\alpha}{1+\alpha}} = \frac{\alpha}{2c}\lambda_1^{\frac{1}{\alpha(\alpha+1)}}(\widetilde{v}_1)^{\frac{1}{1+\alpha}}(\overline{v})^{\frac{\alpha}{1+\alpha}} > \frac{\alpha}{2c}\overline{v} = e^s_2.$$

$$\begin{aligned}\widetilde{e}^s_1 \gtreqless \overline{e}^s &\iff \lambda_1^{-1}\widetilde{v}_1 \gtreqless \overline{v} \\ &\iff \lambda_1^{-1}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right) - \left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \gtreqless 0\end{aligned}$$

Thus

$$\widetilde{e}^s_1 \begin{cases} > \overline{e}^s & \text{if} \quad \lambda_1 < \underline{\lambda} \quad \text{and} \quad \frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2}{\alpha} \\ \leqslant \overline{e}^s & \text{if} \quad \text{either} \quad \lambda_1 \geqslant \underline{\lambda} \quad \text{or} \quad \frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2}{\alpha} \end{cases}$$

where $\underline{\lambda}$ solves $\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\underline{\lambda}^{-\frac{1}{\alpha+1}}\right) = \underline{\lambda}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)$.

Now, assume the equilibrium (e^s_1, e^s_2) satisfies $\lambda_1(e^s_1)^\alpha \leqslant (e^s_2)^\alpha$. In this case, the best response to $e^s_2 = \overline{e}^s$ is the solution to

$$\frac{\alpha\lambda_1}{2}\frac{(e^s_1)^{\alpha-1}}{(\overline{e}^s)^\alpha}\widetilde{v}_1 = c.$$

Substituting $\overline{e}^s$ by $\frac{\alpha}{2c}\overline{v}$ and solving for e_1^s we have

$$\widetilde{e}_1^s = \frac{\alpha}{2c}\lambda_1^{\frac{1}{1-\alpha}} (\widetilde{v}_1)^{\frac{1}{1-\alpha}} (\overline{v})^{-\frac{\alpha}{1-\alpha}} .$$

This is not a feasible solution since it fails to satisfy $\lambda_1(e_1^s)^\alpha \leqslant (e_2^s)^\alpha$. Hence, player 1's equilibrium effort is

$$\widetilde{e}_1^s = \frac{\alpha}{2c}\lambda_1^{-\frac{1}{1+\alpha}} (\widetilde{v}_1)^{\frac{1}{1+\alpha}} (\overline{v})^{\frac{\alpha}{1+\alpha}} .$$

Therefore, player 1's perceived winning probability is

$$\widetilde{p}_{12}^s = 1 - \frac{1}{2}\frac{(\overline{e}^s)^\alpha}{\lambda_1(\widetilde{e}_1^s)^\alpha} = 1 - \frac{1}{2}\lambda_1^{-\frac{1}{\alpha+1}} (\widetilde{v}_1)^{-\frac{\alpha}{1+\alpha}} (\overline{v})^{\frac{\alpha}{1+\alpha}} > \frac{1}{2}.$$

Player 2's perceived winning probability is $\widetilde{p}_{21}^s = 1/2$ since she thinks, mistakenly, player 1 is rational. Player 1's winning probability is

$$p_{12}^s = \begin{cases} 1 - \frac{1}{2}\left(\frac{\overline{e}^s}{\widetilde{e}_1^s}\right)^\alpha = 1 - \frac{1}{2}\lambda_1^{\frac{\alpha}{\alpha+1}} (\widetilde{v}_1)^{-\frac{\alpha}{1+\alpha}} (\overline{v})^{\frac{\alpha}{1+\alpha}} > \frac{1}{2} & \text{if} \quad \lambda_1 < \underline{\lambda} \quad \text{and} \quad \frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2}{\alpha} \\ \frac{1}{2}\left(\frac{\widetilde{e}_1^s}{\overline{e}^s}\right)^\alpha = \frac{1}{2}\lambda_1^{-\frac{\alpha}{\alpha+1}} (\widetilde{v}_1)^{\frac{\alpha}{1+\alpha}} (\overline{v})^{-\frac{\alpha}{1+\alpha}} \leqslant \frac{1}{2} & \text{if} \quad \text{either} \quad \lambda_1 \geqslant \underline{\lambda} \quad \text{or} \quad \frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2}{\alpha} \end{cases}$$

Player 2's winning probability is

$$p_{21}^s = \begin{cases} \frac{1}{2}\lambda_1^{\frac{\alpha}{\alpha+1}} (\widetilde{v}_1)^{-\frac{\alpha}{1+\alpha}} (\overline{v})^{\frac{\alpha}{1+\alpha}} < \frac{1}{2} & \text{if} \quad \lambda_1 < \underline{\lambda} \quad \text{and} \quad \frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2}{\alpha} \\ 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{\alpha+1}} (\widetilde{v}_1)^{\frac{\alpha}{1+\alpha}} (\overline{v})^{-\frac{\alpha}{1+\alpha}} \geqslant \frac{1}{2} & \text{if} \quad \text{either} \quad \lambda_1 \geqslant \underline{\lambda} \quad \text{or} \quad \frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2}{\alpha} \end{cases}$$

The perceived expected utility of player 1 is

$$\widetilde{E}^s(U_{12}) = \widetilde{p}_{12}^s\widetilde{v}_1 - c\widetilde{e}_1^s = \widetilde{v}_1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{1+\alpha}} (\widetilde{v}_1)^{\frac{1}{1+\alpha}} (\overline{v})^{\frac{\alpha}{1+\alpha}}.$$

The perceived expected utility of player 2 is

$$\widetilde{E}^s(U_{21}) = \widetilde{p}_{21}^s\overline{v} - c\overline{e}^s = \frac{1-\alpha}{2}\overline{v}.$$

2.3 Equilibrium Winning Probabilities

Proposition A3 *In a two-stage elimination contest where player 1 is overconfident, players 2, 3, and 4 are rational, and the overconfident player's bias is not observable by the rational players, if $\alpha > \frac{3}{5}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{10}{5\alpha-3}$, then there exist $\lambda_1 \in (1, \underline{\lambda})$ for which the overconfident player has the highest equilibrium winning probability, i.e., $P_1 > P_3 = P_4 > 1/4 > P_2$.*

Proof of Proposition A3

$$\mathbf{P}_1 = p_{13}^f p_{12}^s$$

$$\mathbf{P}_2 = \frac{1}{2}p_{21}^s$$

$$\mathbf{P}_3 = \mathbf{P}_4 = p^s_{12}p^f_{31}p^s_{34} + p^s_{21}p^f_{32}p^s_{34} = p^s_{12}\left(p^f_{31}\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{4} > \frac{1}{4}$$

Since $p^f_{13} < \frac{1}{2}$, a necessary condition for $P_1 > P_3$ is $p^s_{12} > \frac{1}{2}$. Thus throughout the proof of Proposition A3, the parameters are restricted in the range where $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2}{\alpha}$ and $\lambda_1 < \underline{\lambda}$.

$$\mathbf{P}_1 - \mathbf{P}_3 = \frac{3}{2}p^f_{13}p^s_{12} - \frac{1}{4}p^s_{12} - \frac{1}{4}$$

The sign of $\mathbf{P}_1 - \mathbf{P}_3$ is the same as the sign of $6p^f_{13}p^s_{12} - p^s_{12} - 1$. Let $f(\lambda_1) = 6p^f_{13}p^s_{12} - p^s_{12} - 1$

$$\begin{aligned} f(\lambda_1) &= 6\frac{1}{2}\lambda_1^{-\frac{\alpha}{\alpha+1}}\left[1 - \frac{1}{2}\lambda_1^{\frac{\alpha}{\alpha+1}}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}}\right] - \left[1 - \frac{1}{2}\lambda_1^{\frac{\alpha}{\alpha+1}}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}}\right] - 1 \\ &= 3\lambda_1^{-\frac{\alpha}{\alpha+1}} - \frac{3}{2}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}} + \frac{1}{2}\lambda_1^{\frac{\alpha}{\alpha+1}}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}} - 2 \\ &= 3\lambda_1^{-\frac{\alpha}{\alpha+1}} - \frac{3}{2}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)^{-\frac{\alpha}{1+\alpha}}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{1+\alpha}} \\ &\quad + \frac{1}{2}\lambda_1^{\frac{\alpha}{\alpha+1}}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)^{-\frac{\alpha}{1+\alpha}}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{1+\alpha}} \\ &\quad - 2 \end{aligned}$$

We can get that

$$f(\lambda_1 = 1) = 0.$$

$$\begin{aligned} f'(\lambda_1) = &-3\frac{\alpha}{\alpha+1}\lambda_1^{-\frac{\alpha}{\alpha+1}-1} \\ &- \frac{3}{2}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{1+\alpha}}\left(-\frac{\alpha}{1+\alpha}\right) \\ &\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)^{-\frac{\alpha}{1+\alpha}-1}\frac{1+\alpha}{2}\frac{1}{\alpha+1}\lambda_1^{-\frac{1}{\alpha+1}-1} \\ &+ \frac{1}{2}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{1+\alpha}} \\ &\left[\frac{\alpha}{\alpha+1}\lambda_1^{\frac{\alpha}{\alpha+1}-1}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)^{-\frac{\alpha}{1+\alpha}}\right. \\ &\left. + \lambda_1^{\frac{\alpha}{\alpha+1}}\left(-\frac{\alpha}{1+\alpha}\right)\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)^{-\frac{\alpha}{1+\alpha}-1}\frac{1+\alpha}{2}\frac{1}{1+\alpha}\lambda_1^{-\frac{1}{\alpha+1}-1}\right] \end{aligned}$$

$$\begin{aligned} f'(\lambda_1 = 1) &= -3\frac{\alpha}{\alpha+1} + \frac{3}{4}\frac{\alpha}{1+\alpha}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1} + \frac{1}{2}\frac{\alpha}{\alpha+1} \\ &\quad - \frac{1}{4}\frac{\alpha}{\alpha+1}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1} \\ &= -\frac{5}{2}\frac{\alpha}{\alpha+1} + \frac{1}{2}\frac{\alpha}{\alpha+1}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1} \end{aligned}$$

$f'(\lambda_1 = 1) > 0$ when $\alpha > \frac{3}{5}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{10}{5\alpha-3}$. Thus there exist $\lambda_1 \in (1, \underline{\lambda})$ for which $P_1 > P_3$ is satisfied when $\alpha > \frac{3}{5}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{10}{5\alpha-3}$.

3 Three-Stage Elimination Contest

This section shows that our main results extend to a three-stage elimination contest. To do that we consider an elimination contest where eight players compete in four quarterfinals in the first stage, the four first-stage winners compete in two semifinals in the second stage, and the two second-stage winners compete in the final. The winner gets w_1, the runner-up w_2, the two second-stage losers w_3, and the four third-stage losers w_4, where $w_1 > w_2 > w_3 \geqslant w_4 = 0$. In addition, we assume an increasing utility spread as the players move up in the elimination contest, that is, $u(w_1) - u(w_2) > u(w_2) - u(w_3) > u(w_3) - u(w_4)$. Furthermore, we assume that in the top half of the contest, players 1 and 2 are seeded in one quarterfinal, and players 3 and 4 are seeded in the other quarterfinal. Finally, we assume that in the bottom half of the contest, players 5 and 6 are seeded in one quarterfinal, and players 7 and 8 are seeded in the other quarterfinal.

We start by analyzing a three-stage elimination contest with eight rational players. Next, we analyze a three-stage elimination contest with one overconfident player and seven rational players. In both cases we do not analyze the final since it is identical to a final of a two-stage contest. Hence, we solve the three-stage contest backwards, starting with the semifinals and ending with the quarterfinals.

3.1 Eight rational players

Lemma A1 *In a semifinal of a three-stage elimination contest with eight rational players, the equilibrium effort is*

$$\overline{e}^s = \frac{\alpha}{2c}\left(\frac{1-\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)[u(w_1)-u(w_2)]$$

and the equilibrium expected utility is

$$\overline{E}^s(U) = \left[\frac{1-\alpha}{2}\left(\frac{1-\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)}\right][u(w_1)-u(w_2)]$$

Proof of Lemma A1

Since players are identical, we assume players 1 and 3 meet in the top half semifinal and players 5 and 7 meet in the bottom half semifinal. Moreover, we also assume, without loss of generality, player 5 beats 7. $E^s(U_{13})$ can be written as:

$$\begin{aligned} E^s(U_{13}) &= p^s_{13} v^s_1 + (1 - p^s_{13})\, u(w_3) - c e^s_1 \\ &= p^s_{13}\,(v^s_1 - u(w_3)) + u(w_3) - c e^s_1 \end{aligned}$$

Player 1's continuation value in the semifinal of a three-stage elimination contest is the same as that in two-stage:

$$v^s_1 = p^f_{15}[u(w_1) - u(w_2)] + u(w_2) - c e^f_1$$

From Proposition 1 we know that $e^f_1 = \overline{e}^f = \frac{\alpha}{2c}[u(w_1) - u(w_2)]$, $p^f_{15} = \frac{1}{2}$. Plug these values into the equation above we get

$$v^s_1 = \overline{v}^s = \frac{1-\alpha}{2}[u(w_1) - u(w_2)] + u(w_2)$$

Similar to the proof of Proposition 1, we get that the equilibrium effort is

$$\begin{aligned} e_1^s = \overline{e}^s &= \frac{\alpha}{2c}\left(v_1^s - u(w_3)\right) \\ &= \frac{\alpha}{2c}\left(\frac{1-\alpha}{2}[u(w_1)-u(w_2)] + u(w_2) - u(w_3)\right) \\ &= \frac{\alpha}{2c}\left(\frac{1-\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)[u(w_1)-u(w_2)] \end{aligned}$$

Due to symmetry, all the rational players exert effort $\overline{e}^s$ at equilibrium.

The equilibrium winning probabilities are

$$\overline{p}^s = p_{13}^s = p_{31}^s = p_{57}^s = p_{75}^s = \frac{1}{2}$$

The equilibrium expected utility is

$$\overline{E}^s(U) = \left[\frac{1-\alpha}{2}\left(\frac{1-\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)}\right][u(w_1)-u(w_2)]$$

Now we move on to the quarterfinals.

Proposition A4 *In a quarterfinal of a three-stage elimination contest with eight rational players, the equilibrium effort is*

$$\overline{e}^q = \frac{\alpha}{2c}\overline{v}^q = \frac{\alpha}{2c}\left[\frac{1-\alpha}{2}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)}\right][u(w_1)-u(w_2)],$$

and the equilibrium expected utility is

$$\overline{E}^q(U) = \frac{1-\alpha}{2}\left[\frac{1-\alpha}{2}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)}\right][u(w_1)-u(w_2)].$$

Proof of Proposition A4

1. Continuation values

   $E^q(U_{12})$ denotes player 1's expected utility in the quarterfinal when he plays against player 2, and v_1^q denotes his continuation value in the quarterfinal. We can get

   $$E^q(U_{12}) = p_{12}^q v_1^q + (1-p_{12}^q)\times 0 - ce_1^q$$

   where

   $$\begin{aligned} v_1^q &= E^s(U_{13}) \\ &= \overline{v}^q \\ &= \left[\frac{1-\alpha}{2}\left(\frac{1-\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)}\right][u(w_1)-u(w_2)] \end{aligned}$$

2. The equilibrium efforts and winning probabilities

   Similar to the proof of Proposition 1

$$\begin{aligned}\overline{e}^q &= \frac{\alpha}{2c}\overline{v}^q\\ &= \frac{\alpha}{2c}\left[\frac{1-\alpha}{2}\left(\frac{1-\alpha}{2}+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)+\frac{u(w_3)}{u(w_1)-u(w_2)}\right][u(w_1)-u(w_2)]\end{aligned}$$

$$\overline{p}^q = p^q_{12} = p^q_{21} = p^q_{34} = p^q_{43} = p^q_{56} = p^q_{65} = p^q_{78} = p^q_{87} = \frac{1}{2}$$

3. Expected utility of the quarterfinal

$$\begin{aligned}\overline{E}^q(U) &= \frac{1}{2}\overline{v}^q - c\frac{\alpha}{2c}\overline{v}^q\\ &= \frac{1-\alpha}{2}\left[\frac{1-\alpha}{2}\left(\frac{1-\alpha}{2}+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)+\frac{u(w_3)}{u(w_1)-u(w_2)}\right][u(w_1)-u(w_2)]\end{aligned}$$

   Since $0 < \alpha \leqslant 1$ and $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > 1$, we have $\overline{E}^q(U) \geqslant 0$. The participation constraints are satisfied.

3.2 One Overconfident Player and Seven Rational Players

We now show that the results for a two-stage elimination contest with one overconfident player and three rational players generalize to a three-stage elimination contest with one overconfident player and seven rational players. We set player 1 as the overconfident player with $\lambda_1 > \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 1$. We first characterize the equilibrium in the semifinal between the overconfident player 1 and rational player 3.

Lemma A2 *Consider the semifinal between an overconfident player and a rational player of a three-stage elimination contest with eight players where player 1 is overconfident and the other seven players are rational.*
(i) If $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ *and* $\lambda_1 < \hat{\lambda}$ *where* $\hat{\lambda} > 1$ *is given by* $\frac{1+\alpha}{2}\left(1+\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)}\right) = \frac{\hat{\lambda}-1}{\hat{\lambda}-\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}}$*, then the equilibrium efforts and winning probabilities satisfy* $e^s_1 > \overline{e}^s > e^s_3$ *and* $\widetilde{p}^s_{13} > p^s_{13} > 1/2 > p^s_{31}$.
(ii) If either $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ *or* $\lambda_1 \geqslant \hat{\lambda}$*, then the equilibrium efforts and winning probabilities satisfy* $e^s_1 \leqslant e^s_3 \leqslant \overline{e}^s$ *and* $\widetilde{p}^s_{13} > p^s_{31} \geqslant 1/2 \geqslant p^s_{13}$.

Proof of Lemma A2

Since the seven rational players are identical, we assume that player 1 meets 3 in the semifinal and that player 5 enters the final.

1. Continuation values

   Overconfident player 1:

   Player 1's continuation value in the semifinal of a three-stage elimination contest is the same as that of a two-stage

$$\begin{aligned}\widetilde{v}_1^s &= \widetilde{p}_{15}^f[u(w_1) - u(w_2)] + u(w_2) - ce_1^f \\ &= \left(1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)[u(w_1) - u(w_2)] + u(w_2)\end{aligned}$$

Rational player 3:

Since player 3 will meet a rational player in the final, her continuation value is the benchmark

$$v_3^s = \overline{v}^s = \frac{1-\alpha}{2}[u(w_1) - u(w_2)] + u(w_2)$$

We can easily get

$$\widetilde{v}_1^s > v_3^s$$

2. Equilibrium efforts

Overconfident player 1 $\quad max$

$$\begin{aligned}\widetilde{E}^s(U_{13}) &= \widetilde{p}_{13}^s\widetilde{v}_1^s + (1 - \widetilde{p}_{13}^s)u(w_3) - ce_1 \\ &= \widetilde{p}_{13}^s\left(\widetilde{v}_1^s - u(w_3)\right) + u(w_3) - ce_1 \\ &= \begin{cases}\left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)\left(\widetilde{v}_1^s - u(w_3)\right) + u(w_3) - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}\left(\widetilde{v}_1^s - u(w_3)\right) + u(w_3) - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_3^\alpha\end{cases}\end{aligned}$$

Rational player 3 $\quad max$

$$\begin{aligned}E^s(U_{31}) &= p_{31}^s v_3^s + (1 - p_{31}^s)u(w_3) - ce_3 \\ &= p_{31}^s\left(v_3^s - u(w_3)\right) + u(w_3) - ce_3 \\ &= \begin{cases}\left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)\left(v_3^s - u(w_3)\right) + u(w_3) - ce_3 & \text{if} \quad e_3 \geqslant e_1 \\ \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\left(v_3^s - u(w_3)\right) + u(w_3) - ce_3 & \text{if} \quad e_3 \leqslant e_1\end{cases}\end{aligned}$$

There are 4 cases.

$$\begin{cases}\lambda_1 e_1^\alpha \geqslant e_3^\alpha & and \quad e_3 \geqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_3^\alpha & and \quad e_3 \leqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_3^\alpha & and \quad e_3 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_3^\alpha & and \quad e_3 \leqslant e_1\end{cases}$$

Since $\lambda_1 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1$, which corresponds to Lemma A2 (i).

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)\left(\widetilde{v}_1^s - u(w_3)\right) + u(w_3) - ce_1$

Player 3 $\quad max \quad \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\left(v_3^s - u(w_3)\right) + u(w_3) - ce_3$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_3^\alpha}{e_1^{\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right) - c = 0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_3^{\alpha-1}}{e_1^\alpha}\left(v_3^s - u(w_3)\right) - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_3^\alpha}{e_1^{\alpha+2}}\left(\widetilde{v}_1^s - u(w_3)\right) < 0$

$[e_3] \quad \frac{\alpha}{2}(\alpha-1)\frac{e_3^{\alpha-2}}{e_1^\alpha}\left(v_3^s - u(w_3)\right) < 0$

Solve the two F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}\left(\widetilde{v}_1^s - u(w_3)\right)^{1-\alpha}\left(v_3^s - u(w_3)\right)^\alpha$$

$$e_3 = \frac{\alpha}{2c}\lambda_1^{\alpha}\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(v_3^s - u(w_3)\right)^{\alpha+1}$$

$$\frac{e_3}{e_1} = \lambda_1\frac{v_3^s - u(w_3)}{\widetilde{v}_1^s - u(w_3)}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$ and $e_3 \leqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$

As long as $e_1 \geqslant e_3$ is satisfied, $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$ is satisfied.

② $e_3 \leqslant e_1$

$$\begin{aligned} e_1 \geqslant e_3 &\iff \frac{e_1}{e_3} \geqslant 1 \\ &\iff \frac{\widetilde{v}_1^s - u(w_3)}{\lambda_1\left(v_3^s - u(w_3)\right)} \geqslant 1 \\ &\iff \frac{\left(1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)[u(w_1)-u(w_2)]}{\lambda_1\left(1 - \frac{1+\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)[u(w_1)-u(w_2)]} \geqslant 1 \\ &\iff 1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} \geqslant \lambda_1\left(1 - \frac{1+\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right) \end{aligned}$$

Let

$$f(\lambda_1) = \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) - \lambda_1\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right).$$

We can easily get that $f(\lambda_1 = 1) = 0$ and $f(\lambda_1 \to \infty) < 0$.

$$f'(\lambda_1) = \frac{(1+\alpha)^2}{2(2\alpha+1)}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1} - \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)$$

$$
\begin{aligned}
f'(\lambda_1) \lesseqqgtr 0 &\iff \frac{(1+\alpha)^2}{2(2\alpha+1)}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1} \lesseqqgtr 1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2} \\
&\iff \frac{(1+\alpha)^2}{2(2\alpha+1)}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{-1} \lesseqqgtr \lambda_1^{\frac{\alpha+1}{2\alpha+1}+1} \\
&\iff \left[\frac{(1+\alpha)^2}{2(2\alpha+1)}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{-1}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}} \lesseqqgtr \lambda_1
\end{aligned}
$$

Let $g(\alpha)=\left[\frac{(1+\alpha)^2}{2(2\alpha+1)}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{-1}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}}$

a) $g(\alpha)\leqslant 1$

if $g(\alpha)\leqslant 1$, then $f'(\lambda_1)<0$ always holds and thus $f(\lambda_1)<0$ always holds. Therefore $e_1<e_3$ when $g(\alpha)\leqslant 1$.

$$
\begin{aligned}
g(\alpha)\leqslant 1 &\iff \frac{(1+\alpha)^2}{2(2\alpha+1)}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{-1} \leqslant 1 \\
&\iff \frac{(1+\alpha)^2}{2(2\alpha+1)} \leqslant 1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2} \\
&\iff \frac{1+\alpha}{2}\left(\frac{1+\alpha}{2\alpha+1}+1\right)-1 \leqslant \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} \\
&\iff \frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}
\end{aligned}
$$

When $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)}\leqslant\frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$, the condition $e_1\geqslant e_3$ is never satisfied given that $\lambda_1>1$.

b) $g(\alpha)>1$

if $g(\alpha)>1$, then

$$
f'(\lambda_1)\begin{cases} >0 & \text{when } \lambda_1 < \left[\frac{(1+\alpha)^2}{2(2\alpha+1)}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{-1}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}} \\ <0 & \text{when } \lambda_1 > \left[\frac{(1+\alpha)^2}{2(2\alpha+1)}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{-1}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}} \end{cases}
$$

We now show that if $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)}>\frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$, then there exists a unique threshold $\hat{\lambda}>1$ where $f(\lambda_1)=0$, that is,

$$
1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}=\hat{\lambda}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right).
$$

To see this is the case, we rearrange the equality as

$$
\frac{1+\alpha}{2}\left(\hat{\lambda}-\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}\right)=(\hat{\lambda}-1)\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right),
$$

or

$$
\frac{1+\alpha}{2}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)^{-1}=\frac{\hat{\lambda}-1}{\hat{\lambda}-\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}}. \tag{7}
$$

Since $\alpha \in (0,1]$ and $\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} > 0$, the left-hand side of (7) takes a value in the interval $(0,1)$. The right-hand side of (7) is increasing in $\hat{\lambda}$ for $\lambda_1 > 1$, its limit when $\hat{\lambda} \to 1$ is $\frac{2\alpha+1}{3\alpha+2}$, and its limit when $\hat{\lambda} \to \infty$ is 1. Hence, the threshold $\hat{\lambda}$ exists and is unique provided that

$$\frac{1+\alpha}{2}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)^{-1} > \frac{2\alpha+1}{3\alpha+2}.$$

It is easy to show that this inequality is equivalent to

$$\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}.$$

Therefore, if $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$, then there exists a unique value for $\hat{\lambda}$, greater than 1, that satisfies (7). This, in turn, implies:

$$f(\lambda_1)\begin{cases} >0 & \text{when } \lambda_1 < \hat{\lambda} \\ =0 & \text{when } \lambda_1 = \hat{\lambda} \\ <0 & \text{when } \lambda_1 > \hat{\lambda} \end{cases}$$

$$e_1 - e_3\begin{cases} >0 & \text{when } \lambda_1 < \hat{\lambda} \\ =0 & \text{when } \lambda_1 = \hat{\lambda} \\ <0 & \text{when } \lambda_1 > \hat{\lambda} \end{cases}$$

The condition $e_1 \geqslant e_3$ is only satisfied when $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 \leqslant \hat{\lambda}$. And $e_1 > e_3$ is only satisfied when $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$.

Therefore the solution

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}\left(\widetilde{v}_1^s - u(w_3)\right)^{1-\alpha}\left(v_3^s - u(w_3)\right)^{\alpha}$$

$$e_3 = \frac{\alpha}{2c}\lambda_1^{\alpha}\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(v_3^s - u(w_3)\right)^{\alpha+1}$$

only applies when $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$.

(2) case 2: $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$ *and* $e_3 \geqslant e_1$, which corresponds to Lemma A2 (ii).

Player 1 *max* $\left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)\left(\widetilde{v}_1^s - u(w_3)\right) + u(w_3) - ce_1$

Player 3 *max* $\left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)\left(v_3^s - u(w_3)\right) + u(w_3) - ce_3$

F.o.c

$[e_1]$ $\frac{\alpha}{2\lambda_1}\frac{e_3^\alpha}{e_1^{\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right) - c = 0$

$[e_3]$ $\frac{\alpha}{2}\frac{e_1^\alpha}{e_3^{\alpha+1}}\left(v_3^s - u(w_3)\right) - c = 0$

S.o.c

$[e_1]$ $\frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_3^\alpha}{e_1^{\alpha+2}}\left(\widetilde{v}_1^s-u(w_3)\right)<0$

$[e_3]$ $\frac{\alpha}{2}(-\alpha-1)\frac{e_1^\alpha}{e_3^{\alpha+2}}\left(v_3^s-u(w_3)\right)<0$

Solve F.O.C , we get

$$e_1=\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1^s-u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}}\left(v_3^s-u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}$$

$$e_3=\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s-u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s-u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}}$$

$$\frac{e_1}{e_3}=\lambda_1^{-\frac{1}{2\alpha+1}}\left(\widetilde{v}_1^s-u(w_3)\right)^{\frac{1}{2\alpha+1}}\left(v_3^s-u(w_3)\right)^{-\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$ and $e_3 \geqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$

$$\begin{aligned}\lambda_1 e_1^\alpha \geqslant e_3^\alpha &\Longleftrightarrow \frac{\lambda_1 e_1^\alpha}{e_3^\alpha}\geqslant 1\\ &\Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1^s-u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s-u(w_3)\right)^{-\frac{\alpha}{2\alpha+1}}\geqslant 1\end{aligned}$$

Since $\lambda_1>1$ and $\widetilde{v}_1^s>v_3^s$, the inequality is always satisfied. Therefore $\lambda_1 e_1^\alpha > e_3^\alpha$ always holds when $\lambda_1>1$.

② $e_3 \geqslant e_1$

$$\begin{aligned}e_1\leqslant e_3 &\Longleftrightarrow \frac{e_1}{e_3}\leqslant 1\\ &\Longleftrightarrow \lambda_1^{-\frac{1}{2\alpha+1}}\left(\widetilde{v}_1^s-u(w_3)\right)^{\frac{1}{2\alpha+1}}\left(v_3^s-u(w_3)\right)^{-\frac{1}{2\alpha+1}}\leqslant 1\\ &\Longleftrightarrow \left(\frac{\widetilde{v}_1^s-u(w_3)}{\lambda_1\left(v_3^s-u(w_3)\right)}\right)^{\frac{1}{2\alpha+1}}\leqslant 1\\ &\Longleftrightarrow \frac{\widetilde{v}_1^s-u(w_3)}{\lambda_1\left(v_3^s-u(w_3)\right)}\leqslant 1\end{aligned}$$

We have already seen in case (1) that $e_3 \geqslant e_1$ is satisfied when either $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)}\leqslant\frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1\geqslant\hat{\lambda}$.

Therefore the solution

$$e_1=\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1^s-u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}}\left(v_3^s-u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}$$

$$e_3=\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s-u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s-u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}}$$

only applies when either $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)}\leqslant\frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1\geqslant\hat{\lambda}$

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}\left(\widetilde{v}_1^s - u(w_3)\right) + u(w_3) - ce_1$

Player 3 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)\left(v_3^s - u(w_3)\right) + u(w_3) - ce_3$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_3^\alpha}\left(\widetilde{v}_1^s - u(w_3)\right) - c = 0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_3^{\alpha+1}}\left(v_3^s - u(w_3)\right) - c = 0$

divide the two F.O.C , we get

$$\frac{e_3}{e_1} = \frac{v_3^s - u(w_3)}{\lambda_1\left(\widetilde{v}_1^s - u(w_3)\right)} < 1$$

which contradicts the condition that $e_3 \geqslant e_1$

Therefore, the equilibrium in this semifinal:

(1) When $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$, which corresponds to Lemma A2 (i)

$$e_1^s = \frac{\alpha}{2c}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}$$
$$\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}[u(w_1)-u(w_2)]$$

$$e_3^s = \frac{\alpha}{2c}\lambda_1^{\alpha}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}$$
$$\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{1+\alpha}[u(w_1)-u(w_2)]$$

where $e_1^s > e_3^s$.

$$\begin{aligned}
p_{31}^s &= \frac{1}{2}\left(\frac{e_3^s}{e_1^s}\right)^\alpha \\
&= \frac{1}{2}\left(\frac{\lambda_1 v_2}{\widetilde{v}_1}\right)^\alpha \\
&= \frac{1}{2}\lambda_1^\alpha\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}
\end{aligned}$$

$$\begin{aligned}
p_{13}^s &= 1 - p_{31}^s \\
&= 1 - \frac{1}{2}\lambda_1^\alpha\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}
\end{aligned}$$

$$\begin{aligned}
\widetilde{p}_{13}^s &= 1 - \frac{1}{2}\frac{(e_3^s)^\alpha}{\lambda_1 (e_1^s)^\alpha} \\
&= 1 - \frac{1}{2}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}
\end{aligned}$$

We can get $\widetilde{p}_{13}^s > p_{13}^s > \frac{1}{2} > p_{31}^s$

(2) When either $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$, which corresponds to Lemma A2 (ii).

$$e_1^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} [u(w_1)-u(w_2)]$$

$$e_3^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha+1}{2\alpha+1}} [u(w_1)-u(w_2)]$$

where $\lambda_1 (e_1^s)^\alpha > (e_3^s)^\alpha$ and $e_1^s \leqslant e_3^s$.

$$\begin{aligned} p_{13}^s &= \frac{1}{2}\left(\frac{e_1^s}{e_3^s}\right)^\alpha \\ &= \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\frac{\alpha}{2\alpha+1}} \end{aligned}$$

$$\begin{aligned} p_{31}^s &= 1 - p_{13}^s \\ &= 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\frac{\alpha}{2\alpha+1}} \end{aligned}$$

$$\begin{aligned} \widetilde{p}_{13}^s &= 1 - \frac{1}{2}\frac{(e_3^s)^\alpha}{\lambda_1 (e_1^s)^\alpha} \\ &= 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} \end{aligned}$$

We can get $\widetilde{p}_{13}^s > p_{31}^s \geqslant \frac{1}{2} \geqslant p_{13}^s$

3. Participation constraints

(1) When $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$

$$
\begin{aligned}
\widetilde{E}^s(U_{13}) &= \widetilde{p}_{13}^s \left(\widetilde{v}_1^s - u(w_3)\right) - ce_1^s + u(w_3) \\
&> p_{13}^s \left(\widetilde{v}_1^s - u(w_3)\right) - ce_1^s + u(w_3) \\
&= \left(1 - \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha} \left(v_3^s - u(w_3)\right)^\alpha\right) \left(\widetilde{v}_1^s - u(w_3)\right) \\
&\quad - c\frac{\alpha}{2c}\lambda_1^{\alpha-1} \left(\widetilde{v}_1^s - u(w_3)\right)^{1-\alpha} \left(v_3^s - u(w_3)\right)^\alpha + u(w_3) \\
&= \widetilde{v}_1^s - u(w_3) - \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1^s - u(w_3)\right)^{1-\alpha} \left(v_3^s - u(w_3)\right)^\alpha \\
&\quad - \frac{\alpha}{2}\lambda_1^{\alpha-1} \left(\widetilde{v}_1^s - u(w_3)\right)^{1-\alpha} \left(v_3^s - u(w_3)\right)^\alpha + u(w_3) \\
&> \widetilde{v}_1^s - u(w_3) - \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1^s - u(w_3)\right)^{1-\alpha} \left(v_3^s - u(w_3)\right)^\alpha \\
&\quad - \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1^s - u(w_3)\right)^{1-\alpha} \left(v_3^s - u(w_3)\right)^\alpha + u(w_3) \\
&= \widetilde{v}_1^s - u(w_3) - \lambda_1^\alpha \left(\widetilde{v}_1^s - u(w_3)\right)^{1-\alpha} \left(v_3^s - u(w_3)\right)^\alpha + u(w_3) \\
&= \left(\widetilde{v}_1^s - u(w_3)\right)^{1-\alpha} \left[\left(\widetilde{v}_1^s - u(w_3)\right)^\alpha - \lambda_1^\alpha \left(v_3^s - u(w_3)\right)^\alpha\right] + u(w_3) \\
&> 0
\end{aligned}
$$

$$
\begin{aligned}
E^s(U_{31}) &= p_{31}^s \left(v_3^s - u(w_3)\right) - ce_3^s + u(w_3) \\
&= \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha} \left(v_3^s - u(w_3)\right)^\alpha \left(v_3^s - u(w_3)\right) \\
&\quad - c\frac{\alpha}{2c}\lambda_1^\alpha \left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha} \left(v_3^s - u(w_3)\right)^{\alpha+1} + u(w_3) \\
&= \frac{1-\alpha}{2}\lambda_1^\alpha \left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha} \left(v_3^s - u(w_3)\right)^{1+\alpha} + u(w_3) \\
&\geqslant 0
\end{aligned}
$$

(2) When either $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

$$
\widetilde{E}^s(U_{13}) = \widetilde{p}_{13}^s \left(\widetilde{v}_1^s - u(w_3)\right) - ce_1^s + u(w_3)
$$

Since $\widetilde{p}_{13}^s > \frac{1}{2}$, $\widetilde{v}_1^s > \overline{v}^s$ and $e_1^s < \overline{e}^s$, we can get that $\widetilde{E}^s(U_{13}) > \overline{E}^s(U) \geqslant 0$.

$$
\begin{aligned}
E^s(U_{31}) &= p_{31}^s \left(v_3^s - u(w_3)\right) - c e_3^s + u(w_3) \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}} \left(v_3^s - u(w_3)\right)^{-\frac{\alpha}{2\alpha+1}}\right) \left(v_3^s - u(w_3)\right) \\
&\quad - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}} \left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} + u(w_3) \\
&= v_3^s - u(w_3) - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}} \left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&\quad - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}} \left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} + u(w_3) \\
&= v_3^s - u(w_3) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}} \left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} + u(w_3) \\
&= \left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} \left[\left(v_3^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\right] + u(w_3) \\
&\geqslant 0
\end{aligned}
$$

Next we characterize the quarterfinal between the overconfident player 1 and rational player 2. Due to a lack of full characterization of the equilibrium in the quarterfinal between an overconfident player and a rational player, we show that there exist parameter configurations where in equilibrium the overconfident player exerts higher effort than the rational player.

Proposition A5 *Consider a quarterfinal between the overconfident player 1 and the rational player 2 of a three-stage elimination contest where player 1 is overconfident and the other seven players are rational.*
(i) If $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$*, then there exist* $\lambda_1 \in (1, \hat{\lambda}]$*, where* $\hat{\lambda}$ *solves* $1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}} = \hat{\lambda}(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2})$*, for which the equilibrium efforts and winning probabilities satisfy* $e_1^q > \overline{e}^q > e_2^q$ *and* $\widetilde{p}_{12}^q > p_{12}^q > 1/2 > p_{21}^q$.
(ii) If $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ *and* $\frac{u(w_3)}{u(w_1)-u(w_2)} < \frac{\alpha(3\alpha+1)}{2(2\alpha+1)}\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{-7\alpha^4+\alpha^3+8\alpha^2+5\alpha+1}{4(2\alpha+1)^2}$*, then there exist* λ_1 *close to 1 for which the equilibrium efforts and winning probabilities satisfy* $e_1^q > \overline{e}^q > e_2^q$ *and* $\widetilde{p}_{12}^q > p_{12}^q > 1/2 > p_{21}^q$.

Proposition A5 shows that the results in the semifinal between an overconfident player and a rational player of a two-stage elimination contest generalize to the quarterfinal between an overconfident player and a rational player of a three-stage elimination contest. In the quarterfinal between an overconfident player and a rational player, the equilibrium efforts and winning probabilities depend on the utility spread and the overconfidence level. The equilibrium where the overconfident player exerts higher effort than the rational player exists with certainty under either of the two conditions: (i) if the utility spread between the winner and runner-up compared to that between the runner-up and the second stage loser is large ($\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$) and overconfidence level is close to either 1 or $\hat{\lambda}$, (ii) if the utility spread between the winner and runner-up compared to that between the runner-up and the second stage loser is small ($\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$), the utility spread between the winner and runner-up compared to that between the second stage loser and third stage loser is sufficiently large ($\frac{u(w_3)}{u(w_1)-u(w_2)} < \frac{\alpha(3\alpha+1)}{2(2\alpha+1)}\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{-7\alpha^4+\alpha^3+8\alpha^2+5\alpha+1}{4(2\alpha+1)^2}$), and λ_1 is close to 1.

Proof of Proposition A5

1. Continuation values

   Overconfident player 1:

$$\widetilde{E}^q(U_{12}) = \widetilde{p}_{12}^q \widetilde{v}_1^q - ce_1^q$$

   where

$$\begin{aligned}\widetilde{v}_1^q &= \widetilde{E}^s(U_{13}) \\ &= \widetilde{p}_{13}^s \left[\left(1 - \frac{1+\alpha}{2} \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \right) [u(w_1) - u(w_2)] + u(w_2) - u(w_3) \right] + u(w_3) - ce_1^s\end{aligned}$$

   Rational player 2:

   Since the rational player 2 will only meet rational rivals in the semifinal and final, her continuation value is the benchmark.

$$v_2^q = \overline{v}^q$$

   (1) When $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 < \hat{\lambda}$

$$\begin{aligned}\widetilde{v}_1^q - u(w_3) = &\left[1 - \frac{1}{2} \lambda_1^{\alpha-1} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \right)^{-\alpha} \right. \\ &\left. \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \right)^{\alpha} \right] \\ &\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \right) [u(w_1) - u(w_2)] \\ &- c \frac{\alpha}{2c} \lambda_1^{\alpha-1} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \right)^{1-\alpha} \\ &\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \right)^{\alpha} [u(w_1) - u(w_2)] \\ = &\left[\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \right) \right. \\ &- \frac{1+\alpha}{2} \lambda_1^{\alpha-1} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \right)^{1-\alpha} \\ &\left. \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \right)^{\alpha} \right] [u(w_1) - u(w_2)]\end{aligned}$$

$$
\begin{aligned}
&\frac{\widetilde{v}_1^q - u(w_3)}{\overline{v}^q - u(w_3)} \\
&= \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)} \\
&- \frac{\frac{1+\alpha}{2}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)} \\
&= \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}}{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{1-\alpha}} \\
&\quad \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha} - \frac{1+\alpha}{2}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}} \\
&> \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha} - \frac{1+\alpha}{2}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}} \\
&> \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} - \frac{1+\alpha}{2}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}} \\
&= \frac{1 - \frac{1+\alpha}{2}\lambda_1^{\alpha-1}}{1 - \frac{1+\alpha}{2}} \\
&> 1
\end{aligned}
$$

Thus we get

$$\widetilde{v}_1^q > \overline{v}^q$$

(2) When either $\lambda_1 \geqslant \hat{\lambda}$ or $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$

$$
\begin{aligned}
\widetilde{v}_1^q - u(w_3) = & \left[1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}} \right. \\
& \left. \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} \right] \\
& \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)[u(w_1)-u(w_2)] \\
& - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}} \\
& \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} [u(w_1)-u(w_2)] \\
= & \left[\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \right. \\
& \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}} \\
& \left. \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} \right] [u(w_1)-u(w_2)]
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\widetilde{v}_1^q - u(w_3)}{\overline{v}^q - u(w_3)} \\
&= \frac{1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)} \\
&- \frac{\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)} \\
&= \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}}}{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha+1}{2\alpha+1}}} \\
&\quad \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}} \\
&> \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}} \\
&> \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}} \\
&= \frac{1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}}{1 - \frac{1+\alpha}{2}} \\
&> 1
\end{aligned}
$$

Thus we get

$$\widetilde{v}_1^q > \overline{v}^q$$

2. The equilibrium when $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 < \hat{\lambda}$, , which corresponds to Proposition A5 (i)

   Player 1 $\quad max \quad \widetilde{E}^q(U_{12}) = \widetilde{p}_{12}^q \widetilde{v}_1^q - ce_1$

$$
= \begin{cases} \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1^q - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_2^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1^q - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_2^\alpha \end{cases}
$$

   Player 2 $\quad max \quad E^q(U_{21}) = p_{21}^q v_2^q - ce_2$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right) v_2^q - ce_2 & \text{if } e_2 \geqslant e_1 \\ \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha} v_2^q - ce_2 & \text{if } e_2 \leqslant e_1 \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \end{cases}$$

Since $\lambda_1 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right) \widetilde{v}_1^q - ce_1$

Player 2 $\quad max \quad \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha} v_2^q - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1^q - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_2^{\alpha-1}}{e_1^\alpha} v_2^q - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha - 1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1^q < 0$

$[e_2] \quad \frac{\alpha}{2}(\alpha - 1)\frac{e_2^{\alpha-2}}{e_1^\alpha} v_2^q < 0$

Solve the two F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1^q)^{1-\alpha}(v_2^q)^\alpha$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1^q)^{-\alpha}(v_2^q)^{1+\alpha}$$

$$\frac{e_2}{e_1} = \lambda_1 \frac{v_2^q}{\widetilde{v}_1^q}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \leqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

As long as $e_1 \geqslant e_2$ is satisfied, $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ is satisfied.

② $e_2 \leqslant e_1$

$$\frac{e_2}{e_1} = \lambda_1 \frac{v_2^q}{\widetilde{v}_1^q} = \frac{\overline{v}^q}{\lambda_1^{-1}\widetilde{v}_1^q}$$

Let

$$f(\lambda_1) = \frac{\lambda_1^{-1}\widetilde{v}_1^q - \overline{v}^q}{u(w_1) - u(w_2)}$$

$$\begin{aligned} f(\lambda_1) = \lambda_1^{-1}\Bigg[& \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) \\ & - \frac{1+\alpha}{2}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha} \\ & \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} + \frac{u(w_3)}{u(w_1) - u(w_2)}\Bigg] \\ & - \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right] \end{aligned}$$

We can easily get

$$f(\lambda_1 = 1) = 0.$$

Recall that $\hat{\lambda}^{-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}\right) - \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) =$

0, we can get

$$
\begin{aligned}
f(\lambda_1 = \hat{\lambda}) = \hat{\lambda}^{-1} &\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}\right) \\
&- \frac{1+\alpha}{2}\hat{\lambda}^{\alpha-1}\hat{\lambda}^{-1}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{1+\alpha}{2\alpha+1}}\right)^{1-\alpha} \\
&\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} \\
&+ \hat{\lambda}^{-1}\frac{u(w_3)}{u(w_1) - u(w_2)} \\
&- \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right] \\
= &\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\hat{\lambda}^{-1}\hat{\lambda}^{\alpha-1} \\
&\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{1+\alpha}{2\alpha+1}}\right)^{1-\alpha} \\
&+ \frac{u(w_3)}{u(w_1) - u(w_2)}\hat{\lambda}^{-1} \\
&- \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right] \\
= &\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\hat{\lambda}^{-1} \\
&\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{1-\alpha} \\
&+ \frac{u(w_3)}{u(w_1) - u(w_2)}\hat{\lambda}^{-1} \\
&- \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right] \\
= &\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\hat{\lambda}^{-1} + \frac{u(w_3)}{u(w_1) - u(w_2)}\hat{\lambda}^{-1} \\
&- \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right] \\
= &\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left(1 - \frac{1+\alpha}{2}\hat{\lambda}^{-1}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\hat{\lambda}^{-1} \\
&- \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&f(\lambda_1 = \hat{\lambda}) \lesseqqgtr 0 \\
&\iff \\
&\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left(1 - \frac{1+\alpha}{2}\hat{\lambda}^{-1}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\hat{\lambda}^{-1} \lesseqqgtr \\
&\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)} \\
&\iff -\frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\hat{\lambda}^{-1} + \frac{u(w_3)}{u(w_1) - u(w_2)}\hat{\lambda}^{-1} \lesseqqgtr \\
&-\frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)} \\
&\iff \hat{\lambda}^{-1}\left[-\frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right] \lesseqqgtr \\
&-\frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}
\end{aligned}
$$

If $-\frac{1+\alpha}{2}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)} < 0$ always holds then $f(\lambda_1 = \hat{\lambda}) > 0$ always holds.

We show that $-\frac{1+\alpha}{2}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)} < 0$ always holds:

$$
\begin{aligned}
&-\frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)} < 0 \\
&\iff \frac{u(w_3)}{u(w_1) - u(w_2)} < \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) \\
&\iff 1 < \frac{1+\alpha}{2}\frac{u(w_1) - u(w_2)}{u(w_3)}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) \\
&\iff 1 < \frac{1+\alpha}{2}\left[\left(1 - \frac{1+\alpha}{2}\right)\frac{u(w_1) - u(w_2)}{u(w_3)} + \frac{u(w_2) - u(w_3)}{u(w_3)}\right] \\
&\iff 1 < \frac{1+\alpha}{2}\left[\frac{1-\alpha}{2}\frac{u(w_1) - u(w_2)}{u(w_2) - u(w_3)} + 1\right]\frac{u(w_2) - u(w_3)}{u(w_3)}
\end{aligned}
$$

Since $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$,

$$\begin{aligned}
\frac{1+\alpha}{2}\left[\frac{1-\alpha}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)}+1\right] &> \frac{1+\alpha}{2}\left[\frac{1-\alpha}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)}+1\right] \\
&= \frac{1+\alpha}{2}\left(\frac{1-\alpha}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)}+1\right) \\
&= \frac{1+\alpha}{2}\left(\frac{1-\alpha}{\alpha}\frac{2\alpha+1}{3\alpha+1}+1\right) \\
&= \frac{1+\alpha}{2}\frac{-2\alpha^2+\alpha+1+3\alpha^2+\alpha}{\alpha\left(3\alpha+1\right)} \\
&= \frac{1+\alpha}{2}\frac{\alpha^2+2\alpha+1}{\alpha\left(3\alpha+1\right)} \\
&= \frac{\alpha^3+3\alpha^2+3\alpha+1}{2\alpha\left(3\alpha+1\right)}
\end{aligned}$$

If $\frac{\alpha^3+3\alpha^2+3\alpha+1}{2\alpha(3\alpha+1)} > 1$ is satisfied, then $\frac{1+\alpha}{2}\left[\frac{1-\alpha}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)}+1\right] > 1$ is satisfied.

$$\frac{\alpha^3+3\alpha^2+3\alpha+1}{2\alpha\left(3\alpha+1\right)} > 1 \Longleftrightarrow \alpha^3+3\alpha^2+3\alpha+1-2\alpha\left(3\alpha+1\right) > 0$$

Let $t(\alpha) = \alpha^3+3\alpha^2+3\alpha+1-2\alpha\left(3\alpha+1\right)$

$$t(\alpha) = \alpha^3-3\alpha^2+\alpha+1$$

$$t(\alpha=0)=1, \quad t(\alpha=1)=0$$

$$t'(\alpha) = 3\alpha^2-6\alpha+1$$

We can get $t'(\alpha) > 0$ if $0 < \alpha < \frac{3-\sqrt{6}}{3}$ and $t'(\alpha) < 0$ if $\frac{3-\sqrt{6}}{3} < \alpha < 1$. Thus $t(\alpha) > 0$ when $\alpha \in (0,1)$.

Therefore $\frac{1+\alpha}{2}\left[\frac{1-\alpha}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)}+1\right] > 1$ always holds. Since we assume $\frac{u(w_2)-u(w_3)}{u(w_3)} > 1$, $-\frac{1+\alpha}{2}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)+\frac{u(w_3)}{u(w_1)-u(w_2)} < 0$ always holds. Thus $f(\lambda_1=\hat{\lambda}) > 0$ always holds.

$$
\begin{aligned}
f'(\lambda_1) = &-\lambda_1^{-2}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)+\lambda_1^{-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\\
&-\frac{1+\alpha}{2}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\alpha}\\
&\left[(\alpha-2)\lambda_1^{\alpha-3}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{1-\alpha}\right.\\
&+\lambda_1^{\alpha-2}(1-\alpha)\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{-\alpha}\\
&\left.\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\right]-\frac{u(w_3)}{u(w_1)-u(w_2)}\lambda_1^{-2}\\
=\ &\lambda_1^{-2}\left[-\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)+\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right.\\
&-\frac{1+\alpha}{2}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\alpha}\\
&\left((\alpha-2)\lambda_1^{\alpha-1}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{1-\alpha}\right.\\
&\left.+(1-\alpha)\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{-\alpha}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{\alpha-\frac{\alpha+1}{2\alpha+1}-1}\right)\\
&\left.-\frac{u(w_3)}{u(w_1)-u(w_2)}\right]
\end{aligned}
$$

Let

$$
\begin{aligned}
g(\lambda_1) = &-\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)+\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\\
&-\frac{1+\alpha}{2}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\alpha}\\
&\left[(\alpha-2)\lambda_1^{\alpha-1}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{1-\alpha}\right.\\
&\left.+(1-\alpha)\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{-\alpha}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{\alpha-\frac{\alpha+1}{2\alpha+1}-1}\right]\\
&-\frac{u(w_3)}{u(w_1)-u(w_2)}
\end{aligned}
$$

$$
\begin{aligned}
g(\lambda_1 = 1) = &- \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} \\
&\left[(\alpha-2)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{1-\alpha} + (1-\alpha)\right. \\
&\left.\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right] - \frac{u(w_3)}{u(w_1) - u(w_2)} \\
= &- \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)(\alpha-2) \\
&- \frac{1+\alpha}{2}(1-\alpha)\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} - \frac{u(w_3)}{u(w_1) - u(w_2)} \\
= &\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left[-1 - \frac{1+\alpha}{2}(\alpha-2)\right] \\
&+ \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left[1 - \frac{1+\alpha}{2}(1-\alpha)\right] - \frac{u(w_3)}{u(w_1) - u(w_2)} \\
= &\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left[-1 + \frac{1+\alpha}{2}(2-\alpha)\right] \\
&+ \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left(1 - \frac{1-\alpha^2}{2}\right) - \frac{u(w_3)}{u(w_1) - u(w_2)} \\
= &\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\frac{-2 + (-\alpha^2 + \alpha + 2)}{2} \\
&+ \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{2 - (1-\alpha^2)}{2} - \frac{u(w_3)}{u(w_1) - u(w_2)} \\
= &\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\frac{-\alpha^2 + \alpha}{2} \\
&+ \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2} - \frac{u(w_3)}{u(w_1) - u(w_2)}
\end{aligned}
$$

$$
\begin{aligned}
&f'(\lambda_1 = 1) \lesseqqgtr 0 \\
\iff &\left(\frac{1-\alpha}{2} + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)}\right)\frac{-\alpha^2 + \alpha}{2} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2} \lesseqqgtr \frac{u(w_3)}{u(w_1) - u(w_2)}
\end{aligned}
$$

We can easily get that $\left(\frac{1-\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)\frac{-\alpha^2+\alpha}{2} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2} > 0$ always holds. Thus

$$f'(\lambda_1 = 1) \lesseqqgtr 0$$

$$\Longleftrightarrow$$

$$\left[\left(\frac{1-\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)\frac{-\alpha^2+\alpha}{2} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2}\right]\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)}$$

$$\lesseqqgtr \frac{u(w_3)}{u(w_2)-u(w_3)}$$

$$\Longleftrightarrow$$

$$\frac{1-\alpha}{2}\frac{-\alpha^2+\alpha}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} + \frac{-\alpha^2+\alpha}{2}$$

$$\lesseqqgtr \frac{u(w_3)}{u(w_2)-u(w_3)}$$

We show that $\frac{1-\alpha}{2}\frac{-\alpha^2+\alpha}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} + \frac{-\alpha^2+\alpha}{2} > \frac{u(w_3)}{u(w_2)-u(w_3)}$ always holds:

Since $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$,

$$\frac{1-\alpha}{2}\frac{-\alpha^2+\alpha}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} + \frac{-\alpha^2+\alpha}{2}$$
$$> \frac{1-\alpha}{2}\frac{-\alpha^2+\alpha}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)} + \frac{-\alpha^2+\alpha}{2}$$

Let $t(\alpha) = \frac{1-\alpha}{2}\frac{-\alpha^2+\alpha}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)} + \frac{-\alpha^2+\alpha}{2}$,

$$\begin{aligned}
t(\alpha) &= \frac{1-\alpha}{2}\frac{-\alpha^2+\alpha}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)} + \frac{-\alpha^2+\alpha}{2} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)} \\
&= \frac{1-\alpha}{2}(1-\alpha)\frac{2\alpha+1}{3\alpha+1} + \frac{-\alpha^2+\alpha}{2} + \frac{(1+\alpha)(1+\alpha)(1+\alpha^2)}{2\alpha(3\alpha+1)} \\
&= \frac{(1-\alpha)(1-\alpha)\alpha(1+2\alpha) + (-\alpha^2+\alpha)\alpha(3\alpha+1) + (1+2\alpha+\alpha^2)(1+\alpha^2)}{2\alpha(3\alpha+1)} \\
&= \frac{2\alpha^4 - 3\alpha^3 + \alpha - 3\alpha^4 + 2\alpha^3 + \alpha^2 + \alpha^4 + 2\alpha^3 + 2\alpha^2 + 2\alpha + 1}{2\alpha(3\alpha+1)} \\
&= \frac{\alpha^3 + 3\alpha^2 + 3\alpha + 1}{2\alpha(3\alpha+1)}
\end{aligned}$$

We have shown before that $\frac{\alpha^3+3\alpha^2+3\alpha+1}{2\alpha(3\alpha+1)} > 1$ always holds when $\alpha \in (0, 1)$.

Thus we have proved that $f'(\lambda_1 = 1) > 0$ always holds.

Therefore, it is certain that under the conditions $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 < \hat{\lambda}$, there must exist some parameter configurations where $e_1^q > e_2^q$ is satisfied.

When $e_1^q > e_2^{qq}$ is satisfied, $\lambda_1^{-1}\widetilde{v}_1^q > \overline{v}^q$. Thus

$$\begin{aligned} e_1^q &= \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1^q)^{1-\alpha}(v_2^q)^{\alpha} \\ &= \frac{\alpha}{2c}(\lambda_1^{-1}\widetilde{v}_1^q)^{1-\alpha}(v_2^q)^{\alpha} \\ &> \frac{\alpha}{2c}\overline{v}^q = \overline{e}^q \end{aligned}$$

$$\begin{aligned} e_2^q &= \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1^q)^{-\alpha}(v_2^q)^{1+\alpha} \\ &= \frac{\alpha}{2c}(\lambda_1^{-1}\widetilde{v}_1^q)^{-\alpha}(v_2^q)^{1+\alpha} \\ &< \frac{\alpha}{2c}\overline{v}^q = \overline{e}^q \end{aligned}$$

where $e_1^q > \overline{e}^q > e_2^q$.

The equilibrium winning probabilities are

$$p_{21}^q = \frac{1}{2}\left(\frac{e_2^q}{e_1^q}\right)^{\alpha} = \frac{1}{2}\left(\frac{\lambda_1 v_2^q}{\widetilde{v}_1^q}\right)^{\alpha} = \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1^q\right)^{-\alpha}\left(v_2^q\right)^{\alpha}$$

$$p_{12}^q = 1 - p_{21}^q = 1 - \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1^q\right)^{-\alpha}\left(v_2^q\right)^{\alpha}$$

$$\widetilde{p}_{12}^q = 1 - \frac{1}{2}\frac{(e_2^q)^{\alpha}}{\lambda_1\left(e_1^q\right)^{\alpha}} = 1 - \frac{1}{2}\lambda_1^{-1}\left(\lambda_1\left(\widetilde{v}_1^q\right)^{-1}\left(v_2^q\right)\right)^{\alpha} = 1 - \frac{1}{2}\lambda_1^{\alpha-1}\left(\widetilde{v}_1^q\right)^{-\alpha}\left(v_2^q\right)^{\alpha}$$

where $\widetilde{p}_{12}^q > p_{12}^q > \frac{1}{2} > p_{21}^q$.

(2) case 2: $\lambda_1 e_1^{\alpha} \geqslant e_2^{\alpha} \quad and \quad e_2 \geqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^{\alpha}}{\lambda_1 e_1^{\alpha}}\right)\widetilde{v}_1^q - ce_1$

Player 2 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^{\alpha}}{e_2^{\alpha}}\right)v_2^q - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^{\alpha}}{e_1^{\alpha+1}}\widetilde{v}_1^q - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^{\alpha}}{e_2^{\alpha+1}}v_2^q - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^{\alpha}}{e_1^{\alpha+2}}\widetilde{v}_1^q < 0$

$[e_2] \quad \frac{\alpha}{2}(-\alpha-1)\frac{e_1^{\alpha}}{e_2^{\alpha+2}}v_2^q < 0$

Solve F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1^q\right)^{\frac{\alpha+1}{2\alpha+1}}\left(v_2^q\right)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{\frac{\alpha+1}{2\alpha+1}}$$

$$\frac{e_2}{e_1} = \lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1^q)^{-\frac{1}{2\alpha+1}}(v_2^q)^{\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \geqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

$$\lambda_1 e_1^\alpha \geqslant e_2^\alpha \Longleftrightarrow \frac{\lambda_1 e_1^\alpha}{e_2^\alpha} \geqslant 1$$
$$\Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{-\frac{\alpha}{2\alpha+1}} \geqslant 1$$

always holds.

② $e_2 \geqslant e_1$

$$\frac{e_2}{e_1} \geqslant 1 \Longleftrightarrow \lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1^q)^{-\frac{1}{2\alpha+1}}(v_2^q)^{\frac{1}{2\alpha+1}} \geqslant 1$$
$$\Longleftrightarrow \frac{v_2^q}{\lambda_1^{-1}\widetilde{v}_1^q} \geqslant 1$$
$$\Longleftrightarrow \overline{v}^q \geqslant \lambda_1^{-1}\widetilde{v}_1^q$$

According to the results we got from (1), we do not know if there exist some parameter configurations where $e_2 \geqslant e_1$ is satisfied.

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_2^\alpha$ *and* $e_2 \geqslant e_1$

Player 1 *max* $\frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1^q - ce_1$

Player 2 *max* $\left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right)v_2^q - ce_2$

F.o.c

$[e_1]$ $\frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^\alpha}\widetilde{v}_1^q - c = 0$

$[e_2]$ $\frac{\alpha}{2}\frac{e_1^\alpha}{e_2^{\alpha+1}}v_2^q - c = 0$

divide the two F.O.C , we get

$$\frac{e_2}{e_1} = \frac{v_2^q}{\lambda_1\widetilde{v}_1^q} < 1 \quad (\widetilde{v}_1^q > v_2^q)$$

which contradicts the condition that $e_2 \geqslant e_1$

To conclude, when $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 < \hat{\lambda}$, we know that case 1 equilibrium, where $e_1^q > \overline{e}^q > e_2^q$, certainly exists but we are not sure about case 2 equilibrium. It could be the case that when the overconfident player exerts higher effort than the rational opponent in the semifinal, his continuation value in the quarterfinal is so high that the encouraging effect always prevails in the quarterfinal and he always exerts higher effort than the rational opponent in the quarterfinal.

3. The equilibrium when $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 \geqslant \hat{\lambda}$, which corresponds to Proposition A5 (i)

Player 1 $\quad max \quad \widetilde{E}^q(U_{12}) = \widetilde{p}^q_{12}\widetilde{v}^q_1 - ce_1$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}^q_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_2^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}^q_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_2^\alpha \end{cases}$$

Player 2 $\quad max \quad E^q(U_{21}) = p^q_{21}v^q_2 - ce_2$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right)v^q_2 - ce_2 & \text{if}\, e_2 \geqslant e_1 \\ \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha}v^q_2 - ce_2 & \text{if}\, e_2 \leqslant e_1 \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \end{cases}$$

Since $\lambda_1 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}^q_1 - ce_1$

Player 2 $\quad max \quad \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha}v^q_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}^q_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_2^{\alpha-1}}{e_1^\alpha}v^q_2 - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}^q_1 < 0$

$[e_2] \quad \frac{\alpha}{2}(\alpha-1)\frac{e_2^{\alpha-2}}{e_1^\alpha}v^q_2 < 0$

Solve the two F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}^q_1)^{1-\alpha}(v^q_2)^\alpha$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1^q)^{-\alpha}(v_2^q)^{1+\alpha}$$

$$\frac{e_2}{e_1} = \lambda_1 \frac{v_2^q}{\widetilde{v}_1^q}$$

Check the conditions $\lambda_1 e_1^{\alpha} \geqslant e_2^{\alpha}$ and $e_2 \leqslant e_1$:

① $\lambda_1 e_1^{\alpha} \geqslant e_2^{\alpha}$

As long as $e_1 \geqslant e_2$ is satisfied, $\lambda_1 e_1^{\alpha} \geqslant e_2^{\alpha}$ is satisfied.

② $e_2 \leqslant e_1$

$$\frac{e_2}{e_1} = \lambda_1 \frac{v_2^q}{\widetilde{v}_1^q} = \frac{\overline{v}^q}{\lambda_1^{-1}\widetilde{v}_1^q}$$

Let

$$f(\lambda_1) = \frac{\lambda_1^{-1}\widetilde{v}_1^q - \overline{v}^q}{u(w_1) - u(w_2)}$$

$$\begin{aligned} f(\lambda_1) = \lambda_1^{-1} \Bigg[& \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) \\ & - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}} \\ & \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} + \frac{u(w_3)}{u(w_1) - u(w_2)} \Bigg] \\ & - \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right] \end{aligned}$$

Similar to before, we can get that $f(\lambda_1 = \hat{\lambda}) > 0$.

We can easily get that $f(\lambda_1 \to \infty) < 0$.

Therefore there must exist some parameter configurations and domains of overconfidence level where $e_1^q > e_2^q$ is satisfied. When $e_1^q > e_2^q$ is satisfied, we have $e_1^q > \overline{e}^q > e_2^q$.

The equilibrium winning probabilities are

$$p_{21}^q = \frac{1}{2}\left(\frac{e_2^q}{e_1^q}\right)^{\alpha} = \frac{1}{2}\left(\frac{\lambda_1 v_2^q}{\widetilde{v}_1^q}\right)^{\alpha} = \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1^q\right)^{-\alpha}\left(v_2^q\right)^{\alpha}$$

$$p_{12}^q = 1 - p_{21}^q = 1 - \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1^q\right)^{-\alpha}\left(v_2^q\right)^{\alpha}$$

$$\widetilde{p}_{12}^q = 1 - \frac{1}{2}\frac{(e_2^q)^{\alpha}}{\lambda_1 (e_1^q)^{\alpha}} = 1 - \frac{1}{2}\lambda_1^{-1}\left(\lambda_1\left(\widetilde{v}_1^q\right)^{-1}\left(v_2^q\right)\right)^{\alpha} = 1 - \frac{1}{2}\lambda_1^{\alpha-1}\left(\widetilde{v}_1^q\right)^{-\alpha}\left(v_2^q\right)^{\alpha}$$

where $\widetilde{p}_{12}^q > p_{12}^q > \frac{1}{2} > p_{21}^q$.

(2) case 2: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1^q - ce_1$

Player 2 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right)v_2^q - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1^q - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_2^{\alpha+1}}v_2^q - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1^q < 0$

$[e_2] \quad \frac{\alpha}{2}(-\alpha-1)\frac{e_1^\alpha}{e_2^{\alpha+2}}v_2^q < 0$

Solve F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha+1}{2\alpha+1}}(v_2^q)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{\frac{\alpha+1}{2\alpha+1}}$$

$$\frac{e_2}{e_1} = \lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1^q)^{-\frac{1}{2\alpha+1}}(v_2^q)^{\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \geqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

$$\lambda_1 e_1^\alpha \geqslant e_2^\alpha \Longleftrightarrow \frac{\lambda_1 e_1^\alpha}{e_2^\alpha} \geqslant 1$$
$$\Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{-\frac{\alpha}{2\alpha+1}} \geqslant 1$$

always holds.

② $e_2 \geqslant e_1$

$$\frac{e_2}{e_1} \geqslant 1 \Longleftrightarrow \lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1^q)^{-\frac{1}{2\alpha+1}}(v_2^q)^{\frac{1}{2\alpha+1}} \geqslant 1$$
$$\Longleftrightarrow \frac{v_2^q}{\lambda_1^{-1}\widetilde{v}_1^q} \geqslant 1$$
$$\Longleftrightarrow \overline{v}^q \geqslant \lambda_1^{-1}\widetilde{v}_1^q$$

According to the results in (1), there must exist some parameter configurations where $e_2^q \geqslant e_1^q$ is satisfied.

When $e_2^q \geqslant e_1^q$ is satisfied, $\overline{v}^q \geqslant \lambda_1^{-1}\widetilde{v}_1^q$. The equilibrium efforts are

$$e_1^q = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha+1}{2\alpha+1}}(v_2^q)^{\frac{\alpha}{2\alpha+1}}$$

$$
\begin{aligned}
e_2^q &= \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}}\left(v_2^q\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \frac{\alpha}{2c}\left(\lambda_1^{-1}\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}}\left(v_2^q\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&\leqslant \frac{\alpha}{2c}\overline{v}^q = \overline{e}^q
\end{aligned}
$$

where $e_1^q \leqslant e_2^q \leqslant \overline{e}^q$.

The equilibrium winning probabilities are

$$
p_{12}^q = \frac{1}{2}\left(\frac{e_1^q}{e_2^q}\right)^{\alpha} = \frac{1}{2}\left(\frac{\left(\lambda_1^{-1}\widetilde{v}_1^q\right)^{\frac{1}{2\alpha+1}}}{\left(v_2^q\right)^{\frac{1}{2\alpha+1}}}\right)^{\alpha} = \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}}\left(v_2^q\right)^{-\frac{\alpha}{2\alpha+1}}
$$

$$
p_{21}^q = 1 - p_{12}^q = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}}\left(v_2^q\right)^{-\frac{\alpha}{2\alpha+1}}
$$

$$
\widetilde{p}_{12}^q = 1 - \frac{1}{2}\frac{\left(e_2^q\right)^{\alpha}}{\lambda_1\left(e_1^q\right)^{\alpha}} = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1^q\right)^{-\frac{\alpha}{2\alpha+1}}\left(v_2^q\right)^{\frac{\alpha}{2\alpha+1}}
$$

Thus we have $\widetilde{p}_{12}^q > p_{21}^q \geqslant \frac{1}{2} \geqslant p_{12}^q$.

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1^q - ce_1$

Player 2 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right)v_2^q - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^\alpha}\widetilde{v}_1^q - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_2^{\alpha+1}}v_2^q - c = 0$

divide the two F.O.C , we get

$$
\frac{e_2}{e_1} = \frac{v_2^q}{\lambda_1\widetilde{v}_1^q} < 1 \quad (\widetilde{v}_1^q > v_2^q)
$$

which contradicts the condition that $e_2 \geqslant e_1$

To conclude, under the condition that "$\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 \geqslant \hat{\lambda}$", depending on the utility spread and overconfidence level, in equilibrium the overconfident player can exert either lower or higher effort than the rational player. Both situations exist with certainty. We know for sure that $e_1^q > \overline{e}^q > e_2^q$ is satisfied at $\lambda_1 = \hat{\lambda}$.

4. The equilibrium when $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$, which corresponds to Proposition A5(ii)

Player 1 $\quad max \quad \widetilde{E}^q(U_{12}) = \widetilde{p}_{12}^q\widetilde{v}_1^q - ce_1$

$$
= \begin{cases} \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1^q - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_2^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1^q - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_2^\alpha \end{cases}
$$

Player 2 $\quad max \quad E^q(U_{21}) = p_{21}^q v_2^q - ce_2$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right) v_2^q - ce_2 & \text{if } e_2 \geqslant e_1 \\ \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha} v_2^q - ce_2 & \text{if } e_2 \leqslant e_1 \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \end{cases}$$

Since $\lambda_1 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1$

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1^q)^{1-\alpha}(v_2^q)^\alpha$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1^q)^{-\alpha}(v_2^q)^{1+\alpha}$$

$$\frac{e_2}{e_1} = \lambda_1 \frac{v_2^q}{\widetilde{v}_1^q}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \leqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

As long as $e_1 \geqslant e_2$ is satisfied, $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ is satisfied.

② $e_2 \leqslant e_1$

$$\frac{e_2}{e_1} = \lambda_1 \frac{v_2^q}{\widetilde{v}_1^q} = \frac{\overline{v}^q}{\lambda_1^{-1}\widetilde{v}_1^q}$$

Let

$$f(\lambda_1) = \frac{\lambda_1^{-1}\widetilde{v}_1^q - \overline{v}^q}{u(w_1) - u(w_2)}$$

$$\begin{aligned} f(\lambda_1) = \lambda_1^{-1}\Bigg[&\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) \\ &- \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}} \\ &\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} + \frac{u(w_3)}{u(w_1)-u(w_2)}\Bigg] \\ &- \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)}\right] \end{aligned}$$

We can easily get

$$f(\lambda_1 = 1) = 0 \quad \text{and} \quad f(\lambda_1 \to \infty) < 0.$$

$$\begin{aligned}
f'(\lambda_1) = &-\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} + \frac{u(w_3)}{u(w_1) - u(w_2)}\right)\lambda_1^{-2} \\
&+ \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1} + 1\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-2} \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\left[-\left(\frac{\alpha+1}{2\alpha+1} + 1\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}}\right. \\
&+ \lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\frac{\alpha+1}{2\alpha+1}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}} \\
&\left.\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\right] \\
= \lambda_1^{-2}&\left[-\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} + \frac{u(w_3)}{u(w_1) - u(w_2)}\right)\right. \\
&+ \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1} + 1\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\left(-\left(\frac{\alpha+1}{2\alpha+1} + 1\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}}\right. \\
&\left.\left.+ \lambda_1^{-\frac{2\alpha+2}{2\alpha+1}}\frac{1+\alpha}{2}\frac{(\alpha+1)^2}{(2\alpha+1)^2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
f'(\lambda_1 = 1) = &- \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} + \frac{u(w_3)}{u(w_1) - u(w_2)}\right) + \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\left[- \left(\frac{\alpha+1}{2\alpha+1} + 1\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha+1}{2\alpha+1}} \right. \\
&\left. + \frac{\alpha+1}{2\alpha+1}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{-\frac{\alpha}{2\alpha+1}} \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right] \\
= &- \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} + \frac{u(w_3)}{u(w_1) - u(w_2)}\right) + \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} \left(-\left(\frac{\alpha+1}{2\alpha+1} + 1\right)\right) \\
&\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} \frac{\alpha+1}{2\alpha+1} \\
&\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{-\frac{\alpha}{2\alpha+1}} \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
= &- \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} + \frac{u(w_3)}{u(w_1) - u(w_2)}\right) + \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&+ \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&- \frac{1+\alpha}{2}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha+1}{2\alpha+1}
\end{aligned}
$$

$$
\begin{aligned}
&f'(\lambda_1 = 1) \lesseqgtr 0 \\
&\iff - \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} + \frac{u(w_3)}{u(w_1) - u(w_2)}\right) + \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&+ \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&- \frac{1+\alpha}{2}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha+1}{2\alpha+1} \lesseqgtr 0 \\
&\iff \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1} + 1\right) + \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&- \frac{1+\alpha}{2}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha+1}{2\alpha+1} \lesseqgtr 1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} + \frac{u(w_3)}{u(w_1) - u(w_2)} \\
&\iff \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1} + 1\right) + \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&- \frac{1+\alpha}{2}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha+1}{2\alpha+1} - 1 - \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} \lesseqgtr \frac{u(w_3)}{u(w_1) - u(w_2)} \\
&\iff \frac{\alpha(3\alpha+1)}{2(2\alpha+1)}\frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} + \frac{-7\alpha^4 + \alpha^3 + 8\alpha^2 + 5\alpha + 1}{4(2\alpha+1)^2} \lesseqgtr \frac{u(w_3)}{u(w_1) - u(w_2)}
\end{aligned}
$$

If $\frac{u(w_3)}{u(w_1)-u(w_2)} < \frac{\alpha(3\alpha+1)}{2(2\alpha+1)} \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{-7\alpha^4+\alpha^3+8\alpha^2+5\alpha+1}{4(2\alpha+1)^2}$, then $f'(\lambda_1 = 1) > 0$ and thus $f(\lambda_1) > 0$ exists. This means $e_1 \geqslant e_2$ must be satisfied under some domains of overconfidence level given that $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} < \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$. In this situation we are certain that case 1 equilibrium exists under some parameter configurations.

If $\frac{u(w_3)}{u(w_1)-u(w_2)} > \frac{\alpha(3\alpha+1)}{2(2\alpha+1)} \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{-7\alpha^4+\alpha^3+8\alpha^2+5\alpha+1}{4(2\alpha+1)^2}$, then $f'(\lambda_1 = 1) < 0$ and thus $f(\lambda_1) < 0$ exists. In this situation we do not know if $e_1 \geqslant e_2$ will be satisfied given that $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} < \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$. It is not clear whether case 1 equilibrium exists.

(2) case 2: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1$

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha+1}{2\alpha+1}}(v_2^q)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{\frac{\alpha+1}{2\alpha+1}}$$

$$\frac{e_2}{e_1} = \lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1^q)^{-\frac{1}{2\alpha+1}}(v_2^q)^{\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \geqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

$$\lambda_1 e_1^\alpha \geqslant e_2^\alpha \Longleftrightarrow \frac{\lambda_1 e_1^\alpha}{e_2^\alpha} \geqslant 1$$
$$\Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{-\frac{\alpha}{2\alpha+1}} \geqslant 1$$

always holds.

② $e_2 \geqslant e_1$

$$\frac{e_2}{e_1} \geqslant 1 \Longleftrightarrow \lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1^q)^{-\frac{1}{2\alpha+1}}(v_2^q)^{\frac{1}{2\alpha+1}} \geqslant 1$$
$$\Longleftrightarrow \frac{v_2^q}{\lambda_1^{-1}\widetilde{v}_1^q} \geqslant 1$$
$$\Longleftrightarrow \overline{v}^q \geqslant \lambda_1^{-1}\widetilde{v}_1^q$$

Similar to the results in (1):
If $\frac{u(w_3)}{u(w_1)-u(w_2)} < \frac{\alpha(3\alpha+1)}{2(2\alpha+1)} \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{-7\alpha^4+\alpha^3+8\alpha^2+5\alpha+1}{4(2\alpha+1)^2}$, then $f'(\lambda_1 = 1) > 0$. This means $f(\lambda_1) < 0$ and thus $e_1 \leqslant e_2$ are satisfied when λ_1 is extremely large, given that $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$. In this situation we are certain that case 2 equilibrium exists under some parameter configurations.

If $\frac{u(w_3)}{u(w_1)-u(w_2)} > \frac{\alpha(3\alpha+1)}{2(2\alpha+1)} \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{-7\alpha^4+\alpha^3+8\alpha^2+5\alpha+1}{4(2\alpha+1)^2}$, then $f'(\lambda_1 = 1) < 0$ and thus $f(\lambda_1) < 0$ exists with certainty when λ_1 is close to 1 or extremely large. In this situation we are also certain that case 2 equilibrium exists under some parameter configurations.

When $e_2^q \geqslant e_1^q$ is satisfied, $\overline{v}^q \geqslant \lambda_1^{-1}\widetilde{v}_1^q$. The equilibrium efforts are

$$e_1^q = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1^q\right)^{\frac{\alpha+1}{2\alpha+1}}\left(v_2^q\right)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2^q = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}}\left(v_2^q\right)^{\frac{\alpha+1}{2\alpha+1}}$$

where $e_1^q \leqslant e_2^q \leqslant \overline{e}^q$.

$$p_{12}^q = \frac{1}{2}\left(\frac{e_1^q}{e_2^q}\right)^{\alpha} = \frac{1}{2}\left(\frac{\left(\lambda_1^{-1}\widetilde{v}_1^q\right)^{\frac{1}{2\alpha+1}}}{\left(v_2^q\right)^{\frac{1}{2\alpha+1}}}\right)^{\alpha} = \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}}\left(v_2^q\right)^{-\frac{\alpha}{2\alpha+1}}$$

$$p_{21}^q = 1 - p_{12}^q = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}}\left(v_2^q\right)^{-\frac{\alpha}{2\alpha+1}}$$

$$\widetilde{p}_{12}^q = 1 - \frac{1}{2}\frac{\left(e_2^q\right)^{\alpha}}{\lambda_1\left(e_1^q\right)^{\alpha}} = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1^q\right)^{-\frac{\alpha}{2\alpha+1}}\left(v_2^q\right)^{\frac{\alpha}{2\alpha+1}}$$

where $\widetilde{p}_{12}^q > p_{21}^q \geqslant \frac{1}{2} \geqslant p_{12}^q$.

(3) case 3: $\lambda_1 e_1^{\alpha} \leqslant e_2^{\alpha}$ *and* $e_2 \geqslant e_1$

Player 1 max $\frac{1}{2}\frac{\lambda_1 e_1^{\alpha}}{e_2^{\alpha}}\widetilde{v}_1^q - ce_1$

Player 2 max $\left(1 - \frac{1}{2}\frac{e_1^{\alpha}}{e_2^{\alpha}}\right)v_2^q - ce_2$

F.o.c

$[e_1]$ $\frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^{\alpha}}\widetilde{v}_1^q - c = 0$

$[e_2]$ $\frac{\alpha}{2}\frac{e_1^{\alpha}}{e_2^{\alpha+1}}v_2^q - c = 0$

divide the two F.O.C , we get

$$\frac{e_2}{e_1} = \frac{v_2^q}{\lambda_1\widetilde{v}_1^q} < 1 \quad (\widetilde{v}_1^q > v_2^q)$$

which contradicts the condition that $e_2 \geqslant e_1$

To conclude, under the condition $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$, case 2 equilibrium exists with certainty. Whether case 1 equilibrium depends on the relationship between the utility spread in each stage. When $\frac{u(w_3)}{u(w_1)-u(w_2)} < \frac{\alpha(3\alpha+1)}{2(2\alpha+1)}\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{-7\alpha^4+\alpha^3+8\alpha^2+5\alpha+1}{4(2\alpha+1)^2}$ and λ_1 is close to 1, $e_1^q > \overline{e}^q > e_2^q$ is satisfied and the overconfident player exerts higher effort at equilibrium than the rational player and the benchmark.

5. Participation constraints

(1) When $e_1^q > e_2^q$

$$
\begin{aligned}
\widetilde{E}^q(U_{12}) &= \widetilde{p}_{12}^q \widetilde{v}_1^q - c e_1^q \\
&> p_{12}^q \widetilde{v}_1^q - c e_1^q \\
&= \left(1 - \frac{1}{2}\lambda_1^{\alpha} \left(\widetilde{v}_1^q\right)^{-\alpha} \left(v_2^q\right)^{\alpha}\right) \widetilde{v}_1^q - c\frac{\alpha}{2c}\lambda_1^{\alpha-1} \left(\widetilde{v}_1^q\right)^{1-\alpha} \left(v_2^q\right)^{\alpha} \\
&= \widetilde{v}_1^q - \frac{1}{2}\lambda_1^{\alpha} \left(\widetilde{v}_1^q\right)^{1-\alpha} \left(v_2^q\right)^{\alpha} - \frac{\alpha}{2}\lambda_1^{\alpha-1} \left(\widetilde{v}_1^q\right)^{1-\alpha} \left(v_2^q\right)^{\alpha} \\
&> \widetilde{v}_1^q - \frac{1}{2}\lambda_1^{\alpha} \left(\widetilde{v}_1^q\right)^{1-\alpha} \left(v_2^q\right)^{\alpha} - \frac{1}{2}\lambda_1^{\alpha} \left(\widetilde{v}_1^q\right)^{1-\alpha} \left(v_2^q\right)^{\alpha} \\
&= \widetilde{v}_1^q - \lambda_1^{\alpha} \left(\widetilde{v}_1^q\right)^{1-\alpha} \left(v_2^q\right)^{\alpha} \\
&= \left(\widetilde{v}_1^q\right)^{1-\alpha} \left[\left(\widetilde{v}_1^q\right)^{\alpha} - \lambda_1^{\alpha} \left(v_2^q\right)^{\alpha}\right] \\
&> 0
\end{aligned}
$$

$$
\begin{aligned}
E^q(U_{21}) &= p_{21}^q v_2^q - c e_2^q \\
&= \frac{1}{2}\lambda_1^{\alpha} \left(\widetilde{v}_1^q\right)^{-\alpha} \left(v_2^q\right)^{\alpha} v_2^q - c\frac{\alpha}{2c}\lambda_1^{\alpha} \left(\widetilde{v}_1^q\right)^{-\alpha} \left(v_2^q\right)^{\alpha+1} \\
&= \frac{1}{2}\lambda_1^{\alpha} \left(\widetilde{v}_1^q\right)^{-\alpha} \left(v_2^q\right)^{1+\alpha} - \frac{\alpha}{2}\lambda_1^{\alpha} \left(\widetilde{v}_1^q\right)^{-\alpha} \left(v_2^q\right)^{\alpha+1} \\
&= \frac{1-\alpha}{2}\lambda_1^{\alpha} \left(\widetilde{v}_1^q\right)^{-\alpha} \left(v_2^q\right)^{1+\alpha} \\
&\geqslant 0
\end{aligned}
$$

(2) When $e_1^q \leqslant e_2^q$

$$
\widetilde{E}^q(U_{12}) = \widetilde{p}_{12}^q \widetilde{v}_1^q - c e_1^q
$$

Since $\widetilde{p}_{12}^q > \frac{1}{2}$, $\widetilde{v}_1^q > \overline{v}^q$ and $e_1^q \leqslant \overline{e}^q$, we can get that $\widetilde{E}^q(U_{12}) > \overline{E}^q(U) \geqslant 0$.

$$
\begin{aligned}
E^q(U_{21}) &= p_{21}^q v_2^q - c e_2^q \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2^q\right)^{-\frac{\alpha}{2\alpha+1}}\right) v_2^q - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2^q\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2^q\right)^{-\frac{\alpha}{2\alpha+1}}\right) v_2^q - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2^q\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= v_2^q - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2^q\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \left(v_2^q\right)^{\frac{\alpha+1}{2\alpha+1}} \left(\left(v_2^q\right)^{\frac{\alpha}{2\alpha+1}} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}}\right) \\
&\geqslant 0
\end{aligned}
$$

Next we characterize the equilibrium of the quarterfinal between rational players 3 and 4.

Proposition A6 *In a three-stage elimination contest where player 1 is overconfident and the other seven players are rational, consider the quarterfinal between two rational*

players who have the chance of meeting the overconfident player in the semifinal.
(i) If $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ *and* $\lambda_1 < \hat{\lambda}$ *where* $\hat{\lambda} > 1$ *solves* $1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}} = \hat{\lambda}(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2})$*, then the equilibrium efforts and winning probabilities satisfy* $e_3^q = e_4^q < \overline{e}^q$ *and* $p_{34}^q = p_{43}^q = 1/2$*.*
(ii) If either $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ *or* $\lambda_1 \geqslant \hat{\lambda}$*, then the equilibrium efforts and winning probabilities satisfy* $e_3^q = e_4^q \geqslant \overline{e}^q$ *and* $p_{34}^q = p_{43}^q = 1/2$*.*

Proposition A6 shows that, for the rational players who have the chance to meet the overconfident player in the semifinal, their continuation values in the quarterfinal depend on the equilibrium in the semifinal. When the parameter configurations are such that the overconfident player exerts lower (higher) effort than his rational rival in the equilibrium of the semifinal, the rational players' continuation values in the quarterfinal increase (decrease) and thus they both exert more (less) effort.

Proof of Proposition A6

1. Continuation values

Player 3's continuation value is given by

$$v_3^q = p_{12}^q E^s(U_{31}) + p_{21}^q E^s(U_{32})$$

(1) When $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 < \hat{\lambda}$

$$\begin{aligned}
E^s(U_{31}) &= p_{31}^s\left(v_3^s - u(w_3)\right) - ce_3^s + u(w_3) \\
&= \frac{1}{2}\lambda_1^\alpha\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(v_3^s - u(w_3)\right)^\alpha\left(v_3^s - u(w_3)\right) \\
&\quad - c\frac{\alpha}{2c}\lambda_1^\alpha\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(v_3^s - u(w_3)\right)^{\alpha+1} + u(w_3) \\
&= \frac{1}{2}\lambda_1^\alpha\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(v_3^s - u(w_3)\right)^{1+\alpha} \\
&\quad - \frac{\alpha}{2}\lambda_1^\alpha\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(v_3^s - u(w_3)\right)^{\alpha+1} + u(w_3) \\
&= \frac{1-\alpha}{2}\lambda_1^\alpha\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(v_3^s - u(w_3)\right)^{1+\alpha} + u(w_3) \\
&= \frac{1-\alpha}{2}\lambda_1^\alpha\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(\overline{v}^s - u(w_3)\right)^{1+\alpha} + u(w_3) \\
&= \frac{1-\alpha}{2}\left(\frac{\overline{v}^s - u(w_3)}{\lambda_1^{-1}\left(\widetilde{v}_1^s - u(w_3)\right)}\right)^\alpha\left(\overline{v}^s - u(w_3)\right) + u(w_3) \\
&< \overline{E}^s(U)
\end{aligned}$$

$$E^s(U_{32}) = \overline{E}^s(U)$$

Thus we have

$$v_3^q < \overline{v}^q$$

(2) When either $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ or $\lambda_1 \geqslant \hat{\lambda}$

$$\begin{aligned}
E^s(U_{31}) &= p_{31}^s\left(v_3^s - u(w_3)\right) - ce_3^s + u(w_3) \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{-\frac{\alpha}{2\alpha+1}}\right)\left(v_3^s - u(w_3)\right) \\
&\quad - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} + u(w_3) \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{-\frac{\alpha}{2\alpha+1}}\right)\left(v_3^s - u(w_3)\right) \\
&\quad - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} + u(w_3) \\
&= \left(v_3^s - u(w_3)\right) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} + u(w_3) \\
&= \left(\overline{v}^s - u(w_3)\right) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(\overline{v}^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} + u(w_3) \\
&= \left(\overline{v}^s - u(w_3)\right)\left[1 - \frac{1+\alpha}{2}\left(\frac{\lambda_1^{-1}\left(\widetilde{v}_1^s - u(w_3)\right)}{\overline{v}^s - u(w_3)}\right)^{\frac{\alpha}{2\alpha+1}}\right] + u(w_3) \\
&\geqslant \overline{E}^s(U)
\end{aligned}$$

$$E^s(U_{32}) = \overline{E}^s(U)$$

$$v_3^q \geqslant \overline{v}^q$$

Since player 3 and player 4 are identical, $v_4^q = v_3^q$.

2. Equilibrium efforts

$$e_3^q = e_4^q = \frac{\alpha}{2c}v_3^q$$

$$e_3^q = e_4^q \begin{cases} < \overline{e}^q & \text{when} \quad \frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)} \quad \text{and} \quad \lambda_1 < \hat{\lambda} \\ \geqslant \overline{e}^q & \text{when} \quad \text{either} \quad \frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)} \quad \text{or} \quad \lambda_1 \geqslant \hat{\lambda} \end{cases}$$

3. Participation constraints

$$\begin{aligned}
E^q(U_{34}) = E^q(U_{43}) = p_{34}^q v_3^q - ce_3 &= \frac{1}{2}v_3^q - c\frac{\alpha}{2c}v_3^q \\
&= \frac{1-\alpha}{2}v_3^q \\
&\geqslant 0
\end{aligned}$$

Last we characterize the equilibria of the quarterfinals between rational players 5 and 6, and 7 and 8.

Proposition A7 *In a three-stage elimination contest where player 1 is overconfident and the other seven players are rational, consider the quarterfinals between the rational players who have the chance of meeting the overconfident player in the final. The equilibrium efforts and winning probabilities satisfy* $e_5^q = e_6^q = e_7^q = e_8^q > \overline{e}^q$, $p_{56}^q = p_{65}^q = p_{78}^q = p_{87}^q = 1/2$.

Proposition A7 shows that, for the rational players who only have the chance to meet the overconfident player in the final, their continuation values in the quarterfinal are higher and thus their equilibrium efforts in the quarterfinal increase.

Proof of Proposition A7

1. Continuation values

$$\begin{aligned} v_5^q &= E^s(U_{57}) \\ &= p_{57}^s \left(v_5^s - u(w_3)\right) - ce_5^s + u(w_3) \end{aligned}$$

where

$$\begin{aligned} v_5^s &= p_{12}^q p_{13}^s E^f(U_{51}) + \left(1 - p_{12}^q p_{13}^s\right) E^f(U_{53}) \\ &= p_{12}^q p_{13}^s E^f(U_{51}) + \left(1 - p_{12}^q p_{13}^s\right) \overline{E}^f(U) \\ &> \overline{E}^f(U) = \overline{v}^s. \end{aligned}$$

Therefore

$$v_5^q = v_6^q = v_7^q = v_8^q > \overline{v}^q$$

2. Equilibrium efforts

$$e_5^q = e_6^q = e_7^q = e_8^q = \frac{\alpha}{2c} v_5^q > \overline{e}^q$$

3. Participation constraints

$$\begin{aligned} E^q(U_{56}) = E^q(U_{65}) = E^q(U_{78}) = E^q(U_{87}) = p_{56}^q v_5^q - ce_5 &= \frac{1}{2} v_5^q - c\frac{\alpha}{2c} v_5^q \\ &= \frac{1-\alpha}{2} v_5^q \\ &\geqslant 0 \end{aligned}$$

4 Elimination Contest with One Underconfident Player

This section characterizes the equilibrium of a two-stage elimination contest with one underconfident player and three rational players. Throughout we assume player 1 is underconfident with $0 < \lambda_1 < 1$ and players 2, 3, and 4 are rational with $\lambda_2 = \lambda_3 = \lambda_4 = 1$. Players 1 and 2 are paired in one semifinal and players 3 and 4 are paired in the other semifinal.

4.1 Final

We start by analyzing the impact of underconfidence on the final. Since players 3 and 4 are identical, we consider a final with an underconfident player 1 and a rational player 3 without loss of generality.

Proposition A8 *In a final between an underconfident player and a rational player, the equilibrium effort of the underconfident player is*

$$e_1^f = \frac{\alpha}{2c} \lambda_1^{1+\alpha} [u(w_1) - u(w_2)],$$

and the equilibrium effort of the rational player is

$$e_3^f = \frac{\alpha}{2c} \lambda_1^{\alpha} [u(w_1) - u(w_2)].$$

with $e_1^f < e_3^f < \bar{e}^f$*. The perceived equilibrium winning probability of the underconfident player is*

$$\widetilde{p}_{13}^f = \frac{1}{2}\lambda_1^{1+\alpha},$$

and the true equilibrium winning probabilities are

$$p_{13}^f = \frac{1}{2}\lambda_1^{\alpha}$$

$$p_{31}^f = 1 - \frac{1}{2}\lambda_1^{\alpha}$$

with $p_{31}^f > 1/2 > p_{13}^f > \widetilde{p}_{13}^f$*. The perceived equilibrium expected utility of the underconfident player is*

$$\widetilde{E}^f(U_{13}) = \left[\left(1 - \frac{1+\alpha}{2}\right)\lambda_1^{\alpha+1} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right][u(w_1)-u(w_2)],$$

and the equilibrium expected utility of the rational player is

$$E^f(U_{31}) = \left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{\alpha}\right)[u(w_1)-u(w_2)],$$

with $E^f(U_{31}) > \overline{E}^f(U) > \widetilde{E}^f(U_{13})$.

Proposition A8 shows that an underconfident player exerts less effort than a rational rival in the final and that both players exert less effort than if both were rational. It also shows that the underconfident player's perceived and true probabilities of winning the final are decreasing in his bias whereas the rational player's true probability of winning the final is increasing with the bias of the underconfident player. Finally, Proposition A8 shows that the underconfident player's perceived expected utility of the final is decreasing in his bias whereas the rational player's expected utility of the final is increasing in the bias of the underconfident player. Hence, underconfidence makes reaching the final less attractive for an underconfident player and more attractive for a rational rival.

Proof of Proposition A8

The perceived winning probabilities of the players are:

$$\widetilde{p}_{13}^f = \begin{cases} 1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha} & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha} & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

$$p_{31}^f = \begin{cases} 1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha} & \text{if} \quad e_3 \geqslant e_1 \\ \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha} & \text{if} \quad e_3 \leqslant e_1 \end{cases}$$

Underconfident player 1 $\quad max \quad \widetilde{E}^f(U_{13}) = \widetilde{p}_{13}^f[u(w_1)-u(w_2)] - ce_1 + u(w_2)$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)[u(w_1)-u(w_2)] - ce_1 + u(w_2) & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}[u(w_1)-u(w_2)] - ce_1 + u(w_2) & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

Rational player 3 $\quad max \quad E^f(U_{31}) = p_{31}^f[u(w_1) - u(w_2)] - ce_3 + u(w_2)$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)[u(w_1) - u(w_2)] - ce_3 + u(w_2) & \text{if} \quad e_3 \geqslant e_1 \\ \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}[u(w_1) - u(w_2)] - ce_3 + u(w_2) & \text{if} \quad e_3 \leqslant e_1 \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1 \end{cases}$$

Since $\lambda_1 < 1$, the fourth case is impossible.

1. Equilibrium efforts

   (1) case 1 $\lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1$

   Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}[u(w_1) - u(w_2)] - ce_1 + u(w_2)$

   Player 3 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)[u(w_1) - u(w_2)] - ce_3 + u(w_2)$

   F.o.c

   $[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_3^\alpha}[u(w_1) - u(w_2)] - c = 0$

   $[e_3] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_3^{\alpha+1}}[u(w_1) - u(w_2)] - c = 0$

   S.o.c

   $[e_1] \quad \frac{\alpha\lambda_1}{2}(\alpha - 1)\frac{e_1^{\alpha-2}}{e_3^\alpha}[u(w_1) - u(w_2)] < 0$

   $[e_2] \quad \frac{\alpha}{2}(-\alpha - 1)\frac{e_1^\alpha}{e_3^{\alpha+2}}[u(w_1) - u(w_2)] < 0$

   Solve F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha+1}[u(w_1) - u(w_2)]$$

$$e_3 = \frac{\alpha}{2c}\lambda_1^\alpha[u(w_1) - u(w_2)]$$

   Check the conditions $\lambda_1 e_1^\alpha \leqslant e_3^\alpha$ and $e_3 \geqslant e_1$:

   ① $\lambda_1 e_1^\alpha \leqslant e_3^\alpha$

$$\lambda_1 e_1^\alpha \leqslant e_3^\alpha \iff \lambda_1 \left(\frac{e_1}{e_3}\right)^\alpha \leqslant 1$$
$$\iff \lambda_1^{1+\alpha} \leqslant 1$$

   which always holds.

② $e_3 \geqslant e_1$

$$\frac{e_3}{e_1} = \frac{1}{\lambda_1} > 1$$

$e_3 \geqslant e_1$ always holds.

(2) case 2 $\lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}[u(w_1) - u(w_2)] - ce_1 + u(w_2)$

Player 3 $\quad max \quad \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}[u(w_1) - u(w_2)] - ce_3 + u(w_2)$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_3^\alpha}[u(w_1) - u(w_2)] - c = 0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_3^{\alpha-1}}{e_1^\alpha}[u(w_1) - u(w_2)] - c = 0$

divide the two F.O.C , we get

$$\lambda_1 \left(\frac{e_1}{e_3}\right)^{2\alpha-1} = 1$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$ and $e_3 \geqslant e_1$:

$$\frac{e_3}{e_1} = \lambda_1^{\frac{1}{2\alpha-1}}$$

$$\lambda_1 \left(\frac{e_1}{e_3}\right)^{\alpha} = \lambda_1^{\frac{\alpha-1}{2\alpha-1}}$$

Since $\alpha - 1 < 0$, $\lambda_1^{\frac{1}{2\alpha-1}}$ and $\lambda_1^{\frac{\alpha-1}{2\alpha-1}}$ have different signs, one of the conditions must be contradicted. Case 2 does not hold.

(3) case 3 $\lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)[u(w_1) - u(w_2)] - ce_1 + u(w_2)$

Player 3 $\quad max \quad \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}[u(w_1) - u(w_2)] - ce_3 + u(w_2)$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_3^\alpha}{e_1^{\alpha+1}}[u(w_1) - u(w_2)] - c = 0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_3^{\alpha-1}}{e_1^\alpha}[u(w_1) - u(w_2)] - c = 0$

divide the two F.O.C , we get

$$\frac{e_3}{e_1} = \lambda_1 < 1$$

Thus we get

$$\lambda_1 \left(\frac{e_1}{e_3}\right)^{\alpha} = \lambda_1^{1-\alpha} < 1$$

which contradicts the condition that $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$. Case 3 does not hold.

Thus the unique equilibrium is

$$e_1^f = \frac{\alpha}{2c}\lambda_1^{1+\alpha}[u(w_1) - u(w_2)] < \overline{e}^f$$

$$e_3^f = \frac{\alpha}{2c}\lambda_1^{\alpha}[u(w_1) - u(w_2)] < \overline{e}^f$$

where $\lambda_1 e_1^\alpha < e_3^\alpha$ and $e_3 > e_1$.

2. Winning probabilities

$$p_{13}^f = \frac{1}{2}\left(\frac{e_1}{e_3}\right)^\alpha = \frac{1}{2}\lambda_1^\alpha$$

$$p_{31}^f = 1 - p_{13}^f = 1 - \frac{1}{2}\lambda_1^\alpha$$

$$\widetilde{p}_{13}^f = \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha} = \frac{1}{2}\lambda_1^{1+\alpha}$$

where $p_{31}^f > \frac{1}{2} > p_{13}^f > \widetilde{p}_{13}^f$.

3. Perceived expected utilities of the final

$$\begin{aligned}\widetilde{E}^f(U_{13}) &= \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}[u(w_1) - u(w_2)] - ce_1 + u(w_2) \\ &= \frac{1}{2}\lambda_1^{1+\alpha}[u(w_1) - u(w_2)] - \frac{\alpha}{2}\lambda_1^{1+\alpha}[u(w_1) - u(w_2)] + u(w_2) \\ &= \left[\left(1 - \frac{1+\alpha}{2}\right)\lambda_1^{\alpha+1} + \frac{u(w_2)}{u(w_1) - u(w_2)}\right][u(w_1) - u(w_2)] < \overline{E}^f(U)\end{aligned}$$

$$\begin{aligned}E^f(U_{31}) &= \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)[u(w_1) - u(w_2)] - ce_3 + u(w_2) \\ &= \left(1 - \frac{1}{2}\lambda_1^\alpha\right)[u(w_1) - u(w_2)] - \frac{\alpha}{2}\lambda_1^\alpha[u(w_1) - u(w_2)] + u(w_2) \\ &= \left(1 + \frac{u(w_2)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^\alpha\right)[u(w_1) - u(w_2)] > \overline{E}^f(U)\end{aligned}$$

The participation constraints are satisfied.

4.2 Semifinals

We now analyze the impact of underconfidence on the two semifinals. We start with the semifinal with an underconfident and a rational player. Next, we consider the semifinal with two rational players.

Proposition A9 *In a semifinal between an underconfident player and a rational player of a two-stage elimination contest where player 1 is underconfident and players 2, 3, and 4 are rational, the equilibrium efforts and winning probabilities satisfy* $\overline{e}^s > e_2^s > e_1^s$ *and* $p_{21}^s > 1/2 > p_{12}^s$.

Proposition A9 shows that in the semifinal, unlike overconfidence who has opposite effects on overconfident players, underconfidence only has negative effects on the underconfident player since the continuation value of the underconfident player is lower than the benchmark. The underconfident player lowers his effort relative to the benchmark, and the rational player reacts to this by reducing his effort but not as much.

Proof of Proposition A9

1. Continuation values

   Underconfident player 1:

$$\widetilde{v}_1 = p^s_{34}\widetilde{E}^f(U_{13}) + p^s_{43}\widetilde{E}^f(U_{14})$$

   Since players 3 and 4 are identical, $\widetilde{E}^f(U_{13}) = \widetilde{E}^f(U_{14})$

$$\widetilde{v}_1 = \widetilde{E}^f(U_{13}) = \left[\left(1 - \frac{1+\alpha}{2}\right)\lambda_1^{\alpha+1} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right][u(w_1)-u(w_2)] < \overline{v}$$

   Rational player 2:

$$v_2 = p^s_{34}E^f(U_{23}) + p^s_{43}E^f(U_{24})$$

   Since players 3 and 4 are identical, $E^f(U_{23}) = E^f(U_{24})$

$$v_2 = \left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)[u(w_1)-u(w_2)] = \overline{v}$$

2. The equilibrium

   Player 1 $\quad max \quad \widetilde{E}^s(U_{12}) = \widetilde{p}^s_{12}\widetilde{v}_1 - ce_1$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_2^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_2^\alpha \end{cases}$$

   Player 2 $\quad max \quad E^s(U_{21}) = p^s_{21}v_2 - ce_2$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right)v_2 - ce_2 & \text{if}\, e_2 \geqslant e_1 \\ \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha}v_2 - ce_2 & \text{if}\, e_2 \leqslant e_1 \end{cases}$$

   There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \end{cases}$$

   Since $\lambda_1 < 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right) v_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^\alpha}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_2^{\alpha+1}}v_2 - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}(\alpha-1)\frac{e_1^{\alpha-2}}{e_2^\alpha}\widetilde{v}_1 < 0$

$[e_2] \quad \frac{\alpha}{2}(-\alpha-1)\frac{e_1^\alpha}{e_2^{\alpha+2}}v_2 < 0$

Solve the two F.O.C , we get

$$\begin{aligned} e_1 &= \frac{\alpha}{2c}\lambda_1^{\alpha+1}(\widetilde{v}_1)^{1+\alpha}(v_2)^{-\alpha} \\ &= \frac{\alpha}{2c}\lambda_1^{\alpha+1}\left(\left(1-\frac{1+\alpha}{2}\right)\lambda_1^{\alpha+1} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)^{1+\alpha} \\ &\quad \left(1+\frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha}[u(w_1)-u(w_2)] \\ &< \overline{e}^s \end{aligned}$$

$$\begin{aligned} e_2 &= \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1)^{\alpha}(v_2)^{1-\alpha} \\ &= \frac{\alpha}{2c}\lambda_1^{\alpha}\left(\left(1-\frac{1+\alpha}{2}\right)\lambda_1^{\alpha+1} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)^{\alpha} \\ &\quad \left(1+\frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{1-\alpha}[u(w_1)-u(w_2)] \\ &< \overline{e}^s \end{aligned}$$

Check the conditions $\lambda_1 e_1^\alpha \leqslant e_2^\alpha$ and $e_2 \geqslant e_1$:

① $\lambda_1 e_1^\alpha \leqslant e_2^\alpha$

$$\lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad \text{is satisfied as long as } e_2 \geqslant e_1 \text{ holds.}$$

② $e_2 \geqslant e_1$

$$\frac{e_2}{e_1} = \frac{v_2}{\lambda_1\widetilde{v}_1} > 1$$

always holds.

(2) case 2: $\lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha}v_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^\alpha}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_2^{\alpha-1}}{e_1^\alpha}v_2 - c = 0$

Divide the two F.O.C , we get

$$\frac{e_1}{e_2} = \left(\frac{v_2}{\lambda_1\widetilde{v}_1}\right)^{\frac{1}{2\alpha-1}}$$

Since $v_2 > \lambda_1\widetilde{v}_1$, the condition $e_2 \leqslant e_1$ is satisfied if and only if $2\alpha - 1 \geqslant 0$. Now assume that $\alpha \geqslant \frac{1}{2}$ is satisfied.

$$\begin{aligned} e_1 &= \frac{\alpha}{2c}\lambda_1^{\frac{\alpha-1}{2\alpha-1}}(\widetilde{v}_1)^{\frac{\alpha-1}{2\alpha-1}}(v_2)^{\frac{\alpha}{2\alpha-1}} \\ &> \overline{e}^s \end{aligned}$$

$$\begin{aligned} e_2 &= \frac{\alpha}{2c}\lambda_1^{\frac{\alpha}{2\alpha-1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha-1}}(v_2)^{\frac{\alpha-1}{2\alpha-1}} \\ &< \overline{e}^s \end{aligned}$$

Check the condition $\lambda_1 e_1^\alpha \leqslant e_2^\alpha$:

$$\lambda_1\left(\frac{e_1}{e_2}\right)^\alpha = \lambda_1^{\frac{\alpha-1}{2\alpha-1}}(\widetilde{v}_1)^{-\frac{\alpha}{2\alpha-1}}(v_2)^{\frac{\alpha}{2\alpha-1}}$$

$$\begin{aligned} \lambda_1\left(\frac{e_1}{e_2}\right)^\alpha < 1 &\Longleftrightarrow \lambda_1^{\frac{\alpha-1}{2\alpha-1}}(\widetilde{v}_1)^{-\frac{\alpha}{2\alpha-1}}(v_2)^{\frac{\alpha}{2\alpha-1}} < 1 \\ &\Longleftrightarrow (v_2)^{\frac{\alpha}{2\alpha-1}} < \lambda_1^{\frac{1-\alpha}{2\alpha-1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha-1}} \end{aligned}$$

which is not satisfied under the assumption $\alpha \geqslant \frac{1}{2}$.

(3) case 3: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha}v_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_2^{\alpha-1}}{e_1^\alpha}v_2 - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha - 1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1 < 0$

$[e_2] \quad \frac{\alpha}{2}(\alpha-1)\frac{e_2^{\alpha-2}}{e_1^{\alpha}}v_2 < 0$

Divide the two F.O.C , we get

$$\frac{e_2}{e_1} = \lambda_1 \frac{v_2}{\widetilde{v}_1}$$

Check the condition $\lambda_1 e_1^{\alpha} \geqslant e_2^{\alpha}$:

$$\begin{aligned} \lambda_1 e_1^{\alpha} \geqslant e_2^{\alpha} &\iff \frac{\lambda_1 e_1^{\alpha}}{e_2^{\alpha}} \geqslant 1 \\ &\iff \lambda_1^{1-\alpha}(\widetilde{v}_1)^{\alpha}(v_2)^{-\alpha} \geqslant 1 \end{aligned}$$

which contradicts with $\lambda_1 < 1$ and $\widetilde{v}_1 < v_2$

Thus the unique equilibrium is

$$\begin{aligned} e_1^s = &\frac{\alpha}{2c}\lambda_1^{\alpha+1}\left(\frac{1-\alpha}{2}\lambda_1^{1+\alpha} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)^{1+\alpha} \\ &\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha}[u(w_1)-u(w_2)] \end{aligned}$$

$$\begin{aligned} e_2^s = &\frac{\alpha}{2c}\lambda_1^{\alpha}\left(\frac{1-\alpha}{2}\lambda_1^{1+\alpha} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)^{\alpha} \\ &\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{1-\alpha}[u(w_1)-u(w_2)] \end{aligned}$$

where $e_1^s < e_2^s < \overline{e}^s$.

$$\begin{aligned} p_{12}^s &= \frac{1}{2}\left(\frac{e_1^s}{e_2^s}\right)^{\alpha} \\ &= \frac{1}{2}\left(\frac{\lambda_1 \widetilde{v}_1}{v_2}\right)^{\alpha} \\ &= \frac{1}{2}\lambda_1^{\alpha}\left(\frac{1-\alpha}{2}\lambda_1^{1+\alpha} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)^{\alpha}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha} \end{aligned}$$

$$\begin{aligned} p_{21}^s &= 1 - p_{12}^s \\ &= 1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{1-\alpha}{2}\lambda_1^{1+\alpha} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)^{\alpha}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha} \end{aligned}$$

$$\begin{aligned} \widetilde{p}_{12}^s &= \frac{1}{2}\lambda_1\left(\frac{e_1^s}{e_2^s}\right)^{\alpha} \\ &= \frac{1}{2}\lambda_1^{\alpha+1}\left(\frac{1-\alpha}{2}\lambda_1^{1+\alpha} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)^{\alpha}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha} \end{aligned}$$

where $p_{21}^s > \frac{1}{2} > p_{12}^s > \widetilde{p}_{12}^s$.

3. Participation constraints

$$
\begin{aligned}
\widetilde{E}^s(U_{12}) &= \widetilde{p}^s_{12}\widetilde{v}_1 - ce^s_1 \\
&= \frac{1}{2}\lambda_1^{\alpha+1}\left(\widetilde{v}_1\right)^{\alpha}\left(v_2\right)^{-\alpha}\widetilde{v}_1 - c\frac{\alpha}{2c}\lambda_1^{\alpha+1}(\widetilde{v}_1)^{1+\alpha}(v_2)^{-\alpha} \\
&= \frac{1-\alpha}{2}\lambda_1^{\alpha+1}(\widetilde{v}_1)^{1+\alpha}(v_2)^{-\alpha} \\
&\geqslant 0
\end{aligned}
$$

$$
\begin{aligned}
E^s(U_{21}) &= p^s_{21}v_2 - ce^s_2 \\
&= \left(1 - \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1\right)^{\alpha}\left(v_2\right)^{-\alpha}\right)v_2 - c\frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1)^{\alpha}(v_2)^{1-\alpha} \\
&= \left(1 - \frac{1+\alpha}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1\right)^{\alpha}\left(v_2\right)^{-\alpha}\right)v_2 \\
&> 0
\end{aligned}
$$

Next, we characterize the equilibrium of the semifinal between players 3 and 4.

Proposition A10 *In a semifinal between two rational players of a two-stage elimination contest where player 1 is underconfident and players 2, 3, and 4 are rational, the equilibrium efforts and winning probabilities satisfy* $e^s_3 = e^s_4 > \overline{e}^s$ *and* $p^s_{34} = p^s_{43} = 1/2$.

Proposition A10 shows that since playing against an underconfident player in the final raises a rational player's continuation value, the rational players 3 and 4 exert higher efforts in the semifinal.

Proof of Proposition A10

1. Continuation values

   Rational player 3:

$$
\begin{aligned}
v_3 &= p^s_{12}E^f(U_{31}) + p^s_{21}E^f(U_{32}) \\
&= \left[p^s_{12}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{\alpha}\right) + p^s_{21}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)\right] \\
&\quad [u(w_1) - u(w_2)] \\
&> \overline{v}
\end{aligned}
$$

   Rational player 4:

$$
v_4 = v_3 > \overline{v}
$$

2. The equilibrium

$$
\begin{aligned}
e_3^s = e_4^s = {} & \frac{\alpha}{2c} v_3 \\
= {} & \frac{\alpha}{2c}\left[p_{12}^s\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^\alpha\right)+p_{21}^s\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)\right] \\
& [u(w_1)-u(w_2)] \\
= {} & \frac{\alpha}{2c}\Bigg[p_{12}^s\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^\alpha\right) \\
& +(1-p_{12}^s)\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)\Bigg] \\
& [u(w_1)-u(w_2)] \\
= {} & \frac{\alpha}{2c}\left[p_{12}^s\left(\frac{1+\alpha}{2}-\frac{1+\alpha}{2}\lambda_1^\alpha\right)+\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)\right][u(w_1)-u(w_2)] \\
= {} & \frac{\alpha}{2c}\Bigg[\frac{1}{2}\frac{1+\alpha}{2}\lambda_1^\alpha\left(\frac{\left(\left(1-\frac{1+\alpha}{2}\right)\lambda_1^{1+\alpha}+\frac{u(w_2)}{u(w_1)-u(w_2)}\right)}{1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}}\right)^\alpha(1-\lambda_1^\alpha) \\
& +\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)\Bigg][u(w_1)-u(w_2)] \\
> {} & \overline{e}^s
\end{aligned}
$$

$$p_{34}^s = p_{43}^s = \frac{1}{2}$$

3. Participation constraints

$$
\begin{aligned}
E^s(U_{34}) = p_{34}^s v_3 - c e_3 &= \frac{1}{2}v_3 - c\frac{\alpha}{2c}v_3 \\
&= \frac{1-\alpha}{2}v_3 \\
&\geqslant 0
\end{aligned}
$$