Overconfidence and Strategic Behavior in Elimination Contests: Implications for CEO Selection
Online Appendix

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1 Proofs for Elimination Contest with One Overconfident and Three Rational Players

Proof of Lemma 3

Player i's best response in a final with j, $R_i^f(e_j)$, is defined by

$$\begin{cases} \frac{\alpha}{2\lambda_i}\frac{e_j^\alpha}{e_i^{\alpha+1}}\Delta u = c & \text{if } \lambda_i e_i^\alpha \geqslant e_j^\alpha \\ \\ \frac{\alpha}{2}\lambda_i \frac{e_i^{\alpha-1}}{e_j^\alpha}\Delta u = c & \text{if } \lambda_i e_i^\alpha \leqslant e_j^\alpha \end{cases}$$

Hence, the slope of player i's best response in the final is

$$-\frac{\frac{\partial R_i^f(e_j)}{\partial e_j}}{\frac{\partial R_i^f(e_j)}{\partial e_i}} = -\frac{\frac{\partial^2 \widetilde{E}^f(U_{ij})}{\partial e_i \partial e_j}}{\frac{\partial^2 \widetilde{E}^f(U_{ij})}{\partial e_i^2}} = \begin{cases} -\frac{\frac{\alpha^2}{2\lambda_i}\frac{e_j^{\alpha-1}}{e_i^{\alpha+1}}\Delta u}{-(1+\alpha)\frac{\alpha}{2\lambda_i}\frac{e_j^\alpha}{e_i^{\alpha+2}}\Delta u} = \frac{\alpha}{1+\alpha}\frac{e_i}{e_j} & \text{if } \lambda_i e_i^\alpha > e_j^\alpha \\ \\ -\frac{-\frac{\alpha^2}{2}\lambda_i \frac{e_i^{\alpha-1}}{e_j^{\alpha+1}}\Delta u}{-(1-\alpha)\frac{\alpha}{2}\lambda_i \frac{e_i^{\alpha-2}}{e_j^\alpha}\Delta u} = -\frac{\alpha}{1-\alpha}\frac{e_i}{e_j} & \text{if } \lambda_i e_i^\alpha < e_j^\alpha \end{cases}$$

Therefore, the sign of the slope of player i's best response in the final is positive for $\lambda_i e_i^\alpha > e_j^\alpha$ and negative for $\lambda_i e_i^\alpha < e_j^\alpha$. This implies that $R_i^f(e_j)$ increases in e_j for $\lambda_i e_i^\alpha > e_j^\alpha$, reaches the maximum at $\lambda_i e_i^\alpha = e_j^\alpha$, and decreases in e_j for $\lambda_i e_i^\alpha < e_j^\alpha$.

Proof of Lemma 4

From player i's best response in the final (see Lemma 3) we have

$$\frac{\partial R_i^f(e_j)}{\partial \lambda_i} = \begin{cases} -\frac{\alpha}{2\lambda_i^2}\frac{e_j^\alpha}{e_i^{\alpha+1}}\Delta u & \text{if } \lambda_i e_i^\alpha \geqslant e_j^\alpha \\ \\ \frac{\alpha}{2}\frac{e_i^{\alpha-1}}{e_j^\alpha}\Delta u & \text{if } \lambda_i e_i^\alpha \leqslant e_j^\alpha \end{cases}$$

We see that $\partial R_i^f(e_j)/\partial\lambda_i \leqslant 0$ for $\lambda_i e_i^\alpha \geqslant e_j^\alpha$ and $\partial R_i^f(e_j)/\partial\lambda_i \geqslant 0$ for $\lambda_i e_i^\alpha \leqslant e_j^\alpha$. Substituting $e_j^\alpha = \lambda_i e_i^\alpha$ into player i's best response in the final and denoting the maximal effort that i is willing to invest in the final by e_i^{fmax} we obtain

$$\frac{\alpha}{2\lambda_i}\frac{\lambda_i (e_i^{fmax})^\alpha}{(e_i^{fmax})^{\alpha+1}}\Delta u = c$$

or

$$e_i^{fmax} = \frac{\alpha}{2c}\Delta u.$$

This implies that the value of e_i corresponding to the maximum value of player i's best response in the final, e_i^{fmax}, does not depend on λ_i.

Proof of Proposition 2

The final stage

$$p_{13}^f = \begin{cases} 1 - \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha} & \text{if } e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha} & \text{if } e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

$$p_{31}^f = \begin{cases} 1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha} & \text{if } e_3^\alpha \geqslant e_1^\alpha \\ \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha} & \text{if } e_3^\alpha \leqslant e_1^\alpha \end{cases}$$

Rational player 1 $\quad max \quad E^f(U_{13}) = p_{13}^f \Delta u + u(w_2) - ce_1$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\right)\Delta u + u(w_2) - ce_1 & \text{if} \quad e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\Delta u + u(w_2) - ce_1 & \text{if} \quad e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

Rational player 3 $\quad max \quad E^f(U_{31}) = p_{31}^f \Delta u + u(w_2) - ce_3$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)\Delta u + u(w_2) - ce_3 & \text{if } e_3^\alpha \geqslant e_1^\alpha \\ \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\Delta u + u(w_2) - ce_3 & \text{if } e_3^\alpha \leqslant e_1^\alpha \end{cases}$$

There are 2 cases.

$$\begin{cases} e_1^\alpha \geqslant e_3^\alpha \\ e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

1. equilibrium efforts

   (1) case 1: $e_1^\alpha \geqslant e_3^\alpha$

   Player 1 $\quad max \quad E^f(U_{13}) = \left(1 - \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\right)\Delta u + u(w_2) - ce_1$

   Player 3 $\quad max \quad E^f(U_{31}) = \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\Delta u + u(w_2) - ce_3$

   F.o.c

   $[e_1] \quad \frac{\alpha}{2}\frac{e_3^\alpha}{e_1^{\alpha+1}}\Delta u - c = 0$

   $[e_3] \quad \frac{\alpha}{2}\frac{e_3^{\alpha-1}}{e_1^\alpha}\Delta u - c = 0$

   S.o.c

   $[e_1] \quad \frac{\alpha}{2}(-\alpha - 1)\frac{e_3^\alpha}{e_1^{\alpha+2}}\Delta u < 0$

   $[e_3] \quad \frac{\alpha}{2}(\alpha - 1)\frac{e_3^{\alpha-2}}{e_1^\alpha}\Delta u < 0$

   Solve F.O.C , we get $e_1 = e_3 = \frac{\alpha}{2c}\Delta u$

(2) case 2: $e_1^\alpha \leqslant e_3^\alpha$

The same as the previous case.

Thus the unique equilibrium is $\overline{e}^f = e_1^f = e_3^f = \frac{\alpha}{2c}\Delta u$,

2. winning probabilities

The true winning probabilities are

$$\overline{p}^f = p_{13}^f = p_{31}^f = \frac{1}{2}$$

3. expected utilities of final

$$\overline{E}^f(U) = E^f(U_{13}) = E^f(U_{31}) = \frac{1}{2}[u(w_1) + u(w_2)] - c\frac{\alpha}{2c}\Delta u = \frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)$$

Since $0 < \alpha \leqslant 1$, we have $\overline{E}^f(U) \geqslant 0$. The participation constraints are satisfied.

The semifinals stage

1. Expected utilities of reaching the final

Using the expected utility of the final, we can get the expected utility of reaching the final.
Player 1's expected utility of reaching the final is given by

$$v_1 = p_{34}^s E^f(U_{13}) + p_{43}^s E^f(U_{14}) = \overline{E}^f(U) = \frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)$$

Since all 4 players are identical,

$$\overline{v} = v_1 = v_2 = v_3 = v_4 = \frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2).$$

2. Equilibrium efforts in the semifinal

Using the extension of the equilibrium result in the final, we can get that

$$\overline{e}^s = e_1^s = e_2^s = e_3^s = e_4^s = \frac{\alpha}{2c}\overline{v} = \frac{\alpha}{2c}\left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right]$$

3. True winning probabilities

$$\overline{p}^s = p_{12}^s = p_{21}^s = p_{34}^s = p_{43}^s = \frac{1}{2}$$

4. Expected utilities of semifinal

$$\overline{E}^s(U) = \frac{1}{2}\overline{v} - c\frac{\alpha}{2c}\overline{v} = \frac{1-\alpha}{2}\overline{v} = \frac{1-\alpha}{2}\left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right]$$

Since $0 < \alpha \leqslant 1$, we have $\overline{E}^s(U) \geqslant 0$. The participation constraints are satisfied.

5. the prize spread that satisfies $\overline{e}^s < \overline{e}^f$

$$\begin{aligned}
\overline{e}^s < \overline{e}^f &\iff \frac{\alpha}{2c}\left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right] < \frac{\alpha}{2c}\Delta u \\
&\iff (1-\alpha)u(w_1) + (1+\alpha)u(w_2) < 2\Delta u \\
&\iff (3+\alpha)u(w_2) < (1+\alpha)u(w_1) \\
&\iff \frac{u(w_1)}{u(w_2)} > \frac{3+\alpha}{1+\alpha}
\end{aligned}$$

Since $\alpha \in (0,1]$ this inequality is satisfied for all α when $u(w_1) > 3u(w_2)$.

Proof of Proposition 3

The perceived winning probabilities of the players are:

$$\widetilde{p}_{13}^{f} = \begin{cases} 1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha} & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha} & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

$$p_{31}^{f} = \begin{cases} 1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha} & \text{if} \quad e_3^\alpha \geqslant e_1^\alpha \\ \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha} & \text{if} \quad e_3^\alpha \leqslant e_1^\alpha \end{cases}$$

Overconfident player 1 $\quad max \quad \widetilde{E}^f(U_{13}) = \widetilde{p}_{13}^{f}\Delta u + u(w_2) - ce_1$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)\Delta u + u(w_2) - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}\Delta u + u(w_2) - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

Rational player 3 $\quad max \quad E^f(U_{31}) = p_{31}^{f}\Delta u + u(w_2) - ce_3$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)\Delta u + u(w_2) - ce_3 & \text{if} \quad e_3^\alpha \geqslant e_1^\alpha \\ \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\Delta u + u(w_2) - ce_3 & \text{if} \quad e_3^\alpha \leqslant e_1^\alpha \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1 \end{cases}$$

Since $\lambda_1 > 1$, the fourth case is impossible.

1. equilibrium efforts

   (1) case 1: $\lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1$

   Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)\Delta u + u(w_2) - ce_1$

Player 3 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)\Delta u + u(w_2) - ce_3$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_3^\alpha}{e_1^{\alpha+1}}\Delta u - c = 0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_3^{\alpha+1}}\Delta u - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_3^\alpha}{e_1^{\alpha+2}}\Delta u < 0$

$[e_3] \quad \frac{\alpha}{2}(-\alpha-1)\frac{e_1^\alpha}{e_3^{\alpha+2}}\Delta u < 0$

Solve F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\Delta u$$

$$e_3 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$ and $e_3 \geqslant e_1$:

$$\begin{aligned}
\lambda_1 e_1^\alpha \geqslant e_3^\alpha &\Longleftrightarrow \lambda_1\left(\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^\alpha \geqslant \left(\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)^\alpha \\
&\Longleftrightarrow \left(\frac{\alpha}{2c}\right)^\alpha \lambda_1^{-\frac{(\alpha+1)\alpha}{2\alpha+1}+1} \geqslant \left(\frac{\alpha}{2c}\right)^\alpha \lambda_1^{-\frac{\alpha^2}{2\alpha+1}} \\
&\Longleftrightarrow \lambda_1^{-\frac{(\alpha+1)\alpha}{2\alpha+1}+1+\frac{\alpha^2}{2\alpha+1}} \geqslant 1 \\
&\Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}} \geqslant 1 \\
&\Longleftrightarrow \frac{\alpha+1}{2\alpha+1} \geqslant 0
\end{aligned}$$

$$e_3 \geqslant e_1 \Longleftrightarrow \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \geqslant \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \Longleftrightarrow \lambda_1^{-\frac{\alpha}{2\alpha+1}} \geqslant \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \Longleftrightarrow \lambda_1^{\frac{1}{2\alpha+1}} \geqslant 1 \Longleftrightarrow \frac{1}{2\alpha+1} \geqslant 0$$

The conditions are always satisfied.

(2) case 2: $\lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)\Delta u + u(w_2) - ce_1$

Player 3 $\quad max \quad \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\Delta u + u(w_2) - ce_3$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_3^\alpha}{e_1^{\alpha+1}}\Delta u - c = 0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_3^{\alpha-1}}{e_1^\alpha}\Delta u - c = 0$

divide the two F.O.C , we get

$$\frac{e_3}{e_1} = \lambda_1 > 1$$

which contradicts the condition that $e_3 \leqslant e_1$

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}\Delta u + u(w_2) - ce_1$

Player 3 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)\Delta u + u(w_2) - ce_3$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_3^\alpha}\Delta u - c = 0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_3^{\alpha+1}}\Delta u - c = 0$

divide the two F.O.C , we get

$$\frac{e_3}{e_1} = \frac{1}{\lambda_1} < 1$$

which contradicts the condition that $e_3 \geqslant e_1$

Thus the unique equilibrium is

$$e_1^f = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\Delta u$$

$$e_3^f = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u$$

where $\lambda_1 e_1^\alpha > e_3^\alpha$ and $e_3 > e_1$.

We show that $e_1^f < \bar{e}^f$ and $e_3^f < \bar{e}^f$:

$$e_1^f < \bar{e}^f \iff \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\Delta u < \frac{\alpha}{2c}\Delta u \iff \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} < 1$$

$$e_3^f < \bar{e}^f \iff \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u < \frac{\alpha}{2c}\Delta u \iff \lambda_1^{-\frac{\alpha}{2\alpha+1}} < 1$$

2. equilibrium winning probabilities

The true winning probabilities are

$$p_{13}^f = \frac{1}{2}\left(\frac{e_1^f}{e_3^f}\right)^\alpha = \frac{1}{2}\left(\lambda_1^{-\frac{1}{2\alpha+1}}\right)^\alpha = \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}$$

$$p_{31}^f = 1 - p_{13}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}$$

The overconfident player 1's perceived winning probabilities are

$$\widetilde{p}^f_{13} = 1 - \frac{1}{2}\frac{(e^f_3)^\alpha}{\lambda_1 (e^f_1)^\alpha} = 1 - \frac{1}{2}\frac{\left(\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u\right)^\alpha}{\lambda_1\left(\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\Delta u\right)^\alpha} = 1 - \frac{1}{2}\frac{\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)^\alpha}{\lambda_1\left(\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^\alpha} = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}$$

We show that $\widetilde{p}^f_{13} > p^f_{31} > \frac{1}{2} > p^f_{13}$:

$$\widetilde{p}^f_{13} > p^f_{31} \iff 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} > 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \iff \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} > \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \iff \lambda_1^{\frac{1}{2\alpha+1}} > 1$$

$$p^f_{31} > \frac{1}{2} \iff 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} > \frac{1}{2} \iff \lambda_1^{-\frac{\alpha}{2\alpha+1}} < 1$$

$$p^f_{13} < \frac{1}{2} \iff 1 - p^f_{31} < \frac{1}{2} \iff p^f_{31} > \frac{1}{2}$$

3. expected utilities of final

$$\begin{aligned}
\widetilde{E}^f(U_{13}) &= \widetilde{p}^f_{13}u(w_1) + (1-\widetilde{p}^f_{13})u(w_2) - ce^f_1 \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)u(w_1) + \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}u(w_2) - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\Delta u \\
&= u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\Delta u
\end{aligned}$$

$$\begin{aligned}
E^f(U_{31}) &= p^f_{31}u(w_1) + (1-p^f_{31})u(w_2) - ce^f_3 \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)u(w_1) + \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}u(w_2) - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u \\
&= u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u
\end{aligned}$$

We show that $\widetilde{E}^f(U_{13}) > E^f(U_{31}) > \overline{E}^f(U)$:

$$\begin{aligned}
&\widetilde{E}^f(U_{13}) > E^f(U_{31}) \\
&\iff u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\Delta u > u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u \\
&\iff -\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\Delta u > -\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u \\
&\iff \lambda_1^{-\frac{\alpha}{2\alpha+1}} > \lambda_1^{-\frac{\alpha+1}{2\alpha+1}}
\end{aligned}$$

$$\begin{aligned}
&E^f(U_{31}) > \overline{E}^f(U) \\
&\iff u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u > u(w_1) - \frac{1+\alpha}{2}\Delta u \\
&\iff -\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u > -\frac{1+\alpha}{2}\Delta u \\
&\iff 1 > \lambda_1^{-\frac{\alpha}{2\alpha+1}}
\end{aligned}$$

Since $\overline{E}^f(U) \geqslant 0$, the participation constraints of both players are satisfied.

Proof of Lemma 5

The best response of player i in the semifinal with h, $R_i^s(e_h)$, is defined by

$$\begin{cases} \frac{\alpha}{2\lambda_i}\frac{e_h^\alpha}{e_i^{\alpha+1}}\widetilde{v}_i = c & \text{if } \lambda_i e_i^\alpha \geqslant e_h^\alpha \\ \\ \frac{\alpha}{2}\lambda_i \frac{e_i^{\alpha-1}}{e_h^\alpha}\widetilde{v}_i = c & \text{if } \lambda_i e_i^\alpha \leqslant e_h^\alpha \end{cases}$$

Hence,

$$\frac{\partial R_i^s(e_h)}{\partial \lambda_i} = \begin{cases} -\frac{\alpha}{2\lambda_i^2}\frac{e_h^\alpha}{e_i^{\alpha+1}}\widetilde{v}_i + \frac{\alpha}{2\lambda_i}\frac{e_h^\alpha}{e_i^{\alpha+1}}\frac{\partial \widetilde{v}_i}{\partial \lambda_i} = \frac{\alpha}{2\lambda_i^2}\frac{e_h^\alpha}{e_i^{\alpha+1}}\left(-\widetilde{v}_i + \lambda_i \frac{\partial \widetilde{v}_i}{\partial \lambda_i}\right) & \text{if } \lambda_i e_i^\alpha \geqslant e_h^\alpha \\ \\ \frac{\alpha}{2}\frac{e_i^{\alpha-1}}{e_h^\alpha}\widetilde{v}_i + \frac{\alpha}{2}\lambda_i \frac{e_i^{\alpha-1}}{e_h^\alpha}\frac{\partial \widetilde{v}_i}{\partial \lambda_i} = \frac{\alpha}{2}\frac{e_i^{\alpha-1}}{e_h^\alpha}\left(\widetilde{v}_i + \lambda_i \frac{\partial \widetilde{v}_i}{\partial \lambda_i}\right) & \text{if } \lambda_i e_i^\alpha \leqslant e_h^\alpha \end{cases} \tag{1}$$

Since $\frac{\partial \widetilde{v}_i}{\partial \lambda_i} > 0$ it follows from (1) that $\frac{\partial R_i^s}{\partial \lambda_i} > 0$ for $\lambda_i e_i^\alpha \leqslant e_j^\alpha$. Since $\frac{\partial \widetilde{v}_i}{\partial \lambda_i} > 0$ it also follows from (1) that $\frac{\partial \widetilde{v}_i}{\partial \lambda_i}\frac{\lambda_i}{\widetilde{v}_i} < 1 (> 1)$, then $\frac{\partial R_i^s}{\partial \lambda_i} < 0 (> 0)$ for $\lambda_i e_i^\alpha \geqslant e_j^\alpha$. Substituting $e_j^\alpha = \lambda_i e_i^\alpha$ into player i's best response in the semifinal and denoting the maximal effort that i is willing to invest in the semifinal by e_i^{smax} we obtain

$$\frac{\alpha}{2\lambda_i}\frac{\lambda_i (e_i^{smax})^\alpha}{(e_i^{smax})^{\alpha+1}}\widetilde{v}_i = c$$

or

$$e_i^{smax} = \frac{\alpha}{2c}\widetilde{v}_i.$$

Since $\widetilde{v}_i$ increases with λ_i, it follows from the last equality that e_i^{smax} increases with λ_i.

Proof of Proposition 4

1. Perceived expected utilities of reaching the final

   Using Proposition 3, we can get the perceived expected utilities of reaching the final of each player.

   Overconfident player 1:

   $$\widetilde{v}_1 = p_{34}^s \widetilde{E}^f(U_{13}) + p_{43}^s \widetilde{E}^f(U_{14})$$

   Since player 3 and 4 are identical, $\widetilde{E}^f(U_{13}) = \widetilde{E}^f(U_{14})$

   $$\widetilde{v}_1 = u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\Delta u$$

   Since $\widetilde{E}^f(U_{13}) > \overline{E}^f(U)$, we can get $\widetilde{v}_1 > \overline{v}$.

   Rational player 2:

   $$v_2 = p_{34}^s E^f(U_{23}) + p_{43}^s E^f(U_{24})$$

   Since players 3 and 4 are identical, $E^f(U_{23}) = E^f(U_{24})$

   $$v_2 = \frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2) = \overline{v}$$

2. The equilibrium efforts and winning probabilities

Player 1 $\quad max \quad \widetilde{E}^s(U_{12}) = \widetilde{p}^s_{12}\widetilde{v}_1 - ce_1$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_2^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_2^\alpha \end{cases}$$

Player 2 $\quad max \quad E^s(U_{21}) = p^s_{21}v_2 - ce_2$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right)v_2 - ce_2 & \text{if}\, e_2 \geqslant e_1 \\ \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha}v_2 - ce_2 & \text{if}\, e_2 \leqslant e_1 \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \end{cases}$$

Since $\lambda_1 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1$, which corresponds to (i).

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha}v_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_2^{\alpha-1}}{e_1^\alpha}v_2 - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1 < 0$

$[e_2] \quad \frac{\alpha}{2}(\alpha-1)\frac{e_2^{\alpha-2}}{e_1^\alpha}v_2 < 0$

Solve the two F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(v_2)^\alpha$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(v_2)^{\alpha+1}$$

$$\frac{e_2}{e_1} = \lambda_1 \frac{v_2}{\widetilde{v}_1}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \leqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

As long as $e_1 \geqslant e_2$ is satisfied, $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ is satisfied.

② $e_2 \leqslant e_1$

$$\begin{aligned}
e_1 \geqslant e_2 &\Longleftrightarrow \frac{\widetilde{v}_1}{\lambda_1 v_2} \geqslant 1 \\
&\Longleftrightarrow \frac{\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)\Delta u}{\lambda_1\left[\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)\Delta u\right]} \geqslant 1 \\
&\Longleftrightarrow \frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \geqslant \lambda_1\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)
\end{aligned}$$

Let

$$f(\lambda_1) = \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) - \lambda_1\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)$$

we can easily get that $f(\lambda_1 = 1) = 0$ and $f(\lambda_1 \to \infty) < 0$.

$$f'(\lambda_1) = \frac{(1+\alpha)^2}{2(2\alpha+1)}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1} - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)$$

$$\begin{aligned}
f'(\lambda_1) \lesseqqgtr 0 &\Longleftrightarrow \frac{(1+\alpha)^2}{2(2\alpha+1)}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1} \lesseqqgtr \frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \\
&\Longleftrightarrow \frac{(1+\alpha)^2}{2\alpha+1}\frac{u(w_1)-u(w_2)}{(1-\alpha)u(w_1)+(1+\alpha)u(w_2)} \lesseqqgtr \lambda_1^{\frac{\alpha+1}{2\alpha+1}+1} \\
&\Longleftrightarrow \left[\frac{(1+\alpha)^2}{2\alpha+1}\frac{u(w_1)-u(w_2)}{(1-\alpha)u(w_1)+(1+\alpha)u(w_2)}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}} \lesseqqgtr \lambda_1
\end{aligned}$$

Let $g(\alpha) = \left[\frac{(1+\alpha)^2}{2\alpha+1}\frac{u(w_1)-u(w_2)}{(1-\alpha)u(w_1)+(1+\alpha)u(w_2)}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}}$

a) $g(\alpha) \leqslant 1$

if $g(\alpha) \leqslant 1$, then $f'(\lambda_1) < 0$ always holds. Which means $f(\lambda_1) < 0$ always holds, thus $\frac{e_1}{e_2} < 1$ always holds.

$$
\begin{aligned}
g(\alpha) \leqslant 1 &\iff \frac{(1+\alpha)^2}{2\alpha+1} \frac{u(w_1)-u(w_2)}{(1-\alpha)u(w_1)+(1+\alpha)u(w_2)} \leqslant 1 \\
&\iff \frac{(1+\alpha)^2}{2\alpha+1} \leqslant \frac{(1-\alpha)u(w_1)+(1+\alpha)u(w_2)}{u(w_1)-u(w_2)} \\
&\iff (1+\alpha)^2 \Delta u \leqslant (2\alpha+1)[(1-\alpha)u(w_1)+(1+\alpha)u(w_2)] \\
&\iff (\alpha+3\alpha^2)u(w_1) \leqslant (2+5\alpha+3\alpha^2)u(w_2) \\
&\iff \frac{u(w_1)}{u(w_2)} \leqslant \frac{2+5\alpha+3\alpha^2}{\alpha(1+3\alpha)} \\
&\iff \frac{u(w_1)}{u(w_2)} - 1 \leqslant \frac{2+5\alpha+3\alpha^2}{\alpha(1+3\alpha)} - 1 \\
&\iff \frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}
\end{aligned}
$$

When $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$, the condition $e_1 \geqslant e_2$ is never satisfied given that $\lambda_1 > 1$.

b) $g(\alpha) > 1$

if $g(\alpha) > 1$, then

$$
f'(\lambda_1) \begin{cases} > 0 & \text{when } \lambda_1 < \left[\frac{(1+\alpha)^2}{2\alpha+1} \frac{u(w_1)-u(w_2)}{(1-\alpha)u(w_1)+(1+\alpha)u(w_2)}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}} \\ < 0 & \text{when } \lambda_1 > \left[\frac{(1+\alpha)^2}{2\alpha+1} \frac{u(w_1)-u(w_2)}{(1-\alpha)u(w_1)+(1+\alpha)u(w_2)}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}} \end{cases}
$$

We now show that if $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$, then there exists a unique threshold $\hat{\lambda} > 1$ where $f(\lambda_1) = 0$, that is,

$$
\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} = \lambda_1 \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \right)
$$

which is equivalent to

$$
u(w_1) - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}} \Delta u = \hat{\lambda} \left[\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \right) \Delta u \right].
$$

To see this is the case, we rearrange the equality as

$$
u(w_1) - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}} \Delta u = \hat{\lambda} \left(u(w_1) - \frac{1+\alpha}{2} \Delta u \right),
$$

or

$$
\frac{1+\alpha}{2} \left(\hat{\lambda} - \hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}} \right) \Delta u = (\hat{\lambda} - 1) u(w_1),
$$

or

$$
\frac{1+\alpha}{2} \frac{u(w_1)-u(w_2)}{u(w_1)} = \frac{\hat{\lambda} - 1}{\hat{\lambda} - \hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}}. \tag{2}
$$

Since $\alpha \in (0,1]$ and $u(w_1) > u(w_2)$, the left-hand side of (3) takes a value in the interval $(0,1)$. The right-hand side of (3) is increasing in $\hat{\lambda}$

for $\lambda > 1$, its limit when $\hat{\lambda} \to 1$ is $\frac{2\alpha+1}{3\alpha+2}$, and its limit when $\hat{\lambda} \to \infty$ is 1. Hence, the threshold $\hat{\lambda}$ exists and is unique provided that

$$\frac{1+\alpha}{2}\frac{u(w_1)-u(w_2)}{u(w_1)} > \frac{2\alpha+1}{3\alpha+2}.$$

It is easy to show that this inequality is equivalent to

$$\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}.$$

Therefore, if $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$, then there exists a unique value for $\hat{\lambda}$, greater than 1, that satisfies (3). This, in turn, implies:

$$f(\lambda_1)\begin{cases} > 0 & \text{when } \lambda_1 < \hat{\lambda} \\ = 0 & \text{when } \lambda_1 = \hat{\lambda} \\ < 0 & \text{when } \lambda_1 > \hat{\lambda} \end{cases}$$

$$e_1 - e_2 \begin{cases} > 0 & \text{when } \lambda_1 < \hat{\lambda} \\ = 0 & \text{when } \lambda_1 = \hat{\lambda} \\ < 0 & \text{when } \lambda_1 > \hat{\lambda} \end{cases}$$

The condition $e_1 \geqslant e_2$ is only satisfied when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 \leqslant \hat{\lambda}$. And $e_1 > e_2$ is only satisfied when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$.

Therefore the solution

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(v_2)^{\alpha}$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(v_2)^{\alpha+1}$$

only applies when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$.

(2) case 2: $\lambda_1 e_1^{\alpha} \geqslant e_2^{\alpha}$ *and* $e_2 \geqslant e_1$, which corresponds to (ii).

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^{\alpha}}{\lambda_1 e_1^{\alpha}}\right)\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^{\alpha}}{e_2^{\alpha}}\right)v_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^{\alpha}}{e_1^{\alpha+1}}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^{\alpha}}{e_2^{\alpha+1}}v_2 - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^{\alpha}}{e_1^{\alpha+2}}\widetilde{v}_1 < 0$

$[e_2]\quad \frac{\alpha}{2}(-\alpha-1)\frac{e_1^\alpha}{e_2^{\alpha+2}}v_2 < 0$

Solve F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha+1}{2\alpha+1}}(v_2)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(v_2)^{\frac{\alpha+1}{2\alpha+1}}$$

$$\frac{e_2}{e_1} = \lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1)^{-\frac{1}{2\alpha+1}}(v_2)^{\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \geqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

$$\lambda_1 e_1^\alpha \geqslant e_2^\alpha \Longleftrightarrow \frac{\lambda_1 e_1^\alpha}{e_2^\alpha} \geqslant 1 \Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(v_2)^{-\frac{\alpha}{2\alpha+1}} \geqslant 1$$

Since $\lambda_1 > 1$ and $\widetilde{v}_1 > v_2$, the inequality is always satisfied. Therefore $\lambda_1 e_1^\alpha > e_2^\alpha$ always holds when $\lambda_1 > 1$.

② $e_2 \geqslant e_1$

$$\begin{aligned} e_1 \geqslant e_2 &\Longleftrightarrow \frac{e_1}{e_2} \geqslant 1 \\ &\Longleftrightarrow \lambda_1^{-\frac{1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{1}{2\alpha+1}}(v_2)^{-\frac{1}{2\alpha+1}} \geqslant 1 \\ &\Longleftrightarrow \left(\frac{\widetilde{v}_1}{\lambda_1 v_2}\right)^{\frac{1}{2\alpha+1}} \geqslant 1 \Longleftrightarrow \frac{\widetilde{v}_1}{\lambda_1 v_2} \geqslant 1 \end{aligned}$$

We have already seen in case (1) that $e_2 \geqslant e_1$ is satisfied when either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$.

Therefore the solution

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha+1}{2\alpha+1}}(v_2)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(v_2)^{\frac{\alpha+1}{2\alpha+1}}$$

only applies when either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad [1-\frac{1}{2}(\frac{e_1}{e_2})^\alpha]v_2 - ce_2$

F.o.c

$[e_1]\quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^\alpha}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^{\alpha}}{e_2^{\alpha+1}}v_2 - c = 0$

divide the two F.O.C , we get

$$\frac{e_2}{e_1} = \frac{v_2}{\lambda_1\widetilde{v}_1} < 1$$

which contradicts the condition that $e_2 \geqslant e_1$

Therefore, the equilibrium in this semifinal:

(1) When $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$, which corresponds to (i)

$$e_1^s = \frac{\alpha}{2c}\lambda_1^{\alpha-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\Delta u$$

$$e_2^s = \frac{\alpha}{2c}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{1+\alpha}\Delta u$$

where $e_1^s > e_2^s$.

$$\begin{aligned} p_{21}^s &= \frac{1}{2}\left(\frac{e_2^s}{e_1^s}\right)^{\alpha} \\ &= \frac{1}{2}\left(\frac{\lambda_1 v_2}{\widetilde{v}_1}\right)^{\alpha} \\ &= \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} \end{aligned}$$

$$p_{12}^s = 1 - p_{21}^s = 1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}$$

$$\begin{aligned} \widetilde{p}_{12}^s &= 1 - \frac{1}{2}\frac{(e_2^s)^{\alpha}}{\lambda_1(e_1^s)^{\alpha}} \\ &= 1 - \frac{1}{2}\lambda_1^{-1}\left[\lambda_1\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)\right]^{\alpha} \\ &= 1 - \frac{1}{2}\lambda_1^{\alpha-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} \end{aligned}$$

(2) When either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$, which corresponds to (ii).

$$e_1^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}\Delta u$$

$$e_2^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha+1}{2\alpha+1}}\Delta u$$

where $\lambda_1 (e_1^s)^\alpha > (e_2^s)^\alpha$ and $e_1^s \leqslant e_2^s$.

$$\begin{aligned}
p_{12}^s &= \frac{1}{2}\left(\frac{e_1^s}{e_2^s}\right)^\alpha \\
&= \frac{1}{2}\left[\lambda_1^{-\frac{1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\frac{1}{2\alpha+1}}\right]^\alpha \\
&= \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\frac{\alpha}{2\alpha+1}}
\end{aligned}$$

$$\begin{aligned}
p_{21}^s &= 1 - p_{12}^s \\
&= 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\frac{\alpha}{2\alpha+1}}
\end{aligned}$$

$$\begin{aligned}
\widetilde{p}_{12}^s &= 1 - \frac{1}{2}\frac{(e_2^s)^\alpha}{\lambda_1 (e_1^s)^\alpha} \\
&= 1 - \frac{1}{2}\lambda_1^{-1}\lambda_1^{\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} \\
&= 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}
\end{aligned}$$

3. equilibrium efforts compared to benchmark

(i) when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$

We show that $e_1^s > \overline{e}^s > e_2^s$:

$$e_1^s > \overline{e}^s \iff \frac{e_1^s}{\overline{e}^s} > 1 \iff \frac{\frac{\alpha}{2c}\lambda_1^{\alpha-1}\widetilde{v}_1^{\,1-\alpha}v_2^\alpha}{\frac{\alpha}{2c}\overline{v}} > 1 \iff \left(\frac{\widetilde{v}_1}{\lambda_1 v_2}\right)^{1-\alpha} > 1$$

$$\frac{e_2^s}{\overline{e}^s} \iff \frac{e_2^s}{\overline{e}^s} > 1 \iff \frac{\frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(v_2)^{\alpha+1}}{\frac{\alpha}{2c}\overline{v}} > 1 \iff \left(\frac{\lambda_1 v_2}{\widetilde{v}_1}\right)^{\alpha} > 1$$

Since $\frac{\widetilde{v}_1}{\lambda_1 v_2} > 1$, we can get $e_1^s > \overline{e}^s$ and $e_2^s < \overline{e}^s$.

(ii) when either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

We show that $e_1^s \leqslant e_2^s \leqslant \overline{e}^s$:

Since we already showed that $e_1^s \leqslant e_2^s$ is satisfied under this condition, we only have to show $e_2^s \leqslant \overline{e}^s$.

$$\begin{aligned}
e_2^s \leqslant \overline{e}^s &\iff \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\widetilde{v}_1^{\frac{\alpha}{2\alpha+1}}v_2^{\frac{\alpha+1}{2\alpha+1}} \leqslant \frac{\alpha}{2c}\overline{v} \\
&\iff \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\widetilde{v}_1^{\frac{\alpha}{2\alpha+1}}v_2^{\frac{\alpha+1}{2\alpha+1}} \leqslant \frac{\alpha}{2c}v_2 \\
&\iff \widetilde{v}_1^{\frac{\alpha}{2\alpha+1}} \leqslant \lambda_1^{\frac{\alpha}{2\alpha+1}}v_2^{\frac{\alpha}{2\alpha+1}} \\
&\iff \widetilde{v}_1 \leqslant \lambda_1 v_2
\end{aligned}$$

which always holds, thus $e_1^s \leqslant e_2^s \leqslant \overline{e}^s$ always holds.

4. perceived and true winning probabilities compared to benchmark

   (i) when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$

   We show that $p^s_{21} < \frac{1}{2}$ and $\widetilde{p}^s_{12} > p^s_{12} > \frac{1}{2}$:

$$p^s_{21} < \frac{1}{2} \Longleftrightarrow \frac{1}{2}\left(\frac{e^s_2}{e^s_1}\right)^\alpha < \frac{1}{2} \Longleftrightarrow e^s_2 < e^s_1$$

$$p^s_{12} = 1 - p^s_{21} > \frac{1}{2}$$

$$\widetilde{p}^s_{12} > p^s_{12} \Longleftrightarrow 1 - \frac{1}{2}\left(\frac{e^s_2}{\lambda_1 e^s_1}\right)^\alpha > 1 - \frac{1}{2}\left(\frac{e^s_2}{e^s_1}\right)^\alpha \Longleftrightarrow \lambda_1 > 1$$

   (ii) when either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

   We show that $p^s_{12} \leqslant \frac{1}{2}$, $p^s_{21} \geqslant \frac{1}{2}$ and $\widetilde{p}^s_{12} > \frac{1}{2}$:

$$p^s_{12} \leqslant \frac{1}{2} \Longleftrightarrow \frac{1}{2}\left(\frac{e^s_1}{e^s_2}\right)^\alpha \leqslant \frac{1}{2} \Longleftrightarrow e^s_1 \leqslant e^s_2$$

$$p^s_{21} = 1 - p^s_{12} \geqslant \frac{1}{2}$$

$$\begin{aligned}
\widetilde{p}^s_{12} > \frac{1}{2} &\Longleftrightarrow 1 - \frac{1}{2}\left(\frac{e^s_2}{\lambda_1 e^s_1}\right)^\alpha > \frac{1}{2} \\
&\Longleftrightarrow \frac{e^s_2}{\lambda_1 e^s_1} < 1 \\
&\Longleftrightarrow \frac{\lambda_1^{-\frac{\alpha}{2\alpha+1}} (\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}} (v_2)^{\frac{\alpha+1}{2\alpha+1}}}{\lambda_1^{\frac{\alpha}{2\alpha+1}} (\widetilde{v}_1)^{\frac{\alpha+1}{2\alpha+1}} (v_2)^{\frac{\alpha}{2\alpha+1}}} < 1 \\
&\Longleftrightarrow \lambda_1^{-\frac{2\alpha}{2\alpha+1}} (\widetilde{v}_1)^{-\frac{1}{2\alpha+1}} (v_2)^{\frac{1}{2\alpha+1}} < 1 \\
&\Longleftrightarrow \lambda_1^{-\frac{2\alpha}{2\alpha+1}} \left(\frac{v_2}{\widetilde{v}_1}\right)^{\frac{1}{2\alpha+1}} < 1
\end{aligned}$$

5. Participation constraints

(1) When $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$

$$\begin{aligned}
\widetilde{E}^s(U_{12}) &= \widetilde{p}_{12}^s \widetilde{v}_1 - ce_1^s \\
&> p_{12}^s \widetilde{v}_1 - ce_1^s \\
&= \left(1 - \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^\alpha\right) \widetilde{v}_1 - c\frac{\alpha}{2c}\lambda_1^{\alpha-1} \left(\widetilde{v}_1\right)^{1-\alpha} \left(v_2\right)^\alpha \\
&= \widetilde{v}_1 - \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{1-\alpha} \left(v_2\right)^\alpha - \frac{\alpha}{2}\lambda_1^{\alpha-1} \left(\widetilde{v}_1\right)^{1-\alpha} \left(v_2\right)^\alpha \\
&> \widetilde{v}_1 - \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{1-\alpha} \left(v_2\right)^\alpha - \frac{1}{2}\lambda_1^{\alpha-1} \left(\widetilde{v}_1\right)^{1-\alpha} \left(v_2\right)^\alpha \\
&> \widetilde{v}_1 - \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{1-\alpha} \left(v_2\right)^\alpha - \frac{1}{2}\lambda_1^{\alpha} \left(\widetilde{v}_1\right)^{1-\alpha} \left(v_2\right)^\alpha \\
&= \widetilde{v}_1 - \lambda_1^\alpha \left(\widetilde{v}_1\right)^{1-\alpha} \left(v_2\right)^\alpha \\
&= \left(\widetilde{v}_1\right)^{1-\alpha} \left[\left(\widetilde{v}_1\right)^\alpha - \lambda_1^\alpha \left(v_2\right)^\alpha \right] \\
&> 0
\end{aligned}$$

$$\begin{aligned}
E^s(U_{21}) &= p_{21}^s v_2 - ce_2^s \\
&= \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^\alpha v_2 - c\frac{\alpha}{2c}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^{\alpha+1} \\
&= \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^{1+\alpha} - \frac{\alpha}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^{\alpha+1} \\
&= \frac{1-\alpha}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^{1+\alpha} \\
&\geqslant 0
\end{aligned}$$

(2) When either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

$$\widetilde{E}^s(U_{12}) = \widetilde{p}_{12}^s \widetilde{v}_1 - ce_1^s$$

Since $\widetilde{p}_{12}^s > \frac{1}{2}$, $\widetilde{v}_1 > \overline{v}$ and $e_1^s < \overline{e}^s$, we can get that $\widetilde{E}^s(U_{12}) > \overline{E}^s(U) \geqslant 0$.

$$\begin{aligned}
E^s(U_{21}) &= p_{21}^s v_2 - ce_2^s \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{-\frac{\alpha}{2\alpha+1}}\right) v_2 - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= v_2 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= v_2 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} \left(\left(v_2\right)^{\frac{\alpha}{2\alpha+1}} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \right) \\
&\geqslant 0
\end{aligned}$$

Proof of Proposition 5

1. Expected utilities of reaching the final:

   Rational player 3:

$$
\begin{aligned}
v_3 &= p_{12}^s E^f(U_{31}) + p_{21}^s E^f(U_{32}) \\
&= p_{12}^s \left[u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u\right] + p_{21}^s \left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right]
\end{aligned}
$$

   Rational player 4:

$$
\begin{aligned}
v_4 &= p_{12}^s E^f(U_{41}) + p_{21}^s E^f(U_{42}) \\
&= p_{12}^s \left[u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u\right] + p_{21}^s \left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right]
\end{aligned}
$$

   Since $E^f(U_{31}) = E^f(U_{41}) > \overline{E}^f(U) = E^f(U_{32}) = E^f(U_{42})$, we have $v_3 = v_4 > \overline{v}$.

2. The equilibrium

   (1) When $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$

$$
e_3^s = e_4^s = \frac{\alpha}{2c}v_3 = \frac{\alpha}{2c}\left[p_{12}^s \left[u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u\right] + p_{21}^s \left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right]\right]
$$

   where
   $p_{12}^s = 1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}.$

$$
p_{34}^s = p_{43}^s = \frac{1}{2}
$$

   (2) When either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

$$
e_3^s = e_4^s = \frac{\alpha}{2c}v_3 = \frac{\alpha}{2c}\left[p_{12}^s \left[u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u\right] + p_{21}^s \left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right]\right]
$$

   where
   $p_{12}^s = \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}\left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right]^{-\frac{\alpha}{2\alpha+1}}.$

$$
p_{34}^s = p_{43}^s = \frac{1}{2}
$$

   We show that $e_3^s = e_4^s > \overline{e}^s$ holds in both (1) and (2):

$$
e_3^s = e_4^s > \overline{e}^s \Longleftrightarrow \frac{\alpha}{2c}v_3 > \frac{\alpha}{2c}\overline{v} \Longleftrightarrow v_3 > \overline{v}
$$

3. Participation constraints

$$
E^s(U_{34}) = p_{34}^s v_3 - c e_3^s = \frac{1}{2}v_3 - c\frac{\alpha}{2c}v_3 = \frac{1-\alpha}{2}v_3 \geqslant 0
$$

$$
E^s(U_{43}) = E^s(U_{34}) \geqslant 0
$$

Proof of Proposition 6

1. When $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$

   (1) P_1

   $$\begin{aligned}P_1 &= p_{13}^f p_{12}^s \\ &= \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left[1-\frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\alpha}\right]\end{aligned}$$

   Let
   $$f(\lambda_1) = P_1 - \frac{1}{4}.$$

   We can get
   $$f(\lambda_1 = 1) = \frac{1}{2}\times\frac{1}{2}-\frac{1}{4} = 0$$
   $$f(\lambda_1 = \hat{\lambda}) = \frac{1}{2}\hat{\lambda}^{-\frac{\alpha}{2\alpha+1}}\times\frac{1}{2}-\frac{1}{4} < 0$$

   $f(\lambda_1)$ can also be written as the following:

   $$\begin{aligned}f(\lambda_1) &= \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \\ &\quad -\frac{1}{4}\lambda_1^{\alpha-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}-\frac{1}{4}\end{aligned}$$

   Taking derivative of $f(\lambda_1)$ we obtain

   $$\begin{aligned}f'(\lambda_1) = &-\frac{1}{2}\frac{\alpha}{2\alpha+1}\lambda_1^{-\frac{\alpha}{2\alpha+1}-1}-\frac{1}{4}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\alpha} \\ &\left[\left(\alpha-\frac{\alpha}{2\alpha+1}\right)\lambda_1^{\alpha-\frac{\alpha}{2\alpha+1}-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\right. \\ &+\lambda_1^{\alpha-\frac{\alpha}{2\alpha+1}}(-\alpha)\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha-1} \\ &\left.\left(-\frac{\alpha+1}{2}\right)\left(-\frac{\alpha+1}{2\alpha+1}\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\right]\end{aligned}$$

$$\begin{aligned}
f'(\lambda_1 = 1) &= -\frac{1}{2}\frac{\alpha}{2\alpha+1} - \frac{1}{4}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\left[\left(\alpha - \frac{\alpha}{2\alpha+1}\right)\left(\frac{u(w_1)}{u(w_1)-u(w_2)}\right.\right. \\
&\quad \left.\left. - \frac{1+\alpha}{2}\right)^{-\alpha} - \alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha-1}\frac{\alpha+1}{2}\frac{\alpha+1}{2\alpha+1}\right] \\
&= -\frac{1}{2}\frac{\alpha}{2\alpha+1} - \frac{1}{4}\left[\alpha - \frac{\alpha}{2\alpha+1} - \alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}\frac{\alpha+1}{2\alpha+1}\right] \\
&= -\frac{1}{2}\frac{\alpha}{2\alpha+1} - \frac{1}{4}\left[\alpha\left(\frac{2\alpha}{2\alpha+1} - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}\frac{\alpha+1}{2\alpha+1}\right)\right] \\
&= -\frac{1}{2}\frac{\alpha}{2\alpha+1} - \frac{1}{4}\left[\frac{\alpha}{2\alpha+1}\left(2\alpha - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}(1+\alpha)\right)\right] \\
&= -\frac{1}{2}\frac{\alpha}{2\alpha+1}\left[1 + \frac{1}{2}\left(2\alpha - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}(1+\alpha)\right)\right]
\end{aligned}$$

$f'(\lambda_1 = 1)$ and $1+\frac{1}{2}\left(2\alpha - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}(1+\alpha)\right)$ has the opposite sign.

When $1+\frac{1}{2}\left(2\alpha - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}(1+\alpha)\right) < 0$, $f'(\lambda_1 = 1) > 0$.

And since

$$\begin{aligned}
& 1+\frac{1}{2}\left(2\alpha - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}(1+\alpha)\right) < 0 \\
\iff & 2\alpha - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}(1+\alpha) < -2 \\
\iff & 2\alpha + 2 < \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}(1+\alpha) \\
\iff & 2 < \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2} \\
\iff & 4\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) < 1+\alpha \\
\iff & 4 + \frac{4u(w_2)}{u(w_1)-u(w_2)} < 3(1+\alpha) \\
\iff & \frac{4u(w_2)}{u(w_1)-u(w_2)} < 3(1+\alpha) - 4 \\
\iff & \frac{4u(w_2)}{u(w_1)-u(w_2)} < 3\alpha - 1
\end{aligned}$$

$f'(\lambda_1 = 1) > 0$ is only satisfied when $\alpha > \frac{1}{3}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{4}{3\alpha-1}$.

We show $\frac{4}{3\alpha-1} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$:

$$\begin{aligned}\frac{4}{3\alpha-1} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)} &\iff 4\alpha\,(3\alpha+1) > 2(2\alpha+1)\,(3\alpha-1)\\ &\iff 12\alpha^2+4\alpha > 12\alpha^2+2\alpha-2\\ &\iff 2\alpha+2>0\end{aligned}$$

Thus we know that under the conditions $\alpha > \frac{1}{3}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{4}{3\alpha-1}$, $f(\lambda_1)$ is positive when λ_1 is small and close to 1. Therefore there exist parameter configurations where the overconfident player's equilibrium winning probability P_1 is higher than the benchmark.

(2) P_2

We show that $P_2 < \frac{1}{4}$:

$$P_2 = p^f_{23}p^s_{21} = \frac{1}{2}p^s_{21} < \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$$

(3) P_3 and P_4

We show that $P_3 = P_4 > \frac{1}{4}$:

$$\begin{aligned}P_3 = P_4 &= p^s_{12}p^f_{31}p^s_{34} + p^s_{21}p^f_{32}p^s_{34}\\ &= p^s_{12}p^f_{31}\frac{1}{2} + p^s_{21}\frac{1}{2}\frac{1}{2}\\ &= p^s_{12}p^f_{31}\frac{1}{2} + (1-p^s_{12})\frac{1}{2}\frac{1}{2}\\ &= p^s_{12}\left(p^f_{31}\frac{1}{2}-\frac{1}{4}\right)+\frac{1}{4}\\ &> p^s_{12}\left(\frac{1}{2}\frac{1}{2}-\frac{1}{4}\right)+\frac{1}{4} = \frac{1}{4}\end{aligned}$$

(4) compare P_1 and P_3

$$\begin{aligned}P_1 - P_3 &= p^f_{13}p^s_{12} - p^s_{12}p^f_{31}p^s_{34} - p^s_{21}p^f_{32}p^s_{34}\\ &= p^f_{13}p^s_{12} - p^s_{12}p^f_{31}p^s_{34} - (1-p^s_{12})\,p^f_{32}p^s_{34}\\ &= p^f_{13}p^s_{12} - \frac{1}{2}p^s_{12}p^f_{31} - \frac{1}{2}\frac{1}{2}\,(1-p^s_{12})\\ &= p^f_{13}p^s_{12} - \frac{1}{2}p^s_{12}\left(1-p^f_{13}\right) - \frac{1}{4}\,(1-p^s_{12})\\ &= \frac{3}{2}p^f_{13}p^s_{12} - \frac{1}{4}p^s_{12} - \frac{1}{4}\end{aligned}$$

The sign of $P_1 - P_3$ is the same as the sign of $6p^f_{13}p^s_{12} - p^s_{12} - 1$
Let $f(\lambda_1) = 6p^f_{13}p^s_{12} - p^s_{12} - 1$

$$f(\lambda_1 = 1) = 6\times\frac{1}{2}\times\frac{1}{2} - \frac{1}{2} - 1 = 0$$

$$f(\lambda_1 = \hat{\lambda}) = 6\times\frac{1}{2}\hat{\lambda}^{-\frac{\alpha}{2\alpha+1}}\times\frac{1}{2} - \frac{1}{2} - 1 < 0$$

$$f(\lambda_1) = 3\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left[1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\right]$$

$$- \left[1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\right] - 1$$

$$= 3\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \frac{1}{2}\lambda_1^{\alpha}\left(3\lambda_1^{-\frac{\alpha}{2\alpha+1}} - 1\right)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}$$

$$\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} - 2$$

$$f'(\lambda_1) = -3\frac{\alpha}{2\alpha+1}\lambda_1^{-\frac{\alpha}{2\alpha+1}-1} - \frac{1}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}$$

$$\left[\left(3\left(-\frac{\alpha}{2\alpha+1}+\alpha\right)\lambda_1^{-\frac{\alpha}{2\alpha+1}+\alpha-1} - \alpha\lambda_1^{\alpha-1}\right)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\right.$$

$$+ \lambda_1^{\alpha}\left(3\lambda_1^{-\frac{\alpha}{2\alpha+1}} - 1\right)(-\alpha)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha-1}$$

$$\left.\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\right]$$

$$f'(\lambda_1 = 1) = -3\frac{\alpha}{2\alpha+1} - \frac{1}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}$$

$$\left[\left(3\left(-\frac{\alpha}{2\alpha+1}+\alpha\right) - \alpha\right)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha}\right.$$

$$\left.- 2\alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right]$$

$$= -3\frac{\alpha}{2\alpha+1} - \frac{1}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}$$

$$\left[\left(2\alpha - 3\frac{\alpha}{2\alpha+1}\right)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha}\right.$$

$$\left.- 2\alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right]$$

$$= -3\frac{\alpha}{2\alpha+1} - \frac{1}{2}\left(2\alpha - 3\frac{\alpha}{2\alpha+1}\right)$$

$$+ \alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}$$

$$= -\frac{3}{2}\frac{\alpha}{2\alpha+1} - \alpha + \alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}$$

$$= \alpha\left(-\frac{3}{2}\frac{1}{2\alpha+1} - 1 + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\right)$$

$$
\begin{aligned}
f'(\lambda_1 = 1) > 0 &\iff -\frac{3}{2}\frac{1}{2\alpha+1} - 1 + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1} > 0 \\
&\iff \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1} > \frac{3}{2}\frac{1}{2\alpha+1} + 1 \\
&\iff \frac{\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}}{\frac{3}{2}\frac{1}{2\alpha+1}+1} > \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&\iff \frac{\frac{(\alpha+1)^2}{2(2\alpha+1)}}{\frac{3+2(2\alpha+1)}{2(2\alpha+1)}} > \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&\iff \frac{(\alpha+1)^2}{(4\alpha+5)} > \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&\iff \frac{(\alpha+1)^2}{(4\alpha+5)} + \frac{\alpha-1}{2} > \frac{u(w_2)}{u(w_1)-u(w_2)} \\
&\iff \frac{2\left(\alpha+1\right)^2 + \left(\alpha-1\right)\left(4\alpha+5\right)}{2\left(4\alpha+5\right)} > \frac{u(w_2)}{u(w_1)-u(w_2)} \\
&\iff \frac{6\alpha^2+5\alpha-3}{2\left(4\alpha+5\right)} > \frac{u(w_2)}{u(w_1)-u(w_2)}
\end{aligned}
$$

This is satisfied when $\alpha > \frac{-5+\sqrt{97}}{12}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(4\alpha+5)}{6\alpha^2+5\alpha-3}$.

We show $\frac{2(4\alpha+5)}{6\alpha^2+5\alpha-3} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$:

$$
\begin{aligned}
\frac{2\left(4\alpha+5\right)}{6\alpha^2+5\alpha-3} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)} &\iff \frac{4\alpha+5}{6\alpha^2+5\alpha-3} > \frac{2\alpha+1}{\alpha(3\alpha+1)} \\
&\iff \left(4\alpha+5\right)\alpha(3\alpha+1) > \left(2\alpha+1\right)\left(6\alpha^2+5\alpha-3\right) \\
&\iff 12\alpha^3+19\alpha^2+5\alpha > 12\alpha^3+16\alpha^2-\alpha-3 \\
&\iff 3\alpha^2+6\alpha+3 > 0
\end{aligned}
$$

Thus we know that under the conditions $\alpha > \frac{-5+\sqrt{97}}{12}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(4\alpha+5)}{6\alpha^2+5\alpha-3}$, $f(\lambda_1)$ is positive when λ_1 is small and close to 1. Therefore there exist parameter configurations where the overconfident player's equilibrium winning probability P_1 is higher than that of the rational player in the other semifinal P_3.

2. When either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

   (1) P_1

   Since player 3 and player 4 are identical, the equilibrium winning probability of player 1 is

   $$P_1 = p_{13}^f p_{12}^s < \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

   (2) P_2

   $$P_2 = p_{23}^f p_{21}^s = \frac{1}{2} p_{21}^s > \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

   (3) P_3 and P_4

   $P_3 = P_4 > \frac{1}{4}$ still holds.

Proof of Proposition 7

1. Equilibrium expected utility of overconfident player 1 in the semifinal with rational player 2.

   The equilibrium perceived expected utility of reaching the final of overconfident player 1 is

$$
\begin{aligned}
v_1 &= p^s_{34}E^f(U_{13}) + p^s_{43}E^f(U_{14}) \\
&= E^f(U_{13}) \\
&= \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\Delta u - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\Delta u + u(w_2) \\
&= \left(\frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)\Delta u
\end{aligned}
$$

   and his equilibrium expected utility in the semifinal with rational player 2 when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda < \hat{\lambda}$ is

$$
\begin{aligned}
E^s(U_{12}) &= p^s_{12}v_1 - ce^s_1 \\
&= \left[1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\right] \\
&\quad \left(\frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)\Delta u \\
&\quad - \frac{\alpha}{2}\lambda_1^{\alpha-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\Delta u
\end{aligned}
$$

   The benchmark equilibrium expected utility of the semifinal is

$$
\overline{E}^s(U) = \frac{1-\alpha}{2}\overline{v} = \frac{1-\alpha}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)\Delta u
$$

   Let

$$
\begin{aligned}
f(\lambda_1) &= \frac{E^s(U_{12}) - \overline{E}^s(U)}{\Delta u} \\
&= \left[1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\right] \\
&\quad \left(\frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right) \\
&\quad - \frac{\alpha}{2}\lambda_1^{\alpha-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} \\
&\quad - \frac{1-\alpha}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)
\end{aligned}
$$

   We can easily get that $f(\lambda_1 = 1) = 0$.

$$
\begin{aligned}
f'(\lambda_1) = &\left[1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\right] \\
&\left(-\frac{1}{2}\frac{\alpha}{2\alpha+1}\lambda_1^{-\frac{\alpha}{2\alpha+1}-1} + \frac{\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\right) \\
&+ \Bigg[-\frac{1}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\Bigg(\alpha\lambda_1^{\alpha-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha} \\
&+ \lambda_1^{\alpha}(-\alpha)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\Bigg)\Bigg] \\
&\left(\frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right) \\
&- \frac{\alpha}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\Bigg[(\alpha-1)\lambda_1^{\alpha-2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha} \\
&+ \lambda_1^{\alpha-1}(1-\alpha)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
f'(\lambda_1 = 1) = &\frac{1}{2}\frac{\alpha}{2}\frac{\alpha}{2\alpha+1} + \\
&\left[-\frac{\alpha}{2} + \frac{\alpha}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right]\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&- \frac{\alpha}{2}(\alpha-1)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) - \frac{\alpha}{2}(1-\alpha)\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
= &\frac{1}{2}\frac{\alpha}{2}\frac{\alpha}{2\alpha+1} - \frac{\alpha}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{\alpha}{2}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
&- \frac{\alpha}{2}(\alpha-1)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) - \frac{\alpha}{2}(1-\alpha)\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
= &\frac{\alpha}{2}\Bigg[\frac{1}{2}\frac{\alpha}{2\alpha+1} - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
&- (\alpha-1)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) - (1-\alpha)\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\Bigg] \\
= &\frac{\alpha}{2}\left[\frac{1}{2}\frac{\alpha}{2\alpha+1} - \alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) + \alpha\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right] \\
= &\frac{\alpha^2}{2}\left[\frac{1}{2}\frac{1}{2\alpha+1} - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right] \\
= &\frac{\alpha^2}{2}\left[\frac{1}{2}\frac{1}{2\alpha+1} + \frac{1+\alpha}{2}\left(1 + \frac{\alpha+1}{2\alpha+1}\right) - 1 - \frac{u(w_2)}{u(w_1)-u(w_2)}\right]
\end{aligned}
$$

$$= \frac{\alpha^2}{2}\left[\frac{\alpha(3\alpha+1)+1}{2(2\alpha+1)} - \frac{u(w_2)}{u(w_1)-u(w_2)}\right]$$

$\frac{\alpha(3\alpha+1)+1}{2(2\alpha+1)} - \frac{u(w_2)}{u(w_1)-u(w_2)} > 0$ is satisfied when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$.
Therefore there exist $\lambda_1 \in (1, \hat{\lambda})$ for which $E^s(U_{12}) > \overline{E}^s(U)$.

2. Equilibrium expected utility of rational player 2 in the semifinal with overconfident player 1.

(1) When $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 < \hat{\lambda}$

$$\begin{aligned}
E^s(U_{21}) &= p_{21}^s v_2 - c e_2^s \\
&= \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^\alpha v_2 - c\frac{\alpha}{2c}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^{\alpha+1} \\
&= \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^{1+\alpha} - \frac{\alpha}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^{\alpha+1} \\
&= \frac{1-\alpha}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^{1+\alpha} \\
&= \frac{1-\alpha}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(\overline{v}\right)^{1+\alpha} \\
&= \frac{1-\alpha}{2}\left(\frac{\overline{v}}{\lambda_1^{-1}\widetilde{v}_1}\right)^\alpha \overline{v} \\
&< \frac{1-\alpha}{2}\overline{v} = \overline{E}^s(U)
\end{aligned}$$

(2) When either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ or $\lambda_1 \geqslant \hat{\lambda}$

$$\begin{aligned}
E^s(U_{21}) &= p_{21}^s v_2 - c e_2^s \\
&= \left[1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{-\frac{\alpha}{2\alpha+1}}\right] v_2 - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \left[1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{-\frac{\alpha}{2\alpha+1}}\right] v_2 - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= v_2 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \overline{v} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(\overline{v}\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \left[1 - \frac{1+\alpha}{2}\left(\frac{\lambda_1^{-1}\widetilde{v}_1}{\overline{v}}\right)^{\frac{\alpha}{2\alpha+1}}\right]\overline{v} \\
&> \frac{1-\alpha}{2}\overline{v} = \overline{E}^s(U)
\end{aligned}$$

3. Equilibrium expected utility of rational player 3 (4) in the semifinal with rational player 4 (3).

$$E^s(U_{34}) = E^s(U_{43}) = \frac{1-\alpha}{2}v_3 = \frac{1-\alpha}{2}v_4 > \frac{1-\alpha}{2}\overline{v} = \overline{E}^s(U).$$

Proof of Proposition 8

Part (i) follows directly from Propositions 2 and 3. Let's then prove part (ii). We know that the equilibrium efforts in the semifinal between the two rational players are higher than the benchmark, thus the equilibrium aggregate effort in the semifinals stage is higher than that of the benchmark if the equilibrium total effort of the semifinal with an overconfident and a rational player is higher than that of the benchmark. We also know from Proposition 4 that if $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 < \hat{\lambda}$, then total effort in the semifinal with an overconfident and a rational player is given by

$$e_1^s + e_2^s = \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(v_2)^{\alpha} + \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(v_2)^{\alpha+1} = \frac{\alpha}{2c}\overline{v}\left[\lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(\overline{v})^{\alpha-1} + \lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(\overline{v})^{\alpha}\right]$$

Hence, we have

$$\frac{e_1^s + e_2^s}{2\overline{e}^s} = \frac{\frac{\alpha}{2c}\overline{v}\left[\lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(\overline{v})^{\alpha-1} + \lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(\overline{v})^{\alpha}\right]}{2\frac{\alpha}{2c}\overline{v}} = \frac{1}{2}\left[\lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(\overline{v})^{\alpha-1} + \lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(\overline{v})^{\alpha}\right]$$

Let $f(\lambda) = \lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(\overline{v})^{\alpha-1} + \lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(\overline{v})^{\alpha}$,

$$\begin{aligned} f(\lambda) = {} & \lambda_1^{\alpha-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha-1} \\ & + \lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} \end{aligned}$$

We can easily get that $f(\lambda_1 = 1) = 2$.

$$\begin{aligned} f'(\lambda_1) = {} & \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha-1}\Bigg[(\alpha-1)\lambda_1^{\alpha-2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha} \\ & + \lambda_1^{\alpha-1}(1-\alpha)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\Bigg] \\ & + \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\Bigg[\alpha\lambda_1^{\alpha-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha} \\ & + \lambda_1^{\alpha}(-\alpha)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\Bigg] \end{aligned}$$

$$
\begin{aligned}
f'(\lambda_1 = 1) = &\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha-1} \left[(\alpha-1)\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{1-\alpha}\right. \\
&\left. + (1-\alpha)\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha} \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right] \\
&+ \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} \left[\alpha\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha}\right. \\
&\left. + (-\alpha)\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha-1} \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right] \\
= &(\alpha-1) + (1-\alpha)\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{-1} + \alpha \\
&- \alpha\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{-1} \\
= &(2\alpha-1) + (1-2\alpha)\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{-1} \\
= &(1-2\alpha)\left[-1 + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\right]
\end{aligned}
$$

$-1 + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1} > 0$ is equivalent to $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$.
Thus we have
If $\alpha < \frac{1}{2}$, then $f'(\lambda_1 = 1) > 0$.
If $\alpha > \frac{1}{2}$, then $f'(\lambda_1 = 1) < 0$.
Therefore, when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\alpha < \frac{1}{2}$, there exist $\lambda_1 \in (1, \hat{\lambda})$ such that $e_1^s + e_2^s > 2\overline{e}^s$ which implies

$$
\sum_{i=1}^{4} e_i^s > 4\overline{e}^s
$$

2 Elimination Contest with Two Overconfident and Two Rational Players

This section studies the equilibrium of an elimination contest with two overconfident and two rational players. We assume the two overconfident players differ in their confidence levels. This extension enables us to assess if our findings still hold when two overconfident players encounter each other, either in the final or the semifinal.

There are two possible seedings that we need to consider: (i) the overconfident players are seeded in the same semifinal, and (ii) the overconfident players are seeded in different semifinals. These two types of seeding induce different results and hence we study them separately.

2.1 Final

When the overconfident players are seeded in the same semifinal, the final will be played between an overconfident and a rational player and we can apply Proposition 3. In contrast, when the overconfident players are seeded in different semifinals, the final can have two overconfident players. Hence, we start by characterizing the equilibrium of a final with two overconfident players. Without loss of generality we consider a final between players 1 and 3 with $\lambda_1 > \lambda_3 > 1$.

Proposition A1 *In a final between two overconfident players, the equilibrium effort of the more overconfident player is*

$$e_1^f = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\Delta u$$

and the equilibrium effort of the less overconfident player is

$$e_3^f = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\Delta u$$

with $e_1^f < e_3^f < \overline{e}^f$. The perceived equilibrium winning probabilities are

$$\widetilde{p}_{13}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}$$

$$\widetilde{p}_{31}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}$$

with $\widetilde{p}_{13}^f > \widetilde{p}_{31}^f > 1/2$. The equilibrium winning probabilities are

$$p_{13}^f = \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{\frac{\alpha}{2\alpha+1}}$$

$$p_{31}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{\frac{\alpha}{2\alpha+1}}$$

with $p_{13}^f < 1/2 < p_{31}^f$. The perceived equilibrium expected utilities are

$$\widetilde{E}^f(U_{13}) = u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\Delta u,$$

$$\widetilde{E}^f(U_{31}) = u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\Delta u,$$

with $\widetilde{E}^f(U_{13}) > \widetilde{E}^f(U_{31}) > \overline{E}^f(U)$.

Proposition A1 shows that in a final between two overconfident players, the more overconfident player exerts lower effort at equilibrium. As we have seen, the bias lowers an overconfident player's perceived marginal probability of winning the final. The more overconfident a player is, the higher is the drop in his perceived marginal probability of winning the final. Hence, the more overconfident player exerts lower effort at equilibrium. Both players exert lower effort than if both were rational. Each player perceives he has a winning probability greater than 1/2 but, in fact, only the less overconfident player has a true winning probability greater than 1/2. The perceived expected utility of each player is increasing in his own bias as well as in the rival's bias.

Proof of Proposition A1

The perceived winning probabilities of the players are:

$$\widetilde{p}_{13}^f = \begin{cases} 1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha} & \text{if } \lambda_1 e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha} & \text{if } \lambda_1 e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

$$\widetilde{p}_{31}^{f} = \begin{cases} 1 - \frac{1}{2}\frac{e_1^\alpha}{\lambda_3 e_3^\alpha} & \text{if } \lambda_3 e_3^\alpha \geqslant e_1^\alpha \\ \frac{1}{2}\frac{\lambda_3 e_3^\alpha}{e_1^\alpha} & \text{if } \lambda_3 e_3^\alpha \leqslant e_1^\alpha \end{cases}$$

Overconfident player 1 $\quad max \quad \widetilde{E}^f(U_{13}) = \widetilde{p}_{13}^{f}\Delta u + u(w_2) - ce_1$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)\Delta u + u(w_2) - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}\Delta u + u(w_2) - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

Overconfident player 3 $\quad max \quad \widetilde{E}^f(U_{31}) = \widetilde{p}_{31}^{f}\Delta u + u(w_2) - ce_3$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{\lambda_3 e_3^\alpha}\right)\Delta u + u(w_2) - ce_3 & \text{if} \quad \lambda_3 e_3^\alpha \geqslant e_1^\alpha \\ \frac{1}{2}\frac{\lambda_3 e_3^\alpha}{e_1^\alpha}\Delta u + u(w_2) - ce_3 & \text{if} \quad \lambda_3 e_3^\alpha \leqslant e_1^\alpha \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad \lambda_3 e_3^\alpha \geqslant e_1^\alpha \\ \lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad \lambda_3 e_3^\alpha \leqslant e_1^\alpha \\ \lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad \lambda_3 e_3^\alpha \geqslant e_1^\alpha \\ \lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad \lambda_3 e_3^\alpha \leqslant e_1^\alpha \end{cases}$$

Since $\lambda_1 > \lambda_3 > 1$, the fourth case is impossible.

1. equilibrium efforts

   (1) case 1: $\lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad \lambda_3 e_3^\alpha \geqslant e_1^\alpha$

   Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)\Delta u + u(w_2) - ce_1$

   Player 3 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{\lambda_3 e_3^\alpha}\right)\Delta u + u(w_2) - ce_3$

   F.o.c

   $[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_3^\alpha}{e_1^{\alpha+1}}\Delta u - c = 0$

   $[e_3] \quad \frac{\alpha}{2\lambda_3}\frac{e_1^\alpha}{e_3^{\alpha+1}}\Delta u - c = 0$

   S.o.c

   $[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha - 1)\frac{e_3^\alpha}{e_1^{\alpha+2}}\Delta u < 0$

   $[e_3] \quad \frac{\alpha}{2\lambda_3}(-\alpha - 1)\frac{e_1^\alpha}{e_3^{\alpha+2}}\Delta u < 0$

   Solve F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\Delta u$$

$$e_3 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\Delta u$$

Check the condition $\lambda_3 e_3^\alpha \geqslant e_1^\alpha$:

$$\lambda_3 e_3^\alpha \geqslant e_1^\alpha \Longleftrightarrow \frac{\lambda_3 e_3^\alpha}{e_1^\alpha} \geqslant 1 \Longleftrightarrow \lambda_1^{\frac{\alpha}{2\alpha+1}}\lambda_3^{\frac{\alpha+1}{2\alpha+1}} \geqslant 1$$

Check the condition $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$:

$$\lambda_1 e_1^\alpha \geqslant e_3^\alpha \Longleftrightarrow \frac{\lambda_1 e_1^\alpha}{e_3^\alpha} \geqslant 1 \Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}}\lambda_3^{\frac{\alpha}{2\alpha+1}} \geqslant 1$$

which is always satisfied.

(2) case 2: $\lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad \lambda_3 e_3^\alpha \leqslant e_1^\alpha$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)\Delta u + u(w_2) - ce_1$

Player 3 $\quad max \quad \frac{1}{2}\frac{\lambda_3 e_3^\alpha}{e_1^\alpha}\Delta u + u(w_2) - ce_3$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_3^\alpha}{e_1^{\alpha+1}}\Delta u - c = 0$

$[e_3] \quad \frac{\alpha}{2}\lambda_3\frac{e_3^{\alpha-1}}{e_1^\alpha}\Delta u - c = 0$

divide the two F.O.C , we get

$$\frac{e_3}{e_1} = \lambda_1\lambda_3 > 1$$

which contradicts the condition that $\lambda_3 e_3^\alpha \leqslant e_1^\alpha$

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad \lambda_3 e_3^\alpha \geqslant e_1^\alpha$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}\Delta u + u(w_2) - ce_1$

Player 3 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{\lambda_3 e_3^\alpha}\right)\Delta u + u(w_2) - ce_3$

F.o.c

$[e_1] \quad \frac{\alpha}{2}\lambda_1\frac{e_1^{\alpha-1}}{e_3^\alpha}\Delta u - c = 0$

$[e_3] \quad \frac{\alpha}{2\lambda_3}\frac{e_1^\alpha}{e_3^{\alpha+1}}\Delta u - c = 0$

divide the two F.O.C , we get

$$\frac{e_3}{e_1} = \frac{1}{\lambda_1\lambda_3} < 1$$

which contradicts the condition that $\lambda_1 e_1^\alpha \leqslant e_3^\alpha$.

Thus the unique equilibrium is

$$e_1^f = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\Delta u$$

$$e_3^f = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\Delta u$$

Since $\lambda_1 > \lambda_3 > 1$, we can get $e_1^f < e_3^f < \overline{e}^f$.

2. winning probabilities

The true winning probabilities are

$$p_{13}^f = \frac{1}{2}\left(\frac{e_1^f}{e_3^f}\right)^\alpha = \frac{1}{2}\left(\lambda_1^{-\frac{1}{2\alpha+1}}\lambda_3^{\frac{1}{2\alpha+1}}\right)^\alpha = \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{\frac{\alpha}{2\alpha+1}} < \frac{1}{2}$$

$$p_{31}^f = 1 - p_{13}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{\frac{\alpha}{2\alpha+1}} > \frac{1}{2}$$

The perceived winning probabilities are

$$\widetilde{p}_{13}^f = 1 - \frac{1}{2}\frac{(e_3^f)^\alpha}{\lambda_1(e_1^f)^\alpha} = 1 - \frac{1}{2\lambda_1}\left(\lambda_1^{\frac{1}{2\alpha+1}}\lambda_3^{-\frac{1}{2\alpha+1}}\right)^\alpha = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}$$

$$\widetilde{p}_{31}^f = 1 - \frac{1}{2}\frac{(e_1^f)^\alpha}{\lambda_3(e_3^f)^\alpha} = 1 - \frac{1}{2\lambda_3}\left(\lambda_1^{-\frac{1}{2\alpha+1}}\lambda_3^{\frac{1}{2\alpha+1}}\right)^\alpha = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}$$

Thus we have

$$\widetilde{p}_{13}^f > \widetilde{p}_{31}^f > p_{31}^f > \frac{1}{2} > p_{13}^f$$

3. expected utilities of final

$$\begin{aligned}
\widetilde{E}^f(U_{13}) &= \widetilde{p}_{13}^f u(w_1) + (1-\widetilde{p}_{13}^f)u(w_2) - ce_1^f \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)u(w_1) + \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}u(w_2) \\
&\quad - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\Delta u \\
&= u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\Delta u
\end{aligned}$$

$$\begin{aligned}
\widetilde{E}^f(U_{31}) &= \widetilde{p}_{31}^f u(w_1) + (1-\widetilde{p}_{31}^f)u(w_2) - ce_3^f \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)u(w_1) + \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}u(w_2) \\
&\quad - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\Delta u \\
&= u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\Delta u
\end{aligned}$$

Since $\alpha < 1$ and $\lambda_1 > \lambda_3 > 1$, $\widetilde{E}^f(U_{13}) > \widetilde{E}^f(U_{31}) > \overline{E}^f(U)$. The participation constraints are also satisfied.

2.2 Overconfident players seeded in the same semifinal

Assume players 1 and 2, seeded in one semifinal, are overconfident with $\lambda_1 > \lambda_2 > 1$ and players 3 and 4, seeded in the other semifinal, are rational with $\lambda_3 = \lambda_4 = 1$. Note that, under this seeding, the final will involve an overconfident and a rational player and hence we can apply Proposition 3. Note also that since the two rational players are identical, they exert equal efforts in the semifinal and hence, each has an equal probability of winning it. This means that the identity of winner of the semifinal between two rational players does not affect the overconfident players' behavior in their semifinal. However, since the overconfident players' biases differ, the identity of winner of the semifinal between two overconfident players matters for the effort choices of the rational players in their semifinal. Taking this into account, we start by solving the equilibrium of the semifinal with two rational players and then we solve for the equilibrium of the semifinal with two overconfident players.

Proposition A2

In the semifinal between two rational players of a two-stage elimination contest where the overconfident players 1 and 2 are seeded in one semifinal, the rational players 3 and 4 are seeded in the other semifinal, and $\lambda_1 > \lambda_2 > 1 = \lambda_3 = \lambda_4$, the equilibrium efforts and winning probabilities satisfy $e_3^s = e_4^s > \overline{e}^s$ and $p_{34}^s = p_{43}^s = 1/2$.

Proof of Proposition A2

1. Expected utilities of reaching the final

   Rational player 3:

$$\begin{aligned}
v_3 &= p_{12}^s E^f(U_{31}) + p_{21}^s E^f(U_{32}) \\
&= \left[p_{12}^s\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right) + p_{21}^s\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\right)\right]\Delta u \\
&= \left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(\lambda_2^{-\frac{\alpha}{2\alpha+1}} + p_{12}^s\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \lambda_2^{-\frac{\alpha}{2\alpha+1}}\right)\right)\right]\Delta u \\
&> \overline{v}
\end{aligned}$$

   where p_{12}^s is as derived in the proof of Proposition A3.

   Rational player 4:

   Since player 3 and player 4 are identical,

$$v_4 = v_3 > \overline{v}$$

2. The equilibrium

   (1) When $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \leqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}$

$$
\begin{aligned}
e_3^s = e_4^s &= \frac{\alpha}{2c} v_3 \\
&= \frac{\alpha}{2c}\Bigg[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\Bigg(\lambda_2^{-\frac{\alpha}{2\alpha+1}} \\
&\quad + \Bigg(1 - \frac{1}{2}\lambda_1^{\alpha}\lambda_2^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha}\Bigg)\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \lambda_2^{-\frac{\alpha}{2\alpha+1}}\right)\Bigg)\Bigg]\Delta u
\end{aligned}
$$

$$p_{34}^s = p_{43}^s = \frac{1}{2}$$

(2) When $\frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}} \leqslant \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2}$

$$
\begin{aligned}
e_3^s = e_4^s &= \frac{\alpha}{2c} v_3 \\
&= \frac{\alpha}{2c}\Bigg[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\Bigg(\lambda_2^{-\frac{\alpha}{2\alpha+1}} \\
&\quad + \Bigg(1 - \frac{1}{2}\lambda_1^{\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}\Bigg)\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \lambda_2^{-\frac{\alpha}{2\alpha+1}}\right)\Bigg)\Bigg]\Delta u
\end{aligned}
$$

$$p_{34}^s = p_{43}^s = \frac{1}{2}$$

(3) When $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \max\left[\frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2}, \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}\right]$

$$
\begin{aligned}
e_3^s = e_4^s &= \frac{\alpha}{2c} v_3 \\
&= \frac{\alpha}{2c}\Bigg[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\Bigg(\lambda_2^{-\frac{\alpha}{2\alpha+1}} \\
&\quad + \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \lambda_2^{-\frac{\alpha}{2\alpha+1}}\right)\Bigg)\Bigg]\Delta u
\end{aligned}
$$

$$p_{34}^s = p_{43}^s = \frac{1}{2}$$

Since $v_3 = v_4 > \overline{v}$, $e_3^s = e_4^s > \overline{e}^s$ is always satisfied.

3. Participation constraints

$$E^s(U_{34}) = p^s_{34} v_3 - ce_3 = \frac{1-\alpha}{2} v_3 \geqslant 0$$

$$E^s(U_{43}) = E^s(U_{34}) \geqslant 0$$

Proposition A3 *Consider the semifinal between two overconfident players of a two-stage elimination contest where the overconfident players 1 and 2 are seeded in one semifinal, the rational players 3 and 4 are seeded in the other semifinal, and* $\lambda_1 > \lambda_2 > 1 = \lambda_3 = \lambda_4$.

(i) If $\frac{u(w_1)}{u(w_1)-u(w_2)} \frac{2}{1+\alpha} \leqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}$, *then the equilibrium efforts and winning probabilities satisfy* $e^s_1 > e^s_2$, $e^s_1 > \overline{e}^s$, *and* $\widetilde{p}^s_{12} > p^s_{12} > 1/2 \geqslant \widetilde{p}^s_{21} > p^s_{21}$.

(ii) If $\frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}} \leqslant \frac{u(w_1)}{u(w_1)-u(w_2)} \frac{2}{1+\alpha} < \frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1 - \lambda_2}$, *then the equilibrium efforts and winning probabilities satisfy* $e^s_1 > e^s_2 > \overline{e}^s$ *and* $\widetilde{p}^s_{12} > p^s_{12} > 1/2 > p^s_{21}$.

(iii) If $\frac{u(w_1)}{u(w_1)-u(w_2)} \frac{2}{1+\alpha} \geqslant \max \left[\frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1 - \lambda_2}, \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}} \right]$, *then the equilibrium efforts and winning probabilities satisfy* $e^s_1 \leqslant e^s_2$ *and* $\widetilde{p}^s_{12} > \widetilde{p}^s_{21} > p^s_{21} \geqslant 1/2 \geqslant p^s_{12}$.

Proposition A3 reveals that in a semifinal with two overconfident players, both players can exert higher efforts than if both were rational. It also shows that the identity of the player who exerts the highest effort depends on the prize spread, on the confidence gap, $\lambda_1 - \lambda_2$, and the bias of the less overconfident player 2.

Part (i) tells us that the more overconfident player 1 exerts higher effort at equilibrium when the prize spread is large and the confidence gap is moderate.[1] Part (ii) tells us that the more overconfident player 1 exerts higher effort at equilibrium when the prize spread is moderate, the confidence gap is small, and the bias of the less overconfident player 2 is low. In this case both players exert higher effort than if both were rational.[2] Finally, part (iii) tells us that the less overconfident player 2 exerts higher effort at equilibrium when either the prize spread is small, or the confidence gap is large, or the confidence gap is small and the bias of the less overconfident player 2 is large.

Figure 1 illustrates result (ii) in Proposition A3. It depicts the best responses and equilibrium efforts in a semifinal of an elimination contest where $u(w_1) = 11$, $u(w_2) = 1$, $c = 1$, and $\alpha = 0.9$. Point E depicts the equilibrium when both players are rational. Point E' below the 45 degree line depicts the equilibrium when player 1 is overconfident with $\lambda_1 = 1.18$, and player 2 is overconfident with $\lambda_2 = 1.07$. These parameter values satisfy the two inequalities in (ii) and hence the more overconfident player 1 exerts higher effort at equuilibrium than the less overconfident player 2.

[1]When the confidence gap becomes increasingly large, i.e., $\lambda_1 \to \infty$, the right hand side of the inequality in part (i) converges to $\lambda_2^{-\frac{\alpha+1}{2\alpha+1}} < 1$. When the confidence gap becomes increasingly small, i.e., $\lambda_1 \to \lambda_2$, the right hand side of the inequality in part (i) also converges to $\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$ which is less than 1. Hence, since the left hand side of the inequality in part (i) is greater than 1, the inequality cannot be satisfied when the confidence gap is either too large or too small. These two limits are computed at the end of the proof of the proposition.

[2]When $\lambda_1 \to \infty$, the right hand side of the second inequality in part (ii) converges to $\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$ which is less than 1. Hence, since the left hand side of the second inequality in part (ii) is greater than 1, the second inequality in part (ii) cannot be satisfied when the confidence gap is large. When $\lambda_1 \to \lambda_2$, the left hand side of the first inequality in (ii) converges to $\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$ which is less than 1 and the right hand side of the second inequality converges to $\frac{3\alpha+2}{2\alpha+1}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$. Hence, the two inequalities in (ii) can be satisfied when the confidence gap becomes increasingly small as long as the bias of the less overconfident player 2 is low. These two limits are computed at the end of the proof of the proposition.

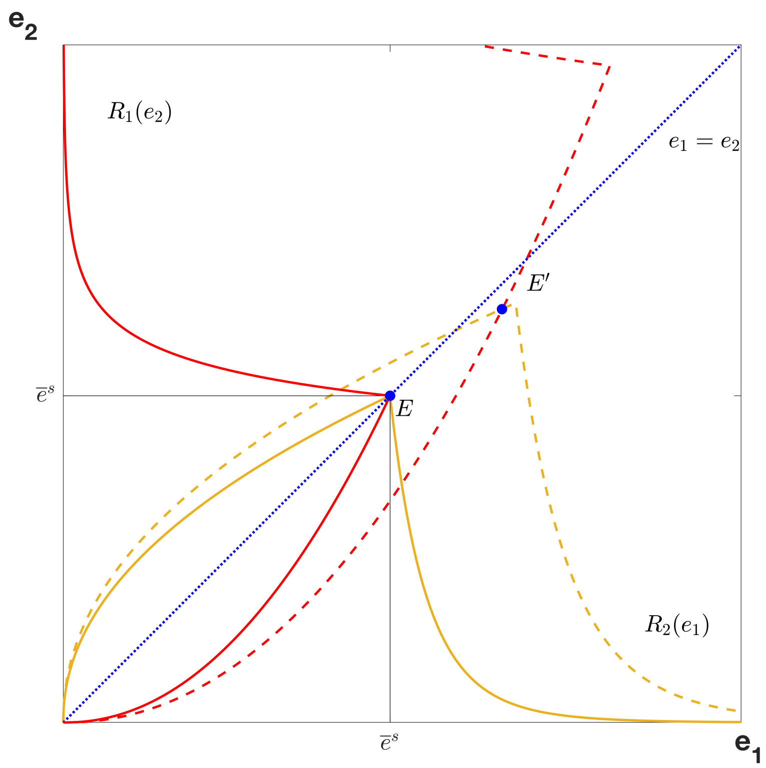

Figure 1: Best Responses and Equilibrium Efforts in a Semifinal with Two Overconfident Players

The equilibrium of the semifinal with two rational players is similar to that derived in Proposition 5. The main difference is that now the expected utility of reaching the final of the rational players takes into account the fact that the overconfident players exert different efforts and hence have different winning probabilities. Still, regardless of the identity of the winner of the semifinal between the two overconfident players, the rational players will have a higher expected utility of reaching the final than if all players were rational. Each rational player knows she will meet an overconfident player in the final which makes reaching the final more attractive. Hence, in the semifinal with two rational players, the equilibrium effort is higher than if all players were rational.

Proposition A3 also shows that, except for a knife-hedge parameter configuration, one of the two overconfident players has a probability of winning his semifinal that is greater than $1/2$. Moreover, some confidence gaps will generate quite large gaps between p^s_{12} and p^s_{21}. We also know that given the equal equilibrium effort, each rational player has an equal probability of winning his semifinal. This means that there will exist parameter configurations where an overconfident player is the one with the highest equilibrium probability of winning the elimination contest.

Hence, the findings obtained for an elimination contest with one overconfident and three rational players extend to an elimination contest where two overconfident players are seeded in one semifinal and two rational players are seeded in the other semifinal.

Proof of Proposition A3

1. Perceived expected utilities of reaching the final

Using Proposition 3, we can get the perceived expected utility of reaching the final of each player.
Overconfident player 1:

$$\widetilde{v}_1 = p^s_{34}\widetilde{E}^f(U_{13}) + p^s_{43}\widetilde{E}^f(U_{14})$$

Since player 3 and 4 are identical, $\widetilde{E}^f(U_{13}) = \widetilde{E}^f(U_{14})$,

$$\widetilde{v}_1 = u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\Delta u$$

We show that $\widetilde{v}_1 > \overline{v}$:

$$\widetilde{v}_1 > \overline{v} \Longleftrightarrow u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\Delta u > u(w_1) - \frac{1+\alpha}{2}\Delta u \Longleftrightarrow \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} < 1$$

Overconfident player 2:

$$\widetilde{v}_2 = p^s_{34}\widetilde{E}^f(U_{23}) + p^s_{43}\widetilde{E}^f(U_{24})$$

Since player 3 and 4 are identical, $\widetilde{E}^f(U_{23}) = \widetilde{E}^f(U_{24})$,

$$\widetilde{v}_2 = u(w_1) - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\Delta u > \overline{v}$$

We can easily get $\widetilde{v}_1 > \widetilde{v}_2$:

$$\widetilde{v}_1 > \widetilde{v}_2 \Longleftrightarrow \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} < \lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$$

Thus we have $\widetilde{v}_1 > \widetilde{v}_2 > \overline{v}$.

2. The equilibrium

Player 1 $\quad max \quad \widetilde{E}^s(U_{12}) = \widetilde{p}^s_{12}\widetilde{v}_1 - ce_1$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_2^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_2^\alpha \end{cases}$$

Player 2 $\quad max \quad \widetilde{E}^s(U_{21}) = \widetilde{p}^s_{21}\widetilde{v}_2 - ce_2$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{\lambda_2 e_2^\alpha}\right)\widetilde{v}_2 - ce_2 & \text{if}\, \lambda_2 e_2^\alpha \geqslant e_1^\alpha \\ \frac{1}{2}\frac{\lambda_2 e_2^\alpha}{e_1^\alpha}\widetilde{v}_2 - ce_2 & \text{if}\, \lambda_2 e_2^\alpha \leqslant e_1^\alpha \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad \lambda_2 e_2^\alpha \leqslant e_1^\alpha \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad \lambda_2 e_2^\alpha \geqslant e_1^\alpha \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad \lambda_2 e_2^\alpha \geqslant e_1^\alpha \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad \lambda_2 e_2^\alpha \leqslant e_1^\alpha \end{cases}$$

Since $\lambda_1 > \lambda_2 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad \lambda_2 e_2^\alpha \leqslant e_1^\alpha$, which corresponds to (i).

Player 1 $\quad max \quad \left(1-\frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \frac{1}{2}\frac{\lambda_2 e_2^\alpha}{e_1^\alpha}\widetilde{v}_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha\lambda_2}{2}\frac{e_2^{\alpha-1}}{e_1^\alpha}\widetilde{v}_2 - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1 < 0$

$[e_2] \quad \frac{\alpha\lambda_2}{2}(\alpha-1)\frac{e_2^{\alpha-2}}{e_1^\alpha}\widetilde{v}_2 < 0$

Solve the two F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}\lambda_2^\alpha(\widetilde{v}_1)^{1-\alpha}(\widetilde{v}_2)^\alpha$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^\alpha\lambda_2^{\alpha+1}(\widetilde{v}_1)^{-\alpha}(\widetilde{v}_2)^{\alpha+1}$$

$$\frac{e_2}{e_1} = \frac{\lambda_2\widetilde{v}_2}{\lambda_1^{-1}\widetilde{v}_1}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $\lambda_2 e_2^\alpha \leqslant e_1^\alpha$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$
When $\lambda_2 e_2^\alpha \leqslant e_1^\alpha$ is satisfied, $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ is satisfied.

② $\lambda_2 e_2^\alpha \leqslant e_1^\alpha$

$$
\begin{aligned}
&\lambda_2 e_2^\alpha \leqslant e_1^\alpha \\
&\Longleftrightarrow \lambda_2 \left(\frac{e_2}{e_1}\right)^\alpha \leqslant 1 \\
&\Longleftrightarrow \lambda_2 \left(\frac{\lambda_1\lambda_2\widetilde{v}_2}{\widetilde{v}_1}\right)^\alpha \leqslant 1 \\
&\Longleftrightarrow \lambda_1^\alpha \lambda_2^{\alpha+1} (\widetilde{v}_1)^{-\alpha} (\widetilde{v}_2)^\alpha \leqslant 1 \\
&\Longleftrightarrow \lambda_2^{\alpha+1} (\widetilde{v}_2)^\alpha \leqslant \lambda_1^{-\alpha} (\widetilde{v}_1)^\alpha \\
&\Longleftrightarrow \lambda_1^\alpha \lambda_2^{\alpha+1} (\widetilde{v}_2)^\alpha \leqslant (\widetilde{v}_1)^\alpha \\
&\Longleftrightarrow \lambda_1 \lambda_2^{\frac{\alpha+1}{\alpha}} \widetilde{v}_2 \leqslant \widetilde{v}_1 \\
&\Longleftrightarrow \lambda_1 \lambda_2^{\frac{\alpha+1}{\alpha}} \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right) \leqslant \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) \\
&\Longleftrightarrow \left(\lambda_1 \lambda_2^{\frac{\alpha+1}{\alpha}} - 1\right) \frac{u(w_1)}{u(w_1)-u(w_2)} \leqslant \frac{1+\alpha}{2}\left(\lambda_1 \lambda_2^{\frac{\alpha+1}{\alpha}-\frac{\alpha+1}{2\alpha+1}} - \lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) \\
&\Longleftrightarrow \frac{u(w_1)}{u(w_1)-u(w_2)} \frac{2}{1+\alpha} \leqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}
\end{aligned}
$$

Therefore the solution

$$e_1^s = \frac{\alpha}{2c}\lambda_1^{\alpha-1}\lambda_2^\alpha (\widetilde{v}_1)^{1-\alpha} (\widetilde{v}_2)^\alpha$$

$$e_2^s = \frac{\alpha}{2c}\lambda_1^{\alpha}\lambda_2^{\alpha+1} (\widetilde{v}_1)^{-\alpha} (\widetilde{v}_2)^{\alpha+1}$$

only applies when $\frac{u(w_1)}{u(w_1)-u(w_2)} \frac{2}{1+\alpha} \leqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}$.

Therefore, in (i), $e_1^s > e_2^s$ is always satisfied since $\lambda_2 e_2^\alpha \leqslant e_1^\alpha$ and $\lambda_2 > 1$.

(2) case 2: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad \lambda_2 e_2^\alpha \geqslant e_1^\alpha$, which corresponds to (ii) and (iii).

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{\lambda_2 e_2^\alpha}\right)\widetilde{v}_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2\lambda_2}\frac{e_1^\alpha}{e_2^{\alpha+1}}\widetilde{v}_2 - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1 < 0$

$[e_2] \quad \frac{\alpha}{2\lambda_2}(-\alpha-1)\frac{e_1^\alpha}{e_2^{\alpha+2}}\widetilde{v}_2 < 0$

Solve F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha+1}{2\alpha+1}}$$

$$\frac{e_2}{e_1} = \left(\frac{\lambda_2^{-1}\widetilde{v}_2}{\lambda_1^{-1}\widetilde{v}_1}\right)^{\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $\lambda_2 e_2^\alpha \geqslant e_1^\alpha$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

$$\begin{aligned}\lambda_1 e_1^\alpha \geqslant e_2^\alpha &\Longleftrightarrow \frac{\lambda_1 e_1^\alpha}{e_2^\alpha} \geqslant 1 \\ &\Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}}\lambda_2^{\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(\widetilde{v}_2)^{-\frac{\alpha}{2\alpha+1}} \geqslant 1 \\ &\Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}} \geqslant \lambda_2^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha}{2\alpha+1}}\end{aligned}$$

this is always satisfied. Therefore $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ always holds.

② $\lambda_2 e_2^\alpha \geqslant e_1^\alpha$

$$\begin{aligned}&\lambda_2 e_2^\alpha \geqslant e_1^\alpha \\ &\Longleftrightarrow \lambda_2\left(\frac{e_2}{e_1}\right)^\alpha \geqslant 1 \\ &\Longleftrightarrow \lambda_2\left(\frac{\lambda_2^{-1}\widetilde{v}_2}{\lambda_1^{-1}\widetilde{v}_1}\right)^{\frac{\alpha}{2\alpha+1}} \geqslant 1 \\ &\Longleftrightarrow \lambda_1^{\frac{\alpha}{2\alpha+1}}\lambda_2^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha}{2\alpha+1}} \geqslant 1 \\ &\Longleftrightarrow \lambda_1^{\frac{\alpha}{2\alpha+1}}\lambda_2^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha}{2\alpha+1}} \geqslant (\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}} \\ &\Longleftrightarrow \lambda_1^\alpha\lambda_2^{\alpha+1}\widetilde{v}_2^\alpha \geqslant \widetilde{v}_1^\alpha \\ &\Longleftrightarrow \lambda_1\lambda_2^{\frac{\alpha+1}{\alpha}}\widetilde{v}_2 \geqslant \widetilde{v}_1 \\ &\Longleftrightarrow \lambda_1\lambda_2^{\frac{\alpha+1}{\alpha}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right) \geqslant \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) \\ &\Longleftrightarrow \left(\lambda_1\lambda_2^{\frac{\alpha+1}{\alpha}} - 1\right)\frac{u(w_1)}{u(w_1)-u(w_2)} \geqslant \frac{1+\alpha}{2}\left(\lambda_1\lambda_2^{\frac{\alpha+1}{\alpha}-\frac{\alpha+1}{2\alpha+1}} - \lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) \\ &\Longleftrightarrow \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}\end{aligned}$$

Thus $\lambda_2 e_2^\alpha \geqslant e_1^\alpha$ is satisfied when $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}$.

Therefore the solution

$$e_1^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha+1}{2\alpha+1}}$$

only applies when $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}$.

When the above condition is satisfied,

$$e_1^s - e_2^s \begin{cases} > 0 & \text{when } \lambda_1^{-1}\widetilde{v}_1 > \lambda_2^{-1}\widetilde{v}_2 \\ = 0 & \text{when } \lambda_1^{-1}\widetilde{v}_1 = \lambda_2^{-1}\widetilde{v}_2 \\ < 0 & \text{when } \lambda_1^{-1}\widetilde{v}_1 < \lambda_2^{-1}\widetilde{v}_2 \end{cases}$$

$$\begin{aligned}
&\lambda_1^{-1}\widetilde{v}_1 \gtreqless \lambda_2^{-1}\widetilde{v}_2 \\
\iff &\lambda_1^{-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) \gtreqless \lambda_2^{-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right) \\
\iff &\frac{1+\alpha}{2}\left(\lambda_2^{-\frac{\alpha+1}{2\alpha+1}-1} - \lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\right) \gtreqless \frac{u(w_1)}{u(w_1)-u(w_2)}\left(\lambda_2^{-1}-\lambda_1^{-1}\right) \\
\iff &\frac{\lambda_2^{-\frac{\alpha+1}{2\alpha+1}-1} - \lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}}{\lambda_2^{-1}-\lambda_1^{-1}} \gtreqless \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \\
\iff &\frac{\lambda_2^{-\frac{\alpha+1}{2\alpha+1}-1} - \lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}}{\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2}} \gtreqless \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \\
\iff &\frac{\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\lambda_1 - \lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2}{\lambda_1-\lambda_2} \gtreqless \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \\
\iff &\frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right) \gtreqless \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}
\end{aligned}$$

Therefore,

$$e_1^s - e_2^s \begin{cases} > 0 & \text{when } \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right) \\ = 0 & \text{when } \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} = \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right) \\ < 0 & \text{when } \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} > \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right) \end{cases}$$

Combining the condition $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}$ we can get

$$e_1^s - e_2^s \begin{cases} > 0 & \text{when } \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}} \leqslant \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right) \\ \leqslant 0 & \text{when } \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \max\left[\frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2}, \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}\right] \end{cases}$$

where the first line corresponds to (ii) and the second line corresponds to (iii).

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad \lambda_2 e_2^\alpha \geqslant e_1^\alpha$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \left(1-\frac{e_1^\alpha}{\lambda_2 e_2^\alpha}\right)\widetilde{v}_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^\alpha}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{\lambda_2 e_2^{\alpha+1}}\widetilde{v}_2 - c = 0$

divide the two F.O.C , we get

$$\frac{e_2}{e_1} = \frac{\widetilde{v}_2}{\lambda_1\lambda_2\widetilde{v}_1} < 1$$

which contradicts the condition that $\lambda_1 e_1^\alpha \leqslant e_2^\alpha$

Therefore, the equilibrium in this semifinal is given by:

(1) Proposition A3 (i): when $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \leqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}$

$$e_1^s = \frac{\alpha}{2c}\lambda_1^{\alpha-1}\lambda_2^\alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha}\Delta u$$

$$e_2^s = \frac{\alpha}{2c}\lambda_1^{\alpha}\lambda_2^{\alpha+1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1+\alpha}\Delta u$$

We show that $e_1^s > \overline{e}^s$:

$$e_1^s = \frac{\alpha}{2c}\left(\lambda_1^{-1}\widetilde{v}_1\right)^{1-\alpha}\lambda_2^\alpha(\widetilde{v}_2)^\alpha > \frac{\alpha}{2c}\left(\lambda_2\widetilde{v}_2\right)^{1-\alpha}\lambda_2^\alpha(\widetilde{v}_2)^\alpha = \frac{\alpha}{2c}\lambda_2\widetilde{v}_2 > \frac{\alpha}{2c}\overline{v} = \overline{e}^s$$

The equilibrium winning probabilities are

$$\begin{aligned} p_{21}^s &= \frac{1}{2}\left(\frac{e_2^s}{e_1^s}\right)^\alpha \\ &= \frac{1}{2}\left(\frac{\lambda_2\widetilde{v}_2}{\lambda_1^{-1}\widetilde{v}_1}\right)^\alpha \\ &= \frac{1}{2}\lambda_1^\alpha\lambda_2^\alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha} \end{aligned}$$

$$\begin{aligned} p_{12}^s &= 1 - p_{21}^s \\ &= 1 - \frac{1}{2}\lambda_1^\alpha\lambda_2^\alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha} \end{aligned}$$

$$\widetilde{p}_{12}^{s} = 1 - \frac{1}{2}\frac{(e_2^s)^\alpha}{\lambda_1 (e_1^s)^\alpha}$$
$$= 1 - \frac{1}{2}\lambda_1^{\alpha-1}\lambda_2^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha}$$

$$\widetilde{p}_{21}^{s} = \frac{1}{2}\frac{\lambda_2 (e_2^s)^\alpha}{(e_1^s)^\alpha}$$
$$= \frac{1}{2}\lambda_2\left(\frac{\lambda_2 \widetilde{v}_2}{\lambda_1^{-1}\widetilde{v}_1}\right)^{\alpha}$$
$$= \frac{1}{2}\lambda_1^{\alpha}\lambda_2^{\alpha+1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha}$$

Since $e_1^s > e_2^s$, we get $p_{12}^s > \frac{1}{2} > p_{21}^s$. Since $\lambda_2 (e_2^s)^\alpha \leqslant (e_1^s)^\alpha$, we get $\widetilde{p}_{21}^s \leqslant \frac{1}{2}$.

Thus in Proposition A3 (i) we have

$$\widetilde{p}_{12}^{s} > p_{12}^{s} > \frac{1}{2} \geqslant \widetilde{p}_{21}^{s} > p_{21}^{s}$$

(2) Proposition A3 (ii) and (iii): when $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}$

The equilibrium efforts are

$$e_1^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}}$$
$$\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}\Delta u$$

$$e_2^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}$$
$$\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}}\Delta u$$

and the efforts satisfy

$$e_1^s \begin{cases} > e_2^s > \bar{e}^s & \text{when } \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}} \leqslant \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right) \\ \leqslant e_2^s & \text{when } \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \max\left[\frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2}, \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}\right] \end{cases}$$

① Proposition A3 (ii): when $\frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}} \leqslant \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right)$

We first show that $e_1^s > e_2^s > \bar{e}^s$ is satisfied under ①.

Since $e_2^s = \frac{\alpha}{2c}\left(\lambda_1^{-1}\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}}\left(\lambda_2^{-1}\widetilde{v}_2\right)^{\frac{\alpha+1}{2\alpha+1}}$ and $e_1^s > e_2^s$, if both $\lambda_1^{-1}\widetilde{v}_1 > \overline{v}$ and $\lambda_2^{-1}\widetilde{v}_2 > \overline{v}$ are satisfied then we can get $e_1^s > e_2^s > \overline{e}^s$.

We show that under the condition of $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right)$, both $\lambda_1^{-1}\widetilde{v}_1 > \overline{v}$ and $\lambda_2^{-1}\widetilde{v}_2 > \overline{v}$ are satisfied:

Let

$$f(\lambda_1) = \lambda_1^{-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)$$

$$\begin{aligned} f'(\lambda_1) &= -\lambda_1^{-2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) + \lambda_1^{-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1} \\ &= \lambda_1^{-2}\left[-\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right] \\ &= \lambda_1^{-2}\left[\frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1}+1\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} - \frac{u(w_1)}{u(w_1)-u(w_2)}\right] \end{aligned}$$

Let $g(\lambda_1) = \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1}+1\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} - \frac{u(w_1)}{u(w_1)-u(w_2)}$, we can easily get $g'(\lambda_1) < 0$.

$$g(\lambda_1 = 1) = \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1}+1\right) - \frac{u(w_1)}{u(w_1)-u(w_2)}$$

$$g(\lambda_1 \to \infty) = -\frac{u(w_1)}{u(w_1)-u(w_2)} < 0$$

(a) When $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$

If $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$, then $g(\lambda_1 = 1) < 0$, $g(\lambda_1) \leqslant 0$ is always satisfied and thus $f'(\lambda_1) < 0$ always holds. Therefore $f(\lambda_1) < f(\lambda_2) < f(1) = \overline{v}$. This contradicts the condition $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right)$ since we showed earlier that this condition is equivalent to $f(\lambda_1) > f(\lambda_2)$ and $e_1^s > e_2^s$.

(b) When $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$

If $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$, then $g(\lambda_1 = 1) > 0$ and thus

$$g(\lambda_1) \quad \text{and} \quad f'(\lambda_1) \begin{cases} > 0 & \text{when } \lambda_1 < \left(\frac{\frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1}+1\right)}{\frac{u(w_1)}{u(w_1)-u(w_2)}}\right)^{\frac{2\alpha+1}{\alpha+1}} \\ < 0 & \text{when } \lambda_1 > \left(\frac{\frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1}+1\right)}{\frac{u(w_1)}{u(w_1)-u(w_2)}}\right)^{\frac{2\alpha+1}{\alpha+1}} \end{cases}$$

$$f(\lambda_1) \begin{cases} > \overline{v} & \text{when } \lambda_1 < \hat{\lambda} \\ = \overline{v} & \text{when } \lambda_1 = \hat{\lambda} \\ < \overline{v} & \text{when } \lambda_1 > \hat{\lambda} \end{cases}$$

where $\hat{\lambda}$ is as derived in the Proposition 4.

When $f(\lambda_1) > f(\lambda_2)$ is satisfied, $f(\lambda_1) > f(\lambda_2) > f(1) = f(\hat{\lambda}) = \overline{v}$ must be true.
If $f(\lambda_2) < \overline{v}$, then $\lambda_2 > \hat{\lambda}$. And since $f'(\lambda_1) < 0$ for $\forall\ \lambda_1 > \hat{\lambda}$, $f(\lambda_1) < f(\lambda_2)$ when $f(\lambda_2) < \overline{v}$. This contradicts $f(\lambda_1) > f(\lambda_2)$. Thus when the condition $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right)$ is satisfied, $e_1 > e_2 > \overline{e}^s$ is always true.

The true equilibrium winning probabilities under ① are

$$
\begin{aligned}
p_{21}^s &= \frac{1}{2}\left(\frac{e_2^s}{e_1^s}\right)^\alpha \\
&= \frac{1}{2}\left(\frac{\lambda_2^{-1}\widetilde{v}_2}{\lambda_1^{-1}\widetilde{v}_1}\right)^{\frac{\alpha}{2\alpha+1}} \\
&= \frac{1}{2}\lambda_1^{\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}
\end{aligned}
$$

$$
\begin{aligned}
p_{12}^s &= 1 - p_{21}^s \\
&= 1 - \frac{1}{2}\lambda_1^{\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}
\end{aligned}
$$

Since $e_1^s > e_2^s$, we can get $p_{12}^s > \frac{1}{2} > p_{21}^s$.
Thus we have $\widetilde{p}_{12}^s > p_{12}^s > \frac{1}{2} > p_{21}^s$.

② Proposition A3 (iii): when $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \max\left[\frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2}, \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}\right]$

The true equilibrium winning probabilities are

$$
\begin{aligned}
p_{12}^s &= \frac{1}{2}\left(\frac{e_1^s}{e_2^s}\right)^\alpha \\
&= \frac{1}{2}\left(\frac{\lambda_1^{-1}\widetilde{v}_1}{\lambda_2^{-1}\widetilde{v}_2}\right)^{\frac{\alpha}{2\alpha+1}} \\
&= \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}
\end{aligned}
$$

$$
\begin{aligned}
p_{21}^s &= 1 - p_{12}^s \\
&= 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}
\end{aligned}
$$

Since $e_1^s \leqslant e_2^s$, we can get $p_{21}^s \geqslant \frac{1}{2} \geqslant p_{12}^s$.

The perceived equilibrium winning probabilities under (2) are:

$$
\begin{aligned}
\widetilde{p}_{12}^s &= 1 - \frac{1}{2}\frac{(e_2^s)^\alpha}{\lambda_1 (e_1^s)^\alpha} \\
&= 1 - \frac{1}{2}\lambda_1^{-1}\left(\lambda_1^{\frac{1}{2\alpha+1}}\lambda_2^{-\frac{1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{1}{2\alpha+1}}\right. \\
&\quad \left.\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{1}{2\alpha+1}}\right)^{\alpha} \\
&= 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{p}_{21}^s &= 1 - \frac{1}{2}\frac{(e_1^s)^\alpha}{\lambda_2 (e_2^s)^\alpha} \\
&= 1 - \frac{1}{2}\lambda_2^{-1}\left(\lambda_1^{-\frac{1}{2\alpha+1}}\lambda_2^{\frac{1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{1}{2\alpha+1}}\right. \\
&\quad \left.\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{1}{2\alpha+1}}\right)^{\alpha} \\
&= 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}
\end{aligned}
$$

We show that $\widetilde{p}_{12}^s > \widetilde{p}_{21}^s$:

$$
\begin{aligned}
&\widetilde{p}_{12}^s > \widetilde{p}_{21}^s \\
&\iff 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1)^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha}{2\alpha+1}} > 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(\widetilde{v}_2)^{-\frac{\alpha}{2\alpha+1}} \\
&\iff \lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(\widetilde{v}_2)^{-\frac{\alpha}{2\alpha+1}} > \lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1)^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha}{2\alpha+1}} \\
&\iff \lambda_1^{\frac{1}{2\alpha+1}}\lambda_2^{-\frac{1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{2\alpha}{2\alpha+1}}(\widetilde{v}_2)^{-\frac{2\alpha}{2\alpha+1}} > 1 \\
&\iff \frac{\lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{2\alpha}{2\alpha+1}}}{\lambda_2^{\frac{1}{2\alpha+1}}(\widetilde{v}_2)^{\frac{2\alpha}{2\alpha+1}}} > 1
\end{aligned}
$$

Thus in Proposition A3 (ii) we have $\widetilde{p}_{12}^s > p_{12}^s > \frac{1}{2} > p_{21}^s$ and in Proposition A3 (iii) we have $\widetilde{p}_{12}^s > \widetilde{p}_{21}^s > p_{21}^s \geqslant \frac{1}{2} \geqslant p_{12}^s$.

3. Participation constraints

(1) When $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \leqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}$

$$
\begin{aligned}
\widetilde{E}^s(U_{12}) &= \widetilde{p}_{12}^s\widetilde{v}_1 - ce_1^s \\
&> p_{12}^s\widetilde{v}_1 - ce_1^s \\
&= \left(1-\frac{1}{2}\lambda_1^{\alpha}\lambda_2^{\alpha}\left(\widetilde{v}_1\right)^{-\alpha}\left(\widetilde{v}_2\right)^{\alpha}\right)\widetilde{v}_1 - c\frac{\alpha}{2c}\lambda_1^{\alpha-1}\lambda_2^{\alpha}\left(\widetilde{v}_1\right)^{1-\alpha}\left(\widetilde{v}_2\right)^{\alpha} \\
&= \widetilde{v}_1 - \frac{1}{2}\lambda_1^{\alpha}\lambda_2^{\alpha}\left(\widetilde{v}_1\right)^{1-\alpha}\left(\widetilde{v}_2\right)^{\alpha} - \frac{\alpha}{2}\lambda_1^{\alpha-1}\lambda_2^{\alpha}\left(\widetilde{v}_1\right)^{1-\alpha}\left(\widetilde{v}_2\right)^{\alpha} \\
&> \widetilde{v}_1 - \frac{1}{2}\lambda_1^{\alpha}\lambda_2^{\alpha}\left(\widetilde{v}_1\right)^{1-\alpha}\left(\widetilde{v}_2\right)^{\alpha} - \frac{\alpha}{2}\lambda_1^{\alpha}\lambda_2^{\alpha}\left(\widetilde{v}_1\right)^{1-\alpha}\left(\widetilde{v}_2\right)^{\alpha} \\
&= \widetilde{v}_1 - \frac{1+\alpha}{2}\lambda_1^{\alpha}\lambda_2^{\alpha}\left(\widetilde{v}_1\right)^{1-\alpha}\left(\widetilde{v}_2\right)^{\alpha} \\
&= \widetilde{v}_1\left[1-\frac{1+\alpha}{2}\lambda_1^{\alpha}\lambda_2^{\alpha}\left(\widetilde{v}_1\right)^{-\alpha}\left(\widetilde{v}_2\right)^{\alpha}\right] \\
&= \widetilde{v}_1\left[1-\frac{1+\alpha}{2}\left(\frac{e_2^s}{e_1^s}\right)^{\alpha}\right] \\
&> 0
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{E}^s(U_{21}) &= \widetilde{p}_{21}^s\widetilde{v}_2 - ce_2^s \\
&= \frac{1}{2}\lambda_1^{\alpha}\lambda_2^{\alpha+1}\left(\widetilde{v}_1\right)^{-\alpha}\left(\widetilde{v}_2\right)^{\alpha}\widetilde{v}_2 - c\frac{\alpha}{2c}\lambda_1^{\alpha}\lambda_2^{\alpha+1}\left(\widetilde{v}_1\right)^{-\alpha}\left(\widetilde{v}_2\right)^{\alpha+1} \\
&= \frac{1}{2}\lambda_1^{\alpha}\lambda_2^{\alpha+1}\left(\widetilde{v}_1\right)^{-\alpha}\left(\widetilde{v}_2\right)^{1+\alpha} - \frac{\alpha}{2}\lambda_1^{\alpha}\lambda_2^{\alpha+1}\left(\widetilde{v}_1\right)^{-\alpha}\left(\widetilde{v}_2\right)^{\alpha+1} \\
&= \frac{1-\alpha}{2}\lambda_1^{\alpha}\lambda_2^{\alpha+1}\left(\widetilde{v}_1\right)^{-\alpha}\left(\widetilde{v}_2\right)^{1+\alpha} \\
&\geqslant 0
\end{aligned}
$$

(2) When $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}$

$$
\begin{aligned}
\widetilde{E}^s(U_{12}) &= \widetilde{p}_{12}^s\widetilde{v}_1 - ce_1^s \\
&= \left(1-\frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_2\right)^{\frac{\alpha}{2\alpha+1}}\right)\widetilde{v}_1 - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_2\right)^{\frac{\alpha}{2\alpha+1}} \\
&= \widetilde{v}_1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_2\right)^{\frac{\alpha}{2\alpha+1}} \\
&= \left(\widetilde{v}_1\right)^{\frac{\alpha+1}{2\alpha+1}}\left[\left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_2\right)^{\frac{\alpha}{2\alpha+1}}\right] \\
&> 0
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{E}^s(U_{21}) &= \widetilde{p}_{21}^s \widetilde{v}_1 - c e_2^s \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \lambda_2^{-\frac{\alpha+1}{2\alpha+1}} (\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}} (\widetilde{v}_2)^{-\frac{\alpha}{2\alpha+1}}\right) \widetilde{v}_2 - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \lambda_2^{-\frac{\alpha+1}{2\alpha+1}} (\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}} (\widetilde{v}_2)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \widetilde{v}_2 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \lambda_2^{-\frac{\alpha+1}{2\alpha+1}} (\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}} (\widetilde{v}_2)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \widetilde{v}_2 \left[1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \lambda_2^{-\frac{\alpha+1}{2\alpha+1}} (\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}} (\widetilde{v}_2)^{-\frac{\alpha}{2\alpha+1}}\right] \\
&= \widetilde{v}_2 \left[1 - \frac{1+\alpha}{2} \frac{(e_1^s)^\alpha}{\lambda_2 (e_2^s)^\alpha}\right] \\
&> 0
\end{aligned}
$$

To complete the proof we derive the following four limits:

$$
\lim_{\lambda_1 \to \infty} \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}} = \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}} = \lambda_2^{-\frac{\alpha+1}{2\alpha+1}}
$$

$$
\lim_{\lambda_1 \to \lambda_2} \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}} = \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_2^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_2^{-1}} = \frac{\lambda_2^{-\frac{3\alpha+2}{2\alpha+1}}\left(\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}+\frac{3\alpha+2}{2\alpha+1}} - 1\right)}{\lambda_2^{-1}\left(\lambda_2^{\frac{\alpha+1}{\alpha}+1} - 1\right)} = \lambda_2^{-\frac{\alpha+1}{2\alpha+1}}
$$

$$
\lim_{\lambda_1 \to \infty} \frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1 - \lambda_2} = \lim_{\lambda_1 \to \infty} \frac{\frac{1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}+1}}}{1 - \frac{\lambda_2}{\lambda_1}} = \frac{\lambda_2^{-\frac{\alpha+1}{2\alpha+1}} - 0}{1 - 0} = \lambda_2^{-\frac{\alpha+1}{2\alpha+1}}
$$

Let $t = \lambda_1 - \lambda_2$,

$$
\begin{aligned}
\lim_{\lambda_1 \to \lambda_2} \frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1 - \lambda_2} &= \lim_{t \to 0} \frac{\frac{\lambda_2+t}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{(\lambda_2+t)^{\frac{\alpha+1}{2\alpha+1}}}}{t} \\
&= \lim_{t \to 0} \frac{\frac{\partial\left(\frac{\lambda_2+t}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{(\lambda_2+t)^{\frac{\alpha+1}{2\alpha+1}}}\right)}{\partial t}}{\frac{\partial t}{\partial t}} \\
&= \lim_{t \to 0} \frac{\lambda_2^{-\frac{\alpha+1}{2\alpha+1}} - \lambda_2\left(-\frac{\alpha+1}{2\alpha+1}\right)(\lambda_2+t)^{-\frac{\alpha+1}{2\alpha+1}-1}}{1} \\
&= \left(\frac{3\alpha+2}{2\alpha+1}\right)\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}
\end{aligned}
$$

2.3 Overconfident players seeded in different semifinals

We continue to assume players 1 and 2 are seeded in one semifinal and players 3 and 4 are seeded in the other semifinal. However, we now assume players 1 and 3 are overconfident with $\lambda_1 > \lambda_3 > 1$, and players 2 and 4 are rational with $\lambda_2 = \lambda_4 = 1$.

In the semifinal between players 1 and 2, the perceived expected utilities of reaching the final are

$$\begin{aligned}\widetilde{v}_1 &= p^s_{34}\widetilde{E}^f(U_{13}) + (1-p^s_{34})\widetilde{E}^f(U_{14})\\ &= u(w_1) - \frac{1+\alpha}{2}\left(1 - p^s_{34} + p^s_{34}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1)-u(w_2)],\end{aligned}$$

and

$$\begin{aligned}v_2 &= p^s_{34}E^f(U_{23}) + (1-p^s_{34})E^f(U_{24})\\ &= u(w_1) - \frac{1+\alpha}{2}\left(1 - p^s_{34} + p^s_{34}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)[u(w_1)-u(w_2)].\end{aligned}$$

In the semifinal between players 3 and 4, the perceived expected utilities of reaching the final are

$$\begin{aligned}\widetilde{v}_3 &= p^s_{12}\widetilde{E}^f(U_{31}) + (1-p^s_{12})\widetilde{E}^f(U_{32})\\ &= u(w_1) - \frac{1+\alpha}{2}\left(1 - p^s_{12} + p^s_{12}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1)-u(w_2)],\end{aligned}$$

and

$$\begin{aligned}v_4 &= p^s_{12}E^f(U_{41}) + (1-p^s_{12})E^f(U_{42})\\ &= u(w_1) - \frac{1+\alpha}{2}\left(1 - p^s_{12} + p^s_{12}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)[u(w_1)-u(w_2)].\end{aligned}$$

The four expressions above shows us that the perceived expected utilities of reaching the final of players seeded in one semifinal depend on the equilibrium winning probabilities of players seeded in the other semifinal. As the equilibrium efforts of one semifinal cannot be solved separately from those of the other semifinal, the equilibrium efforts in the semifinals are jointly determined by the four first-order conditions $mg\widetilde{p}^s_{12}\widetilde{v}_1 = c$, $mgp^s_{21}v_2 = c$, $mg\widetilde{p}^s_{34}\widetilde{v}_3 = c$, and $mgp^s_{43}v_4 = c$.

Still, the findings in Proposition 4 can be applied to both semifinals. In other words, we know that in both semifinals there exist parameter configurations where the overconfident player exerts higher effort than the rational player. Our next result shows that this is indeed the case.

Proposition A4 *Consider the semifinals of a two-stage elimination contest where overconfident player 1 and rational player 2 are seeded in one semifinal, overconfident player 3 and rational player 4 are seeded in the other semifinal, and* $\lambda_1 > \lambda_3 > 1 = \lambda_2 = \lambda_4$. *If* $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ *and* $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{1}{\lambda_1-1}\left(\lambda_1 - \lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)\lambda_3^{-\frac{\alpha}{\alpha+1}}$ *and* $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{1}{\lambda_3-1}\left(\lambda_3 - \lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)\lambda_1^{-\frac{\alpha}{\alpha+1}}$, *then the equilibrium efforts and winning probabilities satisfy* $e^s_1 > e^s_2$, $e^s_3 > e^s_4$, $p^s_{12} > 1/2 > p^s_{21}$, *and* $p^s_{34} > 1/2 > p^s_{43}$.

Proposition A4 shows that in an elimination contest where two overconfident players are seeded in different semifinals, the overconfident players can exert higher effort at equilibrium than their rational rivals. This happens when the prize spread is sufficiently large and the overconfident players are not too confident. In this case, each overconfident player has a higher probability of winning his semifinal than his rational rival.

Hence, the results found for an elimination contest with one overconfident player and three rational players also extend to an elimination contest where one overconfident and one rational player are seeded in each semifinal.

Proof of Proposition A4

1. Perceived expected utilities of reaching the final

   Overconfident player 1:

$$\begin{aligned}
\widetilde{v}_1 &= p^s_{34}\widetilde{E}^f(U_{13}) + p^s_{43}\widetilde{E}^f(U_{14}) \\
&= p^s_{34}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)\Delta u \\
&\quad + p^s_{43}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)\Delta u \\
&= \left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p^s_{34} + p^s_{34}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right]\Delta u
\end{aligned}$$

   Rational player 2:

$$\begin{aligned}
v_2 &= p^s_{34}E^f(U_{23}) + p^s_{43}E^f(U_{24}) \\
&= p^s_{34}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)\Delta u \\
&\quad + p^s_{43}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)\Delta u \\
&= \left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p^s_{34} + p^s_{34}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)\right]\Delta u
\end{aligned}$$

   Overconfident player 3:

$$\begin{aligned}
\widetilde{v}_3 &= p^s_{12}\widetilde{E}^f(U_{31}) + p^s_{21}\widetilde{E}^f(U_{32}) \\
&= p^s_{12}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)\Delta u \\
&\quad + p^s_{21}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)\Delta u \\
&= \left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p^s_{12} + p^s_{12}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right]\Delta u
\end{aligned}$$

   Rational player 4:

$$\begin{aligned}
v_4 &= p^s_{12}E^f(U_{41}) + p^s_{21}E^f(U_{42}) \\
&= p^s_{12}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\Delta u \\
&\quad + p^s_{21}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)\Delta u \\
&= \left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p^s_{12} + p^s_{12}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\right]\Delta u
\end{aligned}$$

2. The equilibrium of the semifinal between player 1 and player 2

   Player 1 $\quad max \quad \widetilde{E}^s(U_{12}) = \widetilde{p}^s_{12}\widetilde{v}_1 - ce_1$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_2^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_2^\alpha \end{cases}$$

Player 2 $\quad max \quad E^s(U_{21}) = p_{21}^s v_2 - ce_2$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right) v_2 - ce_2 & \text{if}\, e_2 \geqslant e_1 \\ \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha} v_2 - ce_2 & \text{if}\, e_2 \leqslant e_1 \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \end{cases}$$

Since $\lambda_1 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \frac{1}{2}(\frac{e_2}{e_1})^\alpha v_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_2^{\alpha-1}}{e_1^\alpha}v_2 - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1 < 0$

$[e_2] \quad \frac{\alpha}{2}(\alpha-1)\frac{e_2^{\alpha-2}}{e_1^\alpha}v_2 < 0$

Solve the two F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(v_2)^\alpha$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(v_2)^{\alpha+1}$$

$$\frac{e_2}{e_1} = \lambda_1\frac{v_2}{\widetilde{v}_1}$$

$$p_{21}^s = \frac{1}{2}\left(\frac{e_2}{e_1}\right)^\alpha = \frac{1}{2}\left(\frac{\lambda_1 v_2}{\widetilde{v}_1}\right)^\alpha$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_1 \geqslant e_2$:

As long as $e_1 \geqslant e_2$ is satisfied, $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ is satisfied. So we only have to check $e_1 \geqslant e_2$.

$$
\begin{aligned}
e_2 \leqslant e_1 &\Longleftrightarrow \frac{e_2}{e_1} \leqslant 1 \\
&\Longleftrightarrow \frac{1}{2}\left(\frac{e_2}{e_1}\right)^\alpha \leqslant \frac{1}{2} \\
&\Longleftrightarrow \frac{e_2}{e_1} \leqslant 1 \\
&\Longleftrightarrow \frac{v_2}{\lambda_1^{-1}\widetilde{v}_1} \leqslant 1 \\
&\Longleftrightarrow \frac{\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p_{34}^s + p_{34}^s \lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)}{\lambda_1^{-1}\left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p_{34}^s + p_{34}^s \lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right]} \leqslant 1
\end{aligned}
$$

Let

$$
\begin{aligned}
f(p_{34}^s) = \lambda_1^{-1}&\left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p_{34}^s + p_{34}^s \lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right] \\
&- \left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p_{34}^s + p_{34}^s \lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)\right]
\end{aligned}
$$

Rearrange the terms we can get

$$
\begin{aligned}
f(p_{34}^s) = \lambda_1^{-1}&\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&+ \frac{1+\alpha}{2}\left(\lambda_3^{-\frac{\alpha}{2\alpha+1}} - 1\right)\left(1 - \lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right) p_{34}^s
\end{aligned}
$$

Since $\left(\lambda_3^{-\frac{\alpha}{2\alpha+1}} - 1\right)\left(1 - \lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right) < 0$ and $p_{34}^s \in [0,1]$, $f(p_{34}^s)$ reaches minimum at $p_{34}^s = 1$. Thus, $e_2 \leqslant e_1$ is always satisfied as long as $f(p_{34}^s = 1) \geqslant 0$.

$$
\begin{aligned}
f(p_{34}^s = 1) = \lambda_1^{-1}&\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&+ \frac{1+\alpha}{2}\left(\lambda_3^{-\frac{\alpha}{2\alpha+1}} - 1\right)\left(1 - \lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right)
\end{aligned}
$$

$$\lambda_1^{-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)-\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)$$
$$+\frac{1+\alpha}{2}\left(\lambda_3^{-\frac{\alpha}{2\alpha+1}}-1\right)\left(1-\lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right)\geqslant 0$$
$$\Longleftrightarrow\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)-\lambda_1\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)$$
$$+\lambda_1\frac{1+\alpha}{2}\left(\lambda_3^{-\frac{\alpha}{2\alpha+1}}-1\right)\left(1-\lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right)\geqslant 0$$
$$\Longleftrightarrow\frac{1+\alpha}{2}\left[\lambda_1\left(1+\left(\lambda_3^{-\frac{\alpha}{2\alpha+1}}-1\right)\left(1-\lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right)\right)-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right]\geqslant(\lambda_1-1)\frac{u(w_1)}{u(w_1)-u(w_2)}$$
$$\Longleftrightarrow\frac{1}{\lambda_1-1}\left[\lambda_1\left(\lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}+\lambda_3^{-\frac{\alpha}{2\alpha+1}}-\lambda_3^{-\frac{\alpha}{2\alpha+1}}\lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right)-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right]\geqslant\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}$$
$$\Longleftrightarrow\frac{1}{\lambda_1-1}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\left(\lambda_1-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)\geqslant\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}$$

To ensure $e_1>e_2$ and $p_{12}^s>p_{21}^s$, we need $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}<\frac{1}{\lambda_1-1}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\left(\lambda_1-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)$.

Since $\frac{\lambda_1-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}}{\lambda_1-1}$ decreasing in λ_1 for $\lambda_1>1$ and its limit when $\lambda_1\to 1$ is $\frac{3\alpha+2}{2\alpha+1}$, we can get $\frac{1}{\lambda_1-1}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\left(\lambda_1-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)<\frac{3\alpha+2}{2\alpha+1}$. Hence, to satisfy the inequality $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}<\frac{1}{\lambda_1-1}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\left(\lambda_1-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)$, we also need $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}<\frac{3\alpha+2}{2\alpha+1}$, which is equivalent to $\frac{u(w_1)-u(w_2)}{u(w_2)}>\frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$.

(2) case 2: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2\geqslant e_1$

Player 1 $\quad max \quad \left(1-\frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1-ce_1$

Player 2 $\quad max \quad \left(1-\frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right)v_2-ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1-c=0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_2^{\alpha+1}}v_2-c=0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1<0$

$[e_2] \quad \frac{\alpha}{2}(-\alpha-1)\frac{e_1^\alpha}{e_2^{\alpha+2}}v_2<0$

Solve F.O.C , we get

$$e_1=\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha+1}{2\alpha+1}}(v_2)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(v_2)^{\frac{\alpha+1}{2\alpha+1}}$$

$$\frac{e_2}{e_1} = \lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1)^{-\frac{1}{2\alpha+1}}(v_2)^{\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \geqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

$$\begin{aligned}\lambda_1 e_1^\alpha \geqslant e_2^\alpha &\Longleftrightarrow \frac{\lambda_1 e_1^\alpha}{e_2^\alpha} \geqslant 1\\ &\Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(v_2)^{-\frac{\alpha}{2\alpha+1}} \geqslant 1\end{aligned}$$

Since $\lambda_1 > 1$ and $\widetilde{v}_1 > v_2$, $\lambda_1 e_1^\alpha > e_2^\alpha$ is always satisfied.

② $e_2 \geqslant e_1$

$e_2 \geqslant e_1$ is always satisfied as long as $f(p_{34}^s = 0) \leqslant 0$.

$$f(p_{34}^s = 0) = \lambda_1^{-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)$$

$$\begin{aligned}f(p_{34}^s = 0) \leqslant 0 &\Longleftrightarrow \frac{u(w_1)}{u(w_1)-u(w_2)}\left(\lambda_1^{-1}-1\right) + \frac{1+\alpha}{2}\left(1-\lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right) \leqslant 0\\ &\Longleftrightarrow \frac{1+\alpha}{2}\left(1-\lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right) \leqslant \frac{u(w_1)}{u(w_1)-u(w_2)}\left(1-\lambda_1^{-1}\right)\\ &\Longleftrightarrow \frac{1-\lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}}{1-\lambda_1^{-1}} \leqslant \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}\end{aligned}$$

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \left(1-\frac{1}{2}(\frac{e_1}{e_2})^\alpha\right)v_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^\alpha}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_2^{\alpha+1}}v_2 - c = 0$

divide the two F.O.C , we get

$$\frac{e_2}{e_1} = \frac{v_2}{\lambda_1\widetilde{v}_1} < 1$$

which contradicts the condition that $e_2 \geqslant e_1$

3. The equilibrium between player 3 and player 4

Player 3 $\quad max \quad \widetilde{E}^s(U_{34}) = \widetilde{p}_{34}^s\widetilde{v}_3 - ce_3$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_4^\alpha}{\lambda_3 e_3^\alpha}\right)\widetilde{v}_3 - ce_3 & \text{if} \quad \lambda_3 e_3^\alpha \geqslant e_4^\alpha \\ \frac{1}{2}\frac{\lambda_3 e_3^\alpha}{e_4^\alpha}\widetilde{v}_3 - ce_3 & \text{if} \quad \lambda_3 e_3^\alpha \leqslant e_4^\alpha \end{cases}$$

Player 4 $\quad max \quad E^s(U_{43}) = p_{43}^s v_4 - ce_4$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_3^\alpha}{e_4^\alpha}\right) v_4 - ce_4 & \text{if}\, e_4 \geqslant e_3 \\ \frac{1}{2}\frac{e_4^\alpha}{e_3^\alpha} v_4 - ce_4 & \text{if}\, e_4 \leqslant e_3 \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_3 e_3^\alpha \geqslant e_4^\alpha \quad and \quad e_4 \leqslant e_3 \\ \lambda_3 e_3^\alpha \geqslant e_4^\alpha \quad and \quad e_4 \geqslant e_3 \\ \lambda_3 e_3^\alpha \leqslant e_4^\alpha \quad and \quad e_4 \geqslant e_3 \\ \lambda_3 e_3^\alpha \leqslant e_4^\alpha \quad and \quad e_4 \leqslant e_3 \end{cases}$$

Since $\lambda_3 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_3 e_3^\alpha \geqslant e_4^\alpha \quad and \quad e_4 \leqslant e_3$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_4^\alpha}{\lambda_3 e_3^\alpha}\right)\widetilde{v}_3 - ce_3$

Player 2 $\quad max \quad \frac{1}{2}(\frac{e_4}{e_3})^\alpha v_4 - ce_4$

F.o.c

$[e_3] \quad \frac{\alpha}{2\lambda_3}\frac{e_4^\alpha}{e_3^{\alpha+1}}\widetilde{v}_3 - c = 0$

$[e_4] \quad \frac{\alpha}{2}\frac{e_4^{\alpha-1}}{e_3^\alpha}v_4 - c = 0$

S.o.c

$[e_3] \quad \frac{\alpha}{2\lambda_3}(-\alpha - 1)\frac{e_4^\alpha}{e_3^{\alpha+2}}\widetilde{v}_3 < 0$

$[e_4] \quad \frac{\alpha}{2}(\alpha - 1)\frac{e_4^{\alpha-2}}{e_3^\alpha}v_4 < 0$

Solve the two F.O.C , we get

$$e_3 = \frac{\alpha}{2c}\lambda_3^{\alpha-1}(\widetilde{v}_3)^{1-\alpha}(v_4)^\alpha$$

$$e_4 = \frac{\alpha}{2c}\lambda_3^{\alpha}(\widetilde{v}_3)^{-\alpha}(v_4)^{\alpha+1}$$

$$\frac{e_4}{e_3} = \lambda_3\frac{v_4}{\widetilde{v}_3}$$

$$
\begin{aligned}
p_{43}^s &= \frac{1}{2}\left(\frac{e_4}{e_3}\right)^{\alpha} \\
&= \frac{1}{2}\left(\frac{\lambda_3 v_4}{\widetilde{v}_3}\right)^{\alpha}
\end{aligned}
$$

Check the conditions $\lambda_3 e_3^\alpha \geqslant e_4^\alpha$ and $e_3 \geqslant e_4$:
As long as $e_3 \geqslant e_4$ is satisfied, $\lambda_3 e_3^\alpha \geqslant e_4^\alpha$ is satisfied. So we only have to check $e_3 \geqslant e_4$.

$$
\begin{aligned}
e_4 \leqslant e_3 &\Longleftrightarrow \frac{e_4}{e_3} \leqslant 1 \\
&\Longleftrightarrow \frac{1}{2}\left(\frac{e_4}{e_3}\right)^{\alpha} \leqslant \frac{1}{2} \\
&\Longleftrightarrow p_{43}^s \leqslant \frac{1}{2} \\
&\Longleftrightarrow \frac{v_4}{\lambda_3^{-1}\widetilde{v}_3} \leqslant 1 \\
&\Longleftrightarrow \frac{\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1-p_{12}^s+p_{12}^s\lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)}{\lambda_3^{-1}\left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1-p_{12}^s+p_{12}^s\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right]} \leqslant 1
\end{aligned}
$$

Let

$$
\begin{aligned}
f(p_{12}^s) = \lambda_3^{-1}&\left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1-p_{12}^s+p_{12}^s\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right] \\
&- \left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1-p_{12}^s+p_{12}^s\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\right]
\end{aligned}
$$

Rearrange the terms we can get

$$
\begin{aligned}
f(p_{12}^s) = \lambda_3^{-1}&\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right) - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&+ \frac{1+\alpha}{2}\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}}-1\right)\left(1-\lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right)p_{12}^s
\end{aligned}
$$

Since $\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}}-1\right)\left(1-\lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right) < 0$ and $p_{12}^s \in [0,1]$, $f(p_{12}^s)$ reaches minimum at $p_{12}^s = 1$. Thus, $e_4 \leqslant e_3$ is always satisfied as long as $f(p_{12}^s = 1) \geqslant 0$.

$$
\begin{aligned}
f(p_{12}^s = 1) = \lambda_3^{-1}&\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right) - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&+ \frac{1+\alpha}{2}\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}}-1\right)\left(1-\lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right)
\end{aligned}
$$

$$\lambda_3^{-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)-\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)$$
$$+\frac{1+\alpha}{2}\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}}-1\right)\left(1-\lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right)\geqslant 0$$
$$\Longleftrightarrow\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)-\lambda_3\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)$$
$$+\lambda_3\frac{1+\alpha}{2}\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}}-1\right)\left(1-\lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right)\geqslant 0$$
$$\Longleftrightarrow\frac{1+\alpha}{2}\left[\lambda_3\left(1-\left(1-\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\left(1-\lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right)\right)-\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right]\geqslant(\lambda_3-1)\frac{u(w_1)}{u(w_1)-u(w_2)}$$
$$\Longleftrightarrow\frac{1}{\lambda_3-1}\left[\lambda_3\left(\lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}+\lambda_1^{-\frac{\alpha}{2\alpha+1}}-\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right)-\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right]\geqslant\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}$$
$$\Longleftrightarrow\frac{1}{\lambda_3-1}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\lambda_3-\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)\geqslant\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}$$

To ensure $e_3>e_4$ and $p^s_{34}>p^s_{43}$, we need $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}<\frac{1}{\lambda_3-1}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\lambda_3-\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)$. Similar to the equilibrium in the semifinal between player 1 and player 2, we also need $\frac{u(w_1)-u(w_2)}{u(w_2)}>\frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$.

(2) case 2: $\lambda_3 e_3^\alpha\geqslant e_4^\alpha \quad and \quad e_4\geqslant e_3$

Player 1 $\quad max \quad \left(1-\frac{1}{2}\frac{e_4^\alpha}{\lambda_3 e_3^\alpha}\right)\widetilde{v}_3-ce_3$

Player 2 $\quad max \quad \left(1-\frac{1}{2}\frac{e_3^\alpha}{e_4^\alpha}\right)v_4-ce_4$

F.o.c

$[e_3] \quad \frac{\alpha}{2\lambda_3}\frac{e_4^\alpha}{e_3^{\alpha+1}}\widetilde{v}_3-c=0$

$[e_4] \quad \frac{\alpha}{2}\frac{e_3^\alpha}{e_4^{\alpha+1}}v_4-c=0$

S.o.c

$[e_3] \quad \frac{\alpha}{2\lambda_3}(-\alpha-1)\frac{e_4^\alpha}{e_3^{\alpha+2}}\widetilde{v}_3<0$

$[e_4] \quad \frac{\alpha}{2}(-\alpha-1)\frac{e_3^\alpha}{e_4^{\alpha+2}}v_4<0$

Solve F.O.C , we get

$$e_3=\frac{\alpha}{2c}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_3)^{\frac{\alpha+1}{2\alpha+1}}(v_4)^{\frac{\alpha}{2\alpha+1}}$$
$$e_4=\frac{\alpha}{2c}\lambda_3^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_3)^{\frac{\alpha}{2\alpha+1}}(v_4)^{\frac{\alpha+1}{2\alpha+1}}$$
$$\frac{e_4}{e_3}=\lambda_3^{\frac{1}{2\alpha+1}}(\widetilde{v}_3)^{-\frac{1}{2\alpha+1}}(v_4)^{\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_3 e_3^\alpha\geqslant e_4^\alpha$ and $e_4\geqslant e_3$:

① $\lambda_3 e_3^\alpha \geqslant e_4^\alpha$

$$\lambda_3 e_3^\alpha \geqslant e_4^\alpha \iff \frac{\lambda_3 e_3^\alpha}{e_4^\alpha} \geqslant 1 \iff \lambda_3^{\frac{\alpha+1}{2\alpha+1}} (\widetilde{v}_3)^{\frac{\alpha}{2\alpha+1}} (v_4)^{-\frac{\alpha}{2\alpha+1}} \geqslant 1$$

Since $\lambda_3 > 1$ and $\widetilde{v}_3 > v_4$, $\lambda_3 e_3^\alpha > e_4^\alpha$ is always satisfied.

② $e_4 \geqslant e_3$

$e_2 \geqslant e_1$ is always satisfied as long as $f(p_{12}^s = 0) \leqslant 0$.

$$f(p_{12}^s = 0) = \lambda_3^{-1} \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \lambda_3^{-\frac{\alpha+1}{2\alpha+1}} \right) - \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \right)$$

$$\begin{aligned} f(p_{12}^s = 0) \leqslant 0 &\iff \frac{u(w_1)}{u(w_1) - u(w_2)} \left(\lambda_3^{-1} - 1 \right) + \frac{1+\alpha}{2} \left(1 - \lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}} \right) \leqslant 0 \\ &\iff \frac{1+\alpha}{2} \left(1 - \lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}} \right) \leqslant \frac{u(w_1)}{u(w_1) - u(w_2)} \left(1 - \lambda_3^{-1} \right) \\ &\iff \frac{1 - \lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}}{1 - \lambda_3^{-1}} \leqslant \frac{u(w_1)}{u(w_1) - u(w_2)} \frac{2}{1+\alpha} \end{aligned}$$

(3) case 3: $\lambda_3 e_3^\alpha \leqslant e_4^\alpha$ *and* $e_4 \geqslant e_3$

Player 1 $\quad max \quad \frac{1}{2} \frac{\lambda_3 e_3^\alpha}{e_4^\alpha} \widetilde{v}_3 - c e_3$

Player 2 $\quad max \quad \left(1 - \frac{1}{2} (\frac{e_3}{e_4})^\alpha \right) v_4 - c e_4$

F.o.c

$[e_3] \quad \frac{\alpha \lambda_3}{2} \frac{e_3^{\alpha-1}}{e_4^\alpha} \widetilde{v}_3 - c = 0$

$[e_4] \quad \frac{\alpha}{2} \frac{e_3^\alpha}{e_4^{\alpha+1}} v_4 - c = 0$

divide the two F.O.C , we get

$$\frac{e_4}{e_3} = \frac{v_4}{\lambda_3 \widetilde{v}_3} < 1$$

which contradicts the condition that $e_4 \geqslant e_3$.

3 Elimination Contest with Unobservable Overconfidence

This section shows that our results also hold when the overconfident player's rivals cannot observe his bias. As in the paper, we assume player 1 is overconfident and players 2, 3, and 4 are rational with $\lambda_1 > 1 = \lambda_2 = \lambda_3 = \lambda_4$. Players 1 and 2 are paired in one semifinal and players 3 and 4 are paired in the other semifinal. The overconfident player's bias is not observable by the rational players.

3.1 Final

Proposition A5 *In a final between an overconfident player and a rational player where the overconfident player's bias is not observable by the rational player, the equilibrium* *effort of the overconfident player is*

$$e_1^f = \frac{\alpha}{2c}\lambda_1^{-\frac{1}{\alpha+1}}\Delta u,$$

and the equilibrium effort of the rational player is

$$e_3^f = \frac{\alpha}{2c}\Delta u.$$

with $e_1^f < e_3^f = \overline{e}^f$*. The perceived equilibrium winning probabilities are*

$$\widetilde{p}_{13}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{1}{\alpha+1}},$$

$$\widetilde{p}_{31}^f = \frac{1}{2}$$

and the true equilibrium winning probabilities are

$$p_{13}^f = \frac{1}{2}\lambda_1^{-\frac{\alpha}{\alpha+1}}$$

$$p_{31}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{\alpha+1}}$$

with $\widetilde{p}_{13}^f > p_{31}^f > 1/2 = \widetilde{p}_{31}^f > p_{13}^f$*. The perceived equilibrium expected utilities are*

$$\widetilde{E}^f(U_{13}) = \left(1 + \frac{u(w_2)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)\Delta u,$$

$$\widetilde{E}^f(U_{31}) = \left(1 + \frac{u(w_2)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\Delta u$$

with $\widetilde{E}^f(U_{13}) > \widetilde{E}^f(U_{31}) = \overline{E}^f(U)$*.*

Proof of Proposition A5

Since the rational player 3 is unaware that player 1 is overconfident, she chooses the benchmark effort $e_3^f = \overline{e}^f$. The overconfident player 1 chooses a best response to $e_3^f = \overline{e}^f$. Assume the equilibrium satisfies $\lambda_1(e_1^f)^\alpha \geqslant (e_3^f)^\alpha$. In this case, the best response to $e_3^f = \overline{e}^f$ is the solution to

$$\frac{\alpha}{2\lambda_1}\frac{(\overline{e}^f)^\alpha}{(e_1^f)^{\alpha+1}}\Delta u = c.$$

Substituting $\overline{e}^f$ by $\frac{\alpha}{2c}\Delta u$ and solving for e_1^f we have

$$\widetilde{e}_1^f = \frac{\alpha}{2c}\lambda_1^{-\frac{1}{\alpha+1}}\Delta u.$$

Note that this solution satisfies $\lambda_1(e_1^f)^\alpha \geqslant (e_3^f)^\alpha$ since

$$(\lambda_1)^{\frac{1}{\alpha}}\widetilde{e}_1^f = (\lambda_1)^{\frac{1}{\alpha}}\frac{\alpha}{2c}\lambda_1^{-\frac{1}{\alpha+1}}\Delta u = \frac{\alpha}{2c}\lambda_1^{\frac{1}{\alpha(\alpha+1)}}\Delta u > \frac{\alpha}{2c}\Delta u = \overline{e}^f.$$

Now, assume the equilibrium satisfies $\lambda_1(e_1^f)^\alpha \leqslant (e_3^f)^\alpha$. In this case, the best response to $e_3^f = \overline{e}^f$ is the solution to

$$\frac{\alpha\lambda_1}{2}\frac{(e_1^f)^{\alpha-1}}{(\overline{e}^f)^\alpha}\Delta u = c.$$

Substituting $\overline{e}^f$ by $\frac{\alpha}{2c}\Delta u$ and solving for e_1^f we have

$$\widetilde{e}_1^f = \frac{\alpha}{2c}\lambda_1^{\frac{1}{1-\alpha}}\Delta u.$$

This is not a feasible solution since it fails to satisfy $\lambda_1(e_1^f)^\alpha \leqslant (e_3^f)^\alpha$. Hence, player 1' equilibrium effort is

$$\widetilde{e}_1^f = \frac{\alpha}{2c}\lambda_1^{-\frac{1}{\alpha+1}}\Delta u.$$

Therefore, player 1's perceived winning probability is

$$\widetilde{p}_{13}^f = 1 - \frac{1}{2}\frac{(\overline{e}^f)^\alpha}{\lambda_1(\widetilde{e}_1^f)^\alpha} = 1 - \frac{1}{2}\frac{\left(\frac{\alpha}{2c}\Delta u\right)^\alpha}{\lambda_1\left(\frac{\alpha}{2c}\lambda_1^{-\frac{1}{\alpha+1}}\Delta u\right)^\alpha} = 1 - \frac{1}{2}\lambda_1^{-\frac{1}{\alpha+1}} > \frac{1}{2}.$$

Player 3's perceived winning probability is $\widetilde{p}_{31}^f = 1/2$ since she thinks, mistakenly, player 1 is rational. Player 1's true winning probability is

$$p_{13}^f = \frac{1}{2}\frac{(\widetilde{e}_1^f)^\alpha}{(\overline{e}^f)^\alpha} = \frac{1}{2}\frac{\left(\frac{\alpha}{2c}\lambda_1^{-\frac{1}{\alpha+1}}\Delta u\right)^\alpha}{\left(\frac{\alpha}{2c}\Delta u\right)^\alpha} = \frac{1}{2}\lambda_1^{-\frac{\alpha}{\alpha+1}} < \frac{1}{2}.$$

Player 3's true winning probability is

$$p_{31}^f = 1 - p_{13}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{\alpha+1}} > \frac{1}{2}.$$

The perceived expected utility of player 1 is

$$\widetilde{E}^f(U_{13}) = \widetilde{p}_{13}^f\Delta u - c\widetilde{e}_1^f + u(w_2) = \left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)\Delta u.$$

The perceived expected utility of player 3 is

$$\widetilde{E}^f(U_{31}) = \widetilde{p}_{31}^f\Delta u - c\overline{e}^f + u(w_2) = \left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)\Delta u.$$

3.2 Semifinals

Proposition A6 *Consider a semifinal between an overconfident player and a rational player of a two-stage elimination contest where player 1 is overconfident, players 2, 3 and 4 are rational, and the overconfident player's bias is not observable by the rational players.*
(i) If $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2}{\alpha}$ *and* $\lambda_1 < \underline{\lambda}$ *where* $\underline{\lambda}$ *solves* $\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\underline{\lambda}^{-\frac{1}{\alpha+1}}\right) = \underline{\lambda}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)$*, then the equilibrium efforts and winning probabilities satisfy* $\widetilde{e}_1^s > \widetilde{e}_2^s = \overline{e}^s$ *and* $\widetilde{p}_{12}^s > p_{12}^s > 1/2 = \widetilde{p}_{21}^s > p_{21}^s$.
(ii) If either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2}{\alpha}$ *or* $\lambda_1 \geqslant \underline{\lambda}$*, then the equilibrium efforts and winning probabilities satisfy* $\widetilde{e}_1^s \leqslant \widetilde{e}_2^s = \overline{e}^s$ *and* $\widetilde{p}_{12}^s > p_{21}^s \geqslant \frac{1}{2} = \widetilde{p}_{21}^s \geqslant p_{12}^s$.

Proof of Proposition A6

The perceived expected utility of reaching the final of the overconfident player 1 is

$$\widetilde{v}_1 = p^f_{34}\widetilde{E}^f(U_{13}) + p^f_{43}\widetilde{E}^f(U_{14}) = \widetilde{E}^f(U_{13}) = \left(1 + \frac{u(w_2)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)\Delta u > \overline{v}$$

The expected utility of reaching the final of the rational player 2 is

$$v_2 = p^f_{34}E^f(U_{23}) + p^f_{43}E^f(U_{24}) = E^f(U_{23}) = \left(1 + \frac{u(w_2)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\Delta u = \overline{v}$$

Since the rational player 2 is unaware that player 1 is overconfident, she chooses the benchmark effort $e^s_2 = \overline{e}^s$. The overconfident player 1 chooses a best response to $e^s_2 = \overline{e}^s$. Assume the equilibrium (e^s_1, e^s_2) satisfies $\lambda_1(e^s_1)^\alpha \geqslant (e^s_2)^\alpha$. In this case, the best response to $e^s_2 = \overline{e}^s$ is the solution to

$$\frac{\alpha}{2\lambda_1}\frac{(\overline{e}^s)^\alpha}{(e^s_1)^{\alpha+1}}\widetilde{v}_1 = c.$$

Substituting $\overline{e}^s$ by $\frac{\alpha}{2c}\overline{v}$ and solving for e^s_1 we have

$$\widetilde{e}^s_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{1}{1+\alpha}}\left(\widetilde{v}_1\right)^{\frac{1}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}}.$$

Note that this solution satisfies $\lambda_1(e^s_1)^\alpha \geqslant (e^s_2)^\alpha$ since

$$(\lambda_1)^{\frac{1}{\alpha}}\widetilde{e}^s_1 = (\lambda_1)^{\frac{1}{\alpha}}\frac{\alpha}{2c}\lambda_1^{-\frac{1}{1+\alpha}}\left(\widetilde{v}_1\right)^{\frac{1}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}} = \frac{\alpha}{2c}\lambda_1^{\frac{1}{\alpha(\alpha+1)}}\left(\widetilde{v}_1\right)^{\frac{1}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}} > \frac{\alpha}{2c}\overline{v} = e^s_2.$$

$$\widetilde{e}^s_1 \gtreqless \overline{e}^s \Longleftrightarrow \lambda_1^{-1}\widetilde{v}_1 \gtreqless \overline{v}$$
$$\Longleftrightarrow \lambda_1^{-1}\left(1 + \frac{u(w_2)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right) - \left(1 + \frac{u(w_2)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) \gtreqless 0$$

Thus

$$\widetilde{e}^s_1 \begin{cases} > \overline{e}^s & \text{if} \quad \lambda_1 < \underline{\lambda} \quad \text{and} \quad \frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2}{\alpha} \\ \leqslant \overline{e}^s & \text{if} \quad \text{either} \quad \lambda_1 \geqslant \underline{\lambda} \quad \text{or} \quad \frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2}{\alpha} \end{cases}$$

where $\underline{\lambda}$ solves $\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\underline{\lambda}^{-\frac{1}{\alpha+1}}\right) = \underline{\lambda}\left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)$.

Now, assume the equilibrium (e^s_1, e^s_2) satisfies $\lambda_1(e^s_1)^\alpha \leqslant (e^s_2)^\alpha$. In this case, the best response to $e^s_2 = \overline{e}^s$ is the solution to

$$\frac{\alpha\lambda_1}{2}\frac{(e^s_1)^{\alpha-1}}{(\overline{e}^s)^\alpha}\widetilde{v}_1 = c.$$

Substituting $\overline{e}^s$ by $\frac{\alpha}{2c}\overline{v}$ and solving for e^s_1 we have

$$\widetilde{e}^s_1 = \frac{\alpha}{2c}\lambda_1^{\frac{1}{1-\alpha}}\left(\widetilde{v}_1\right)^{\frac{1}{1-\alpha}}\left(\overline{v}\right)^{-\frac{\alpha}{1-\alpha}}.$$

This is not a feasible solution since it fails to satisfy $\lambda_1(e^s_1)^\alpha \leqslant (e^s_2)^\alpha$. Hence, player 1's equilibrium effort is

$$\widetilde{e}^s_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{1}{1+\alpha}}\left(\widetilde{v}_1\right)^{\frac{1}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}}.$$

Therefore, player 1's perceived winning probability is

$$\widetilde{p}_{12}^{s} = 1 - \frac{1}{2}\frac{(\overline{e}^{s})^{\alpha}}{\lambda_1(\widetilde{e}_1^{s})^{\alpha}} = 1 - \frac{1}{2}\lambda_1^{-\frac{1}{\alpha+1}}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}} > \frac{1}{2}.$$

Player 2's perceived winning probability is $\widetilde{p}_{21}^{s} = 1/2$ since she thinks, mistakenly, player 1 is rational. Player 1's winning probability is

$$p_{12}^{s} = \begin{cases} 1 - \frac{1}{2}\left(\frac{\overline{e}^{s}}{\widetilde{e}_1^{s}}\right)^{\alpha} = 1 - \frac{1}{2}\lambda_1^{\frac{\alpha}{\alpha+1}}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}} > \frac{1}{2} & \text{if} \quad \lambda_1 < \underline{\lambda} \quad \text{and} \quad \frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2}{\alpha} \\ \frac{1}{2}\left(\frac{\widetilde{e}_1^{s}}{\overline{e}^{s}}\right)^{\alpha} = \frac{1}{2}\lambda_1^{-\frac{\alpha}{\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha}{1+\alpha}}\left(\overline{v}\right)^{-\frac{\alpha}{1+\alpha}} \leqslant \frac{1}{2} & \text{if} \quad \text{either} \quad \lambda_1 \geqslant \underline{\lambda} \quad \text{or} \quad \frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2}{\alpha} \end{cases}$$

Player 2's winning probability is

$$p_{21}^{s} = \begin{cases} \frac{1}{2}\lambda_1^{\frac{\alpha}{\alpha+1}}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}} < \frac{1}{2} & \text{if} \quad \lambda_1 < \underline{\lambda} \quad \text{and} \quad \frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2}{\alpha} \\ 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha}{1+\alpha}}\left(\overline{v}\right)^{-\frac{\alpha}{1+\alpha}} \geqslant \frac{1}{2} & \text{if} \quad \text{either} \quad \lambda_1 \geqslant \underline{\lambda} \quad \text{or} \quad \frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2}{\alpha} \end{cases}$$

The perceived expected utility of player 1 is

$$\widetilde{E}^{s}(U_{12}) = \widetilde{p}_{12}^{s}\widetilde{v}_1 - c\widetilde{e}_1^{s} = \widetilde{v}_1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{1}{1+\alpha}}\left(\widetilde{v}_1\right)^{\frac{1}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}}.$$

The perceived expected utility of player 2 is

$$\widetilde{E}^{s}(U_{21}) = \widetilde{p}_{21}^{s}\overline{v} - c\overline{e}^{s} = \frac{1-\alpha}{2}\overline{v}.$$

3.3 Equilibrium Winning Probabilities

Proposition A7 *In a two-stage elimination contest where player 1 is overconfident, players 2, 3, and 4 are rational, and the overconfident player's bias is not observable by the rational players, if $\alpha > \frac{3}{5}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{10}{5\alpha-3}$, then there exist $\lambda_1 \in (1, \underline{\lambda})$ for which the overconfident player has the highest equilibrium winning probability, i.e., $P_1 > P_3 = P_4 > 1/4 > P_2$.*

Proof of Proposition A7

$$\mathbf{P}_1 = p_{13}^{f}p_{12}^{s}$$

$$\mathbf{P}_2 = \frac{1}{2}p_{21}^{s}$$

$$\mathbf{P}_3 = \mathbf{P}_4 = p_{12}^{s}p_{31}^{f}p_{34}^{s} + p_{21}^{s}p_{32}^{f}p_{34}^{s} = p_{12}^{s}\left(p_{31}^{f}\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{4} > \frac{1}{4}$$

Since $p_{13}^{f} < \frac{1}{2}$, a necessary condition for $P_1 > P_3$ is $p_{12}^{s} > \frac{1}{2}$. Thus, from Proposition A6, for $p_{12}^{s} > \frac{1}{2}$ we must have that $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2}{\alpha}$ and $\lambda_1 < \underline{\lambda}$. The restrictions on the parameters in the statement of Proposition 7 satisfy these conditions since $\frac{10}{5\alpha-3} > \frac{2}{\alpha}$.

$$\mathbf{P}_1 - \mathbf{P}_3 = \frac{3}{2}p_{13}^{f}p_{12}^{s} - \frac{1}{4}p_{12}^{s} - \frac{1}{4}$$

The sign of $\mathbf{P}_1-\mathbf{P}_3$ is the same as the sign of $6p^f_{13}p^s_{12}-p^s_{12}-1$. Let $f(\lambda_1)=6p^f_{13}p^s_{12}-p^s_{12}-1$

$$
\begin{aligned}
f(\lambda_1) &= 6\frac{1}{2}\lambda_1^{-\frac{\alpha}{\alpha+1}}\left[1-\frac{1}{2}\lambda_1^{\frac{\alpha}{\alpha+1}}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}}\right]-\left[1-\frac{1}{2}\lambda_1^{\frac{\alpha}{\alpha+1}}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}}\right]-1\\
&= 3\lambda_1^{-\frac{\alpha}{\alpha+1}}-\frac{3}{2}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}}+\frac{1}{2}\lambda_1^{\frac{\alpha}{\alpha+1}}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{1+\alpha}}\left(\overline{v}\right)^{\frac{\alpha}{1+\alpha}}-2\\
&= 3\lambda_1^{-\frac{\alpha}{\alpha+1}}-\frac{3}{2}\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)^{-\frac{\alpha}{1+\alpha}}\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\frac{\alpha}{1+\alpha}}\\
&\quad+\frac{1}{2}\lambda_1^{\frac{\alpha}{\alpha+1}}\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)^{-\frac{\alpha}{1+\alpha}}\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\frac{\alpha}{1+\alpha}}\\
&\quad-2
\end{aligned}
$$

We can get that

$$f(\lambda_1=1)=0.$$

$$
\begin{aligned}
f'(\lambda_1) &= -3\frac{\alpha}{\alpha+1}\lambda_1^{-\frac{\alpha}{\alpha+1}-1}\\
&\quad-\frac{3}{2}\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\frac{\alpha}{1+\alpha}}\left(-\frac{\alpha}{1+\alpha}\right)\\
&\quad\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)^{-\frac{\alpha}{1+\alpha}-1}\frac{1+\alpha}{2}\frac{1}{\alpha+1}\lambda_1^{-\frac{1}{\alpha+1}-1}\\
&\quad+\frac{1}{2}\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\frac{\alpha}{1+\alpha}}\\
&\quad\left[\frac{\alpha}{\alpha+1}\lambda_1^{\frac{\alpha}{\alpha+1}-1}\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)^{-\frac{\alpha}{1+\alpha}}\right.\\
&\quad\left.+\lambda_1^{\frac{\alpha}{\alpha+1}}\left(-\frac{\alpha}{1+\alpha}\right)\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1}{\alpha+1}}\right)^{-\frac{\alpha}{1+\alpha}-1}\frac{1+\alpha}{2}\frac{1}{1+\alpha}\lambda_1^{-\frac{1}{\alpha+1}-1}\right]
\end{aligned}
$$

$$
\begin{aligned}
f'(\lambda_1=1) &= -3\frac{\alpha}{\alpha+1}+\frac{3}{4}\frac{\alpha}{1+\alpha}\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{-1}+\frac{1}{2}\frac{\alpha}{\alpha+1}\\
&\quad-\frac{1}{4}\frac{\alpha}{\alpha+1}\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{-1}\\
&= -\frac{5}{2}\frac{\alpha}{\alpha+1}+\frac{1}{2}\frac{\alpha}{\alpha+1}\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{-1}
\end{aligned}
$$

$f'(\lambda_1=1)>0$ when $\alpha>\frac{3}{5}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)}>\frac{10}{5\alpha-3}$. Thus there exist $\lambda_1\in(1,\underline{\lambda})$ for which $P_1>P_3$ is satisfied when $\alpha>\frac{3}{5}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)}>\frac{10}{5\alpha-3}$.

4 Three-Stage Elimination Contest

This section shows that our main results extend to a three-stage elimination contest. To do that we consider an elimination contest where eight players compete in four quarterfinals in the first stage, the four first-stage winners compete in two semifinals in the second stage, and the two second-stage winners compete in the final. The winner gets w_1, the runner-up w_2, the two second-stage losers w_3, and the four third-stage losers w_4, where $w_1 > w_2 > w_3 \geqslant w_4 = 0$. In addition, we assume an increasing utility spread as the players move up in the elimination contest, that is, $u(w_1) - u(w_2) > u(w_2) - u(w_3) > u(w_3) - u(w_4)$. Furthermore, we assume that in the top half of the contest, players 1 and 2 are seeded in one quarterfinal, and players 3 and 4 are seeded in the other quarterfinal. Finally, we assume that in the bottom half of the contest, players 5 and 6 are seeded in one quarterfinal, and players 7 and 8 are seeded in the other quarterfinal.

We start by analyzing a three-stage elimination contest with eight rational players. Next, we analyze a three-stage elimination contest with one overconfident player and seven rational players. In both cases we do not analyze the final since it is identical to a final of a two-stage contest. Hence, we solve the three-stage contest backwards, starting with the semifinals and ending with the quarterfinals.

4.1 Eight rational players

Lemma A1 *In a semifinal of a three-stage elimination contest with eight rational players, the equilibrium effort is*

$$\overline{e}^s = \frac{\alpha}{2c}\left(\frac{1-\alpha}{2} + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)}\right)\Delta u$$

and the equilibrium expected utility is

$$\overline{E}^s(U) = \left[\frac{1-\alpha}{2}\left(\frac{1-\alpha}{2} + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right][u(w_1) - u(w_2)]$$

Proof of Lemma A1

Since players are identical, we assume players 1 and 3 meet in the top half semifinal and players 5 and 7 meet in the bottom half semifinal. Moreover, we also assume, without loss of generality, player 5 beats 7. $E^s(U_{13})$ can be written as:

$$E^s(U_{13}) = p_{13}^s v_1^s + (1 - p_{13}^s)\, u(w_3) - ce_1^s = p_{13}^s\left(v_1^s - u(w_3)\right) + u(w_3) - ce_1^s$$

Player 1's expected utility of reaching the final of a three-stage elimination contest is the same as that in two-stage:

$$v_1^s = p_{15}^f[u(w_1) - u(w_2)] + u(w_2) - ce_1^f$$

From Proposition 2 we know that $e_1^f = \overline{e}^f = \frac{\alpha}{2c}[u(w_1) - u(w_2)]$, $p_{15}^f = \frac{1}{2}$. Plug these values into the equation above we get

$$v_1^s = \overline{v}^s = \frac{1-\alpha}{2}[u(w_1) - u(w_2)] + u(w_2)$$

Similar to the proof of Proposition 2, we get that the equilibrium effort is

$$\begin{aligned}
e_1^s = \overline{e}^s &= \frac{\alpha}{2c}\left(v_1^s - u(w_3)\right) \\
&= \frac{\alpha}{2c}\left(\frac{1-\alpha}{2}[u(w_1) - u(w_2)] + u(w_2) - u(w_3)\right) \\
&= \frac{\alpha}{2c}\left(\frac{1-\alpha}{2} + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)}\right)[u(w_1) - u(w_2)]
\end{aligned}$$

Due to symmetry, all the rational players exert effort $\overline{e}^s$ at equilibrium.

The equilibrium winning probabilities are

$$\overline{p}^s = p^s_{13} = p^s_{31} = p^s_{57} = p^s_{75} = \frac{1}{2}$$

The equilibrium expected utility is

$$\overline{E}^s(U) = \left[\frac{1-\alpha}{2}\left(\frac{1-\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)}\right][u(w_1)-u(w_2)]$$

Now we move on to the quarterfinals.

Proposition A8 *In a quarterfinal of a three-stage elimination contest with eight rational players, the equilibrium effort is*

$$\overline{e}^q = \frac{\alpha}{2c}\overline{v}^q = \frac{\alpha}{2c}\left[\frac{1-\alpha}{2}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)}\right]\Delta u,$$

and the equilibrium expected utility is

$$\overline{E}^q(U) = \frac{1-\alpha}{2}\left[\frac{1-\alpha}{2}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)}\right]\Delta u.$$

Proof of Proposition A8

1. Expected utility of reaching the semifinal

   $E^q(U_{12})$ denotes player 1's expected utility in the quarterfinal when he plays against player 2, and v^q_1 denotes his expected utility of reaching the semifinal. We can get

   $$E^q(U_{12}) = p^q_{12}v^q_1 + (1-p^q_{12}) \times 0 - ce^q_1$$

   where

   $$v^q_1 = E^s(U_{13}) = \overline{v}^q = \left[\frac{1-\alpha}{2}\left(\frac{1-\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)}\right][u(w_1)-u(w_2)]$$

2. The equilibrium efforts and winning probabilities

   Similar to the proof of Proposition 2

   $$\overline{e}^q = \frac{\alpha}{2c}\overline{v}^q = \frac{\alpha}{2c}\left[\frac{1-\alpha}{2}\left(\frac{1-\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)}\right][u(w_1)-u(w_2)]$$

   $$\overline{p}^q = p^q_{12} = p^q_{21} = p^q_{34} = p^q_{43} = p^q_{56} = p^q_{65} = p^q_{78} = p^q_{87} = \frac{1}{2}$$

3. Expected utility of the quarterfinal

   $$\overline{E}^q(U) = \frac{1}{2}\overline{v}^q - c\frac{\alpha}{2c}\overline{v}^q = \frac{1-\alpha}{2}\left[\frac{1-\alpha}{2}\left(\frac{1-\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)}\right]\Delta u$$

   Since $0 < \alpha \leqslant 1$ and $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > 1$, we have $\overline{E}^q(U) \geqslant 0$. The participation constraints are satisfied.

4.2 One Overconfident Player and Seven Rational Players

We now show that the results for a two-stage elimination contest with one overconfident player and three rational players generalize to a three-stage elimination contest with one overconfident player and seven rational players. We set player 1 as the overconfident player with $\lambda_1 > \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 1$. We first characterize the equilibrium in the semifinal between the overconfident player 1 and rational player 3.

Lemma A2 *Consider the semifinal between an overconfident player and a rational player of a three-stage elimination contest with eight players where player 1 is overconfident and the other seven players are rational.*
(i) If $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ *and* $\lambda_1 < \hat{\lambda}$ *where* $\hat{\lambda} > 1$ *is given by* $\frac{1+\alpha}{2}\left(1 + \frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)}\right) = \frac{\hat{\lambda}-1}{\hat{\lambda}-\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}}$, *then the equilibrium efforts and winning probabilities satisfy* $e_1^s > \overline{e}^s > e_3^s$ *and* $\widetilde{p}_{13}^s > p_{13}^s > 1/2 > p_{31}^s$.
(ii) If either $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ *or* $\lambda_1 \geqslant \hat{\lambda}$, *then the equilibrium efforts and winning probabilities satisfy* $e_1^s \leqslant e_3^s \leqslant \overline{e}^s$ *and* $\widetilde{p}_{13}^s > p_{31}^s \geqslant 1/2 \geqslant p_{13}^s$.

Proof of Lemma A2

Since the seven rational players are identical, we assume that player 1 meets 3 in the semifinal and that player 5 enters the final.

1. Perceived expected utilities of reaching the final

   Overconfident player 1:

   Player 1's perceived expected utility of reaching the final of a three-stage elimination contest is the same as that of a two-stage

$$\widetilde{v}_1^s = \widetilde{p}_{15}^f \Delta u + u(w_2) - ce_1^f = \left(1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)\Delta u + u(w_2)$$

   Rational player 3:

   Since player 3 will meet a rational player in the final, her expected utility of reaching the final is the benchmark

$$v_3^s = \overline{v}^s = \frac{1-\alpha}{2}\Delta u + u(w_2)$$

   We can easily get

$$\widetilde{v}_1^s > v_3^s$$

2. Equilibrium efforts

   Overconfident player 1 *max*

$$\begin{aligned}
\widetilde{E}^s(U_{13}) &= \widetilde{p}_{13}^s \widetilde{v}_1^s + (1-\widetilde{p}_{13}^s)u(w_3) - ce_1 \\
&= \widetilde{p}_{13}^s\left(\widetilde{v}_1^s - u(w_3)\right) + u(w_3) - ce_1 \\
&= \begin{cases} \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)\left(\widetilde{v}_1^s - u(w_3)\right) + u(w_3) - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}\left(\widetilde{v}_1^s - u(w_3)\right) + u(w_3) - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_3^\alpha \end{cases}
\end{aligned}$$

Rational player 3 $\quad max$

$$\begin{aligned} E^s(U_{31}) &= p_{31}^s v_3^s + (1-p_{31}^s)u(w_3) - ce_3 \\ &= p_{31}^s\left(v_3^s - u(w_3)\right) + u(w_3) - ce_3 \\ &= \begin{cases} \left(1-\frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)\left(v_3^s - u(w_3)\right) + u(w_3) - ce_3 & \text{if} \quad e_3 \geqslant e_1 \\ \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\left(v_3^s - u(w_3)\right) + u(w_3) - ce_3 & \text{if} \quad e_3 \leqslant e_1 \end{cases} \end{aligned}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1 \end{cases}$$

Since $\lambda_1 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1$, which corresponds to (i).

Player 1 $\quad max \quad \left(1-\frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)\left(\widetilde{v}_1^s - u(w_3)\right) + u(w_3) - ce_1$

Player 3 $\quad max \quad \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\left(v_3^s - u(w_3)\right) + u(w_3) - ce_3$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_3^\alpha}{e_1^{\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right) - c = 0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_3^{\alpha-1}}{e_1^\alpha}\left(v_3^s - u(w_3)\right) - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_3^\alpha}{e_1^{\alpha+2}}\left(\widetilde{v}_1^s - u(w_3)\right) < 0$

$[e_3] \quad \frac{\alpha}{2}(\alpha-1)\frac{e_3^{\alpha-2}}{e_1^\alpha}\left(v_3^s - u(w_3)\right) < 0$

Solve the two F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}\left(\widetilde{v}_1^s - u(w_3)\right)^{1-\alpha}\left(v_3^s - u(w_3)\right)^\alpha$$

$$e_3 = \frac{\alpha}{2c}\lambda_1^{\alpha}\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(v_3^s - u(w_3)\right)^{\alpha+1}$$

$$\frac{e_3}{e_1} = \lambda_1\frac{v_3^s - u(w_3)}{\widetilde{v}_1^s - u(w_3)}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$ and $e_3 \leqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$

As long as $e_1 \geqslant e_3$ is satisfied, $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$ is satisfied.

② $e_3 \leqslant e_1$

$$
\begin{aligned}
e_1 \geqslant e_3 &\Longleftrightarrow \frac{e_1}{e_3} \geqslant 1 \\
&\Longleftrightarrow \frac{\widetilde{v}_1^s - u(w_3)}{\lambda_1 \left(v_3^s - u(w_3)\right)} \geqslant 1 \\
&\Longleftrightarrow \frac{\left(1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)\Delta u}{\lambda_1 \left(1 - \frac{1+\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)\Delta u} \geqslant 1 \\
&\Longleftrightarrow 1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} \geqslant \lambda_1 \left(1 - \frac{1+\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)
\end{aligned}
$$

Let

$$f(\lambda_1) = \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) - \lambda_1 \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right).$$

We can easily get that $f(\lambda_1 = 1) = 0$ and $f(\lambda_1 \to \infty) < 0$.

$$f'(\lambda_1) = \frac{(1+\alpha)^2}{2(2\alpha+1)}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1} - \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)$$

$$
\begin{aligned}
f'(\lambda_1) \lesseqqgtr 0 &\Longleftrightarrow \frac{(1+\alpha)^2}{2(2\alpha+1)}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1} \lesseqqgtr 1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \\
&\Longleftrightarrow \frac{(1+\alpha)^2}{2(2\alpha+1)}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1} \lesseqqgtr \lambda_1^{\frac{\alpha+1}{2\alpha+1}+1} \\
&\Longleftrightarrow \left[\frac{(1+\alpha)^2}{2(2\alpha+1)}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}} \lesseqqgtr \lambda_1
\end{aligned}
$$

Let $g(\alpha) = \left[\frac{(1+\alpha)^2}{2(2\alpha+1)}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}}$

a) $g(\alpha) \leqslant 1$

if $g(\alpha) \leqslant 1$, then $f'(\lambda_1) < 0$ always holds and thus $f(\lambda_1) < 0$ always holds. Therefore $e_1 < e_3$ when $g(\alpha) \leqslant 1$.

$$
\begin{aligned}
g(\alpha) \leqslant 1 &\Longleftrightarrow \frac{(1+\alpha)^2}{2(2\alpha+1)}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1} \leqslant 1 \\
&\Longleftrightarrow \frac{(1+\alpha)^2}{2(2\alpha+1)} \leqslant 1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \\
&\Longleftrightarrow \frac{1+\alpha}{2}\left(\frac{1+\alpha}{2\alpha+1} + 1\right) - 1 \leqslant \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} \\
&\Longleftrightarrow \frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}
\end{aligned}
$$

When $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$, the condition $e_1 \geqslant e_3$ is never satisfied given that $\lambda_1 > 1$.

b) $g(\alpha) > 1$

if $g(\alpha) > 1$, then

$$f'(\lambda_1)\begin{cases} > 0 & \text{when } \lambda_1 < \left[\frac{(1+\alpha)^2}{2(2\alpha+1)}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{-1}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}} \\ < 0 & \text{when } \lambda_1 > \left[\frac{(1+\alpha)^2}{2(2\alpha+1)}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{-1}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}} \end{cases}$$

We now show that if $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$, then there exists a unique threshold $\hat{\lambda} > 1$ where $f(\lambda_1) = 0$, that is,

$$1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}} = \hat{\lambda}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right).$$

To see this is the case, we rearrange the equality as

$$\frac{1+\alpha}{2}\left(\hat{\lambda}-\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}\right) = (\hat{\lambda}-1)\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right),$$

or

$$\frac{1+\alpha}{2}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)^{-1} = \frac{\hat{\lambda}-1}{\hat{\lambda}-\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}}. \tag{3}$$

Since $\alpha \in (0,1]$ and $\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} > 0$, the left-hand side of (3) takes a value in the interval $(0,1)$. The right-hand side of (3) is increasing in $\hat{\lambda}$ for $\lambda_1 > 1$, its limit when $\hat{\lambda} \to 1$ is $\frac{2\alpha+1}{3\alpha+2}$, and its limit when $\hat{\lambda} \to \infty$ is 1. Hence, the threshold $\hat{\lambda}$ exists and is unique provided that

$$\frac{1+\alpha}{2}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)^{-1} > \frac{2\alpha+1}{3\alpha+2}.$$

It is easy to show that this inequality is equivalent to

$$\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}.$$

Therefore, if $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$, then there exists a unique value for $\hat{\lambda}$, greater than 1, that satisfies (3). This, in turn, implies:

$$f(\lambda_1)\begin{cases} > 0 & \text{when } \lambda_1 < \hat{\lambda} \\ = 0 & \text{when } \lambda_1 = \hat{\lambda} \\ < 0 & \text{when } \lambda_1 > \hat{\lambda} \end{cases}$$

$$e_1 - e_3\begin{cases} > 0 & \text{when } \lambda_1 < \hat{\lambda} \\ = 0 & \text{when } \lambda_1 = \hat{\lambda} \\ < 0 & \text{when } \lambda_1 > \hat{\lambda} \end{cases}$$

The condition $e_1 \geqslant e_3$ is only satisfied when $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 \leqslant \hat{\lambda}$. And $e_1 > e_3$ is only satisfied when $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$.

Therefore the solution

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}\left(\widetilde{v}_1^s - u(w_3)\right)^{1-\alpha}\left(v_3^s - u(w_3)\right)^{\alpha}$$

$$e_3 = \frac{\alpha}{2c}\lambda_1^{\alpha}\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(v_3^s - u(w_3)\right)^{\alpha+1}$$

only applies when $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$.

(2) case 2: $\lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1$, which corresponds to (ii).

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)\left(\widetilde{v}_1^s - u(w_3)\right) + u(w_3) - ce_1$

Player 3 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)\left(v_3^s - u(w_3)\right) + u(w_3) - ce_3$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_3^\alpha}{e_1^{\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right) - c = 0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_3^{\alpha+1}}\left(v_3^s - u(w_3)\right) - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_3^\alpha}{e_1^{\alpha+2}}\left(\widetilde{v}_1^s - u(w_3)\right) < 0$

$[e_3] \quad \frac{\alpha}{2}(-\alpha-1)\frac{e_1^\alpha}{e_3^{\alpha+2}}\left(v_3^s - u(w_3)\right) < 0$

Solve F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}$$

$$e_3 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}}$$

$$\frac{e_1}{e_3} = \lambda_1^{-\frac{1}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{1}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{-\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$ and $e_3 \geqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$

$$\lambda_1 e_1^\alpha \geqslant e_3^\alpha \Longleftrightarrow \frac{\lambda_1 e_1^\alpha}{e_3^\alpha} \geqslant 1 \Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{-\frac{\alpha}{2\alpha+1}} \geqslant 1$$

Since $\lambda_1 > 1$ and $\widetilde{v}_1^s > v_3^s$, the inequality is always satisfied. Therefore $\lambda_1 e_1^\alpha > e_3^\alpha$ always holds when $\lambda_1 > 1$.

② $e_3 \geqslant e_1$

$$
\begin{aligned}
e_1 \leqslant e_3 &\Longleftrightarrow \frac{e_1}{e_3} \leqslant 1 \\
&\Longleftrightarrow \lambda_1^{-\frac{1}{2\alpha+1}} (\widetilde{v}_1^s - u(w_3))^{\frac{1}{2\alpha+1}} (v_3^s - u(w_3))^{-\frac{1}{2\alpha+1}} \leqslant 1 \\
&\Longleftrightarrow \left(\frac{\widetilde{v}_1^s - u(w_3)}{\lambda_1 (v_3^s - u(w_3))}\right)^{\frac{1}{2\alpha+1}} \leqslant 1 \\
&\Longleftrightarrow \frac{\widetilde{v}_1^s - u(w_3)}{\lambda_1 (v_3^s - u(w_3))} \leqslant 1
\end{aligned}
$$

We have already seen in case (1) that $e_3 \geqslant e_1$ is satisfied when either $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$.

Therefore the solution

$$
e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} (\widetilde{v}_1^s - u(w_3))^{\frac{\alpha+1}{2\alpha+1}} (v_3^s - u(w_3))^{\frac{\alpha}{2\alpha+1}}
$$

$$
e_3 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}} (\widetilde{v}_1^s - u(w_3))^{\frac{\alpha}{2\alpha+1}} (v_3^s - u(w_3))^{\frac{\alpha+1}{2\alpha+1}}
$$

only applies when either $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha} (\widetilde{v}_1^s - u(w_3)) + u(w_3) - ce_1$

Player 3 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right) (v_3^s - u(w_3)) + u(w_3) - ce_3$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_3^\alpha} (\widetilde{v}_1^s - u(w_3)) - c = 0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_3^{\alpha+1}} (v_3^s - u(w_3)) - c = 0$

divide the two F.O.C , we get

$$
\frac{e_3}{e_1} = \frac{v_3^s - u(w_3)}{\lambda_1 (\widetilde{v}_1^s - u(w_3))} < 1
$$

which contradicts the condition that $e_3 \geqslant e_1$

Therefore, the equilibrium in this semifinal:

(1) When $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$, which corresponds to Lemma A2 (i)

$$
e_1^s = \frac{\alpha}{2c}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\Delta u
$$

$$
e_3^s = \frac{\alpha}{2c}\lambda_1^{\alpha}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{1+\alpha}\Delta u
$$

where $e_1^s > e_3^s$.

$$\begin{aligned}
p_{31}^s &= \frac{1}{2}\left(\frac{e_3^s}{e_1^s}\right)^{\alpha} \\
&= \frac{1}{2}\left(\frac{\lambda_1 v_2}{\widetilde{v}_1}\right)^{\alpha} \\
&= \frac{1}{2}\lambda_1^{\alpha}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\alpha}
\end{aligned}$$

$$\begin{aligned}
p_{13}^s &= 1-p_{31}^s \\
&= 1-\frac{1}{2}\lambda_1^{\alpha}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\alpha}
\end{aligned}$$

$$\begin{aligned}
\widetilde{p}_{13}^s &= 1-\frac{1}{2}\frac{(e_3^s)^{\alpha}}{\lambda_1 (e_1^s)^{\alpha}} \\
&= 1-\frac{1}{2}\lambda_1^{\alpha-1}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\alpha}
\end{aligned}$$

We can get $\widetilde{p}_{13}^s > p_{13}^s > \frac{1}{2} > p_{31}^s$

(2) When either $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$, which corresponds to Lemma A2 (ii).

$$\begin{aligned}
e_1^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}&\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}\Delta u
\end{aligned}$$

$$\begin{aligned}
e_3^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}&\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\frac{\alpha+1}{2\alpha+1}}\Delta u
\end{aligned}$$

where $\lambda_1 (e_1^s)^{\alpha} > (e_3^s)^{\alpha}$ and $e_1^s \leqslant e_3^s$.

$$\begin{aligned}
p_{13}^s &= \frac{1}{2}\left(\frac{e_1^s}{e_3^s}\right)^{\alpha} \\
&= \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\quad\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{-\frac{\alpha}{2\alpha+1}}
\end{aligned}$$

$$
\begin{aligned}
p_{31}^{s} &= 1 - p_{13}^{s} \\
&= 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\quad \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\frac{\alpha}{2\alpha+1}}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{p}_{13}^{s} &= 1 - \frac{1}{2}\frac{(e_3^s)^{\alpha}}{\lambda_1 (e_1^s)^{\alpha}} \\
&= 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}} \\
&\quad \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}
\end{aligned}
$$

We can get $\widetilde{p}_{13}^{s} > p_{31}^{s} \geqslant \frac{1}{2} \geqslant p_{13}^{s}$

3. Participation constraints

(1) When $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$

$$
\begin{aligned}
\widetilde{E}^{s}(U_{13}) &= \widetilde{p}_{13}^{s}\left(\widetilde{v}_1^{s} - u(w_3)\right) - ce_1^{s} + u(w_3) \\
&> p_{13}^{s}\left(\widetilde{v}_1^{s} - u(w_3)\right) - ce_1^{s} + u(w_3) \\
&= \left(1 - \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1^{s} - u(w_3)\right)^{-\alpha}\left(v_3^{s} - u(w_3)\right)^{\alpha}\right)\left(\widetilde{v}_1^{s} - u(w_3)\right) \\
&\quad - c\frac{\alpha}{2c}\lambda_1^{\alpha-1}\left(\widetilde{v}_1^{s} - u(w_3)\right)^{1-\alpha}\left(v_3^{s} - u(w_3)\right)^{\alpha} + u(w_3) \\
&= \widetilde{v}_1^{s} - u(w_3) - \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1^{s} - u(w_3)\right)^{1-\alpha}\left(v_3^{s} - u(w_3)\right)^{\alpha} \\
&\quad - \frac{\alpha}{2}\lambda_1^{\alpha-1}\left(\widetilde{v}_1^{s} - u(w_3)\right)^{1-\alpha}\left(v_3^{s} - u(w_3)\right)^{\alpha} + u(w_3) \\
&> \widetilde{v}_1^{s} - u(w_3) - \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1^{s} - u(w_3)\right)^{1-\alpha}\left(v_3^{s} - u(w_3)\right)^{\alpha} \\
&\quad - \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1^{s} - u(w_3)\right)^{1-\alpha}\left(v_3^{s} - u(w_3)\right)^{\alpha} + u(w_3) \\
&= \widetilde{v}_1^{s} - u(w_3) - \lambda_1^{\alpha}\left(\widetilde{v}_1^{s} - u(w_3)\right)^{1-\alpha}\left(v_3^{s} - u(w_3)\right)^{\alpha} + u(w_3) \\
&= \left(\widetilde{v}_1^{s} - u(w_3)\right)^{1-\alpha}\left[\left(\widetilde{v}_1^{s} - u(w_3)\right)^{\alpha} - \lambda_1^{\alpha}\left(v_3^{s} - u(w_3)\right)^{\alpha}\right] + u(w_3) \\
&> 0
\end{aligned}
$$

$$
\begin{aligned}
E^{s}(U_{31}) &= p_{31}^{s}\left(v_3^{s} - u(w_3)\right) - ce_3^{s} + u(w_3) \\
&= \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1^{s} - u(w_3)\right)^{-\alpha}\left(v_3^{s} - u(w_3)\right)^{\alpha}\left(v_3^{s} - u(w_3)\right) \\
&\quad - c\frac{\alpha}{2c}\lambda_1^{\alpha}\left(\widetilde{v}_1^{s} - u(w_3)\right)^{-\alpha}\left(v_3^{s} - u(w_3)\right)^{\alpha+1} + u(w_3) \\
&= \frac{1-\alpha}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1^{s} - u(w_3)\right)^{-\alpha}\left(v_3^{s} - u(w_3)\right)^{1+\alpha} + u(w_3) \\
&\geqslant 0
\end{aligned}
$$

(2) When either $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

$$\widetilde{E}^s(U_{13}) = \widetilde{p}_{13}^s\left(\widetilde{v}_1^s - u(w_3)\right) - ce_1^s + u(w_3)$$

Since $\widetilde{p}_{13}^s > \frac{1}{2}$, $\widetilde{v}_1^s > \overline{v}^s$ and $e_1^s < \overline{e}^s$, we can get that $\widetilde{E}^s(U_{13}) > \overline{E}^s(U) \geqslant 0$.

$$\begin{aligned}
E^s(U_{31}) &= p_{31}^s\left(v_3^s - u(w_3)\right) - ce_3^s + u(w_3)\\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{-\frac{\alpha}{2\alpha+1}}\right)\left(v_3^s - u(w_3)\right)\\
&\quad - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} + u(w_3)\\
&= v_3^s - u(w_3) - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}}\\
&\quad - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} + u(w_3)\\
&= v_3^s - u(w_3) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} + u(w_3)\\
&= \left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}}\left[\left(v_3^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\right] + u(w_3)\\
&\geqslant 0
\end{aligned}$$

Next we characterize the quarterfinal between the overconfident player 1 and rational player 2. Due to a lack of full characterization of the equilibrium in the quarterfinal between an overconfident player and a rational player, we show that there exist parameter configurations where in equilibrium the overconfident player exerts higher effort than the rational player.

Proposition A9 *Consider a quarterfinal between the overconfident player 1 and the rational player 2 of a three-stage elimination contest where player 1 is overconfident and the other seven players are rational.*
(i) If $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$, *then there exist* $\lambda_1 \in (1, \hat{\lambda}]$, *where* $\hat{\lambda}$ *solves* $1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}} = \hat{\lambda}(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2})$, *for which the equilibrium efforts and winning probabilities satisfy* $e_1^q > \overline{e}^q > e_2^q$ *and* $\widetilde{p}_{12}^q > p_{12}^q > 1/2 > p_{21}^q$.
(ii) If $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ *and* $\frac{u(w_3)}{u(w_1)-u(w_2)} < \frac{\alpha(3\alpha+1)}{2(2\alpha+1)}\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{-7\alpha^4+\alpha^3+8\alpha^2+5\alpha+1}{4(2\alpha+1)^2}$, *then there exist* λ_1 *close to 1 for which the equilibrium efforts and winning probabilities satisfy* $e_1^q > \overline{e}^q > e_2^q$ *and* $\widetilde{p}_{12}^q > p_{12}^q > 1/2 > p_{21}^q$.

Proposition A9 shows that the results in the semifinal between an overconfident player and a rational player of a two-stage elimination contest generalize to the quarterfinal between an overconfident player and a rational player of a three-stage elimination contest. In the quarterfinal between an overconfident player and a rational player, the equilibrium efforts and winning probabilities depend on the utility spread and the overconfidence level. The equilibrium where the overconfident player exerts higher effort than the rational player exists with certainty under either of the two conditions: (i) if the utility spread between the winner and runner-up compared to that between the runner-up and the second stage loser is large ($\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$) and overconfidence level is close to either 1 or $\hat{\lambda}$, (ii) if the utility spread between the winner and runner-up compared to that between the runner-up and the second stage loser is small ($\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$), the utility spread between the winner and runner-up compared to that between the second stage loser and third stage loser is sufficiently large ($\frac{u(w_3)}{u(w_1)-u(w_2)} < \frac{\alpha(3\alpha+1)}{2(2\alpha+1)}\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{-7\alpha^4+\alpha^3+8\alpha^2+5\alpha+1}{4(2\alpha+1)^2}$), and λ_1 is close to 1.

Proof of Proposition A9

1. Perceived expected utilities of reaching the semifinal

Overconfident player 1:

$$\widetilde{E}^q(U_{12}) = \widetilde{p}^q_{12}\widetilde{v}^q_1 - ce^q_1$$

where

$$\widetilde{v}^q_1 = \widetilde{E}^s(U_{13}) = \widetilde{p}^s_{13}\left[\left(1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)\Delta u + u(w_2) - u(w_3)\right] + u(w_3) - ce^s_1$$

Rational player 2:

Since the rational player 2 will only meet rational rivals in the semifinal and final, her expected utility of reaching the semifinal is the benchmark.

$$v^q_2 = \overline{v}^q$$

(1) When $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 < \hat{\lambda}$

$$\begin{aligned}
\widetilde{v}^q_1 - u(w_3) = &\left[1 - \frac{1}{2}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\right.\\
&\left.\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\right]\\
&\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)\Delta u\\
&- c\frac{\alpha}{2c}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}\\
&\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\Delta u\\
= &\left[\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)\right.\\
&- \frac{1+\alpha}{2}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}\\
&\left.\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\right]\Delta u
\end{aligned}$$

$$
\begin{aligned}
&\frac{\widetilde{v}_1^q - u(w_3)}{\overline{v}^q - u(w_3)} \\
&= \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)} \\
&- \frac{\frac{1+\alpha}{2}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)} \\
&= \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}}{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{1-\alpha}} \\
&\quad \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha} - \frac{1+\alpha}{2}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}} \\
&> \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha} - \frac{1+\alpha}{2}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}} \\
&> \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} - \frac{1+\alpha}{2}\lambda_1^{\alpha-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}} \\
&= \frac{1 - \frac{1+\alpha}{2}\lambda_1^{\alpha-1}}{1 - \frac{1+\alpha}{2}} \\
&> 1
\end{aligned}
$$

Thus we get

$$\widetilde{v}_1^q > \overline{v}^q$$

(2) When either $\lambda_1 \geqslant \hat{\lambda}$ or $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$

$$\begin{aligned}
\widetilde{v}_1^q - u(w_3) = &\left[1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}\right. \\
&\left.\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}\right] \\
&\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)\Delta u \\
&- c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}\Delta u \\
= &\left[\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right. \\
&\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&\left.\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}\right]\Delta u
\end{aligned}$$

$$
\begin{aligned}
&\frac{\widetilde{v}_1^q - u(w_3)}{\overline{v}^q - u(w_3)} \\
&= \frac{1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)} \\
&\quad - \frac{\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)} \\
&= \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}}}{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha+1}{2\alpha+1}}} \\
&\quad \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}} \\
&> \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}} \\
&> \frac{\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}}{\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}} \\
&= \frac{1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}}{1 - \frac{1+\alpha}{2}} \\
&> 1
\end{aligned}
$$

Thus we get

$$\widetilde{v}_1^q > \overline{v}^q$$

2. The equilibrium when $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 < \hat{\lambda}$, , which corresponds to (i)

Player 1 $\quad max \quad \widetilde{E}^q(U_{12}) = \widetilde{p}_{12}^q \widetilde{v}_1^q - ce_1$

$$
= \begin{cases} \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1^q - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_2^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1^q - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_2^\alpha \end{cases}
$$

Player 2 $\quad max \quad E^q(U_{21}) = p_{21}^q v_2^q - ce_2$

$$
= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right) v_2^q - ce_2 & \text{if}\, e_2 \geqslant e_1 \\ \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha} v_2^q - ce_2 & \text{if}\, e_2 \leqslant e_1 \end{cases}
$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \end{cases}$$

Since $\lambda_1 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1^q - ce_1$

Player 2 $\quad max \quad \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha}v_2^q - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1^q - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_2^{\alpha-1}}{e_1^\alpha}v_2^q - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1^q < 0$

$[e_2] \quad \frac{\alpha}{2}(\alpha-1)\frac{e_2^{\alpha-2}}{e_1^\alpha}v_2^q < 0$

Solve the two F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1^q)^{1-\alpha}(v_2^q)^\alpha$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1^q)^{-\alpha}(v_2^q)^{1+\alpha}$$

$$\frac{e_2}{e_1} = \lambda_1\frac{v_2^q}{\widetilde{v}_1^q}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \leqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

As long as $e_1 \geqslant e_2$ is satisfied, $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ is satisfied.

② $e_2 \leqslant e_1$

$$\frac{e_2}{e_1} = \lambda_1\frac{v_2^q}{\widetilde{v}_1^q} = \frac{\overline{v}^q}{\lambda_1^{-1}\widetilde{v}_1^q}$$

Let

$$f(\lambda_1) = \frac{\lambda_1^{-1}\widetilde{v}_1^q - \overline{v}^q}{u(w_1) - u(w_2)}$$

$$
\begin{aligned}
f(\lambda_1) = \lambda_1^{-1} \Bigg[& \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) \\
& - \frac{1+\alpha}{2} \lambda_1^{\alpha-1} \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha} \\
& \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} + \frac{u(w_3)}{u(w_1) - u(w_2)} \Bigg] \\
& - \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right]
\end{aligned}
$$

We can easily get

$$f(\lambda_1 = 1) = 0.$$

Recall $\hat{\lambda}^{-1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}\right) - \left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) = 0$, we can get

$$
\begin{aligned}
f(\lambda_1 = \hat{\lambda}) = {} & \hat{\lambda}^{-1}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}\right) \\
& - \frac{1+\alpha}{2}\hat{\lambda}^{\alpha-1}\hat{\lambda}^{-1}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{1+\alpha}{2\alpha+1}}\right)^{1-\alpha} \\
& \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} \\
& + \hat{\lambda}^{-1}\frac{u(w_3)}{u(w_1) - u(w_2)} \\
& - \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right] \\
= {} & \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) \\
& - \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\hat{\lambda}^{-1}\hat{\lambda}^{\alpha-1} \\
& \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{1+\alpha}{2\alpha+1}}\right)^{1-\alpha} \\
& + \frac{u(w_3)}{u(w_1) - u(w_2)}\hat{\lambda}^{-1} \\
& - \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right] \\
= {} & \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) \\
& - \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\hat{\lambda}^{-1} \\
& \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{1-\alpha} \\
& + \frac{u(w_3)}{u(w_1) - u(w_2)}\hat{\lambda}^{-1} \\
& - \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right] \\
= {} & \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) \\
& - \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\hat{\lambda}^{-1} + \frac{u(w_3)}{u(w_1) - u(w_2)}\hat{\lambda}^{-1} \\
& - \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right] \\
= {} & \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left(1 - \frac{1+\alpha}{2}\hat{\lambda}^{-1}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\hat{\lambda}^{-1} \\
& - \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&f(\lambda_1 = \hat{\lambda}) \lesseqqgtr 0 \\
&\Longleftrightarrow \\
&\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left(1 - \frac{1+\alpha}{2}\hat{\lambda}^{-1}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\hat{\lambda}^{-1} \lesseqqgtr \\
&\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)} \\
&\Longleftrightarrow -\frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\hat{\lambda}^{-1} + \frac{u(w_3)}{u(w_1) - u(w_2)}\hat{\lambda}^{-1} \lesseqqgtr \\
&-\frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)} \\
&\Longleftrightarrow \hat{\lambda}^{-1}\left[-\frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right] \lesseqqgtr \\
&-\frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}
\end{aligned}
$$

If $-\frac{1+\alpha}{2}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)} < 0$ always holds then $f(\lambda_1 = \hat{\lambda}) > 0$ always holds.

We show that $-\frac{1+\alpha}{2}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1)-u(w_2)} < 0$ always holds:

$$
\begin{aligned}
&-\frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)} < 0 \\
&\Longleftrightarrow \frac{u(w_3)}{u(w_1) - u(w_2)} < \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) \\
&\Longleftrightarrow 1 < \frac{1+\alpha}{2}\frac{u(w_1) - u(w_2)}{u(w_3)}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) \\
&\Longleftrightarrow 1 < \frac{1+\alpha}{2}\left[\left(1 - \frac{1+\alpha}{2}\right)\frac{u(w_1) - u(w_2)}{u(w_3)} + \frac{u(w_2) - u(w_3)}{u(w_3)}\right] \\
&\Longleftrightarrow 1 < \frac{1+\alpha}{2}\left[\frac{1-\alpha}{2}\frac{u(w_1) - u(w_2)}{u(w_2) - u(w_3)} + 1\right]\frac{u(w_2) - u(w_3)}{u(w_3)}
\end{aligned}
$$

Since $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$,

$$\begin{aligned}
\frac{1+\alpha}{2}\left[\frac{1-\alpha}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)}+1\right] &> \frac{1+\alpha}{2}\left[\frac{1-\alpha}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)}+1\right] \\
&= \frac{1+\alpha}{2}\left(\frac{1-\alpha}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)}+1\right) \\
&= \frac{1+\alpha}{2}\left(\frac{1-\alpha}{\alpha}\frac{2\alpha+1}{3\alpha+1}+1\right) \\
&= \frac{1+\alpha}{2}\frac{-2\alpha^2+\alpha+1+3\alpha^2+\alpha}{\alpha\left(3\alpha+1\right)} \\
&= \frac{1+\alpha}{2}\frac{\alpha^2+2\alpha+1}{\alpha\left(3\alpha+1\right)} \\
&= \frac{\alpha^3+3\alpha^2+3\alpha+1}{2\alpha\left(3\alpha+1\right)}
\end{aligned}$$

If $\frac{\alpha^3+3\alpha^2+3\alpha+1}{2\alpha(3\alpha+1)} > 1$ is satisfied, then $\frac{1+\alpha}{2}\left[\frac{1-\alpha}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)}+1\right] > 1$ is satisfied.

$$\frac{\alpha^3+3\alpha^2+3\alpha+1}{2\alpha\left(3\alpha+1\right)} > 1 \Longleftrightarrow \alpha^3+3\alpha^2+3\alpha+1-2\alpha\left(3\alpha+1\right) > 0$$

Let $t(\alpha) = \alpha^3+3\alpha^2+3\alpha+1-2\alpha\left(3\alpha+1\right)$

$$t(\alpha) = \alpha^3-3\alpha^2+\alpha+1$$

$$t(\alpha=0)=1, \quad t(\alpha=1)=0$$

$$t'(\alpha) = 3\alpha^2-6\alpha+1$$

We can get $t'(\alpha) > 0$ if $0 < \alpha < \frac{3-\sqrt{6}}{3}$ and $t'(\alpha) < 0$ if $\frac{3-\sqrt{6}}{3} < \alpha < 1$. Thus $t(\alpha) > 0$ when $\alpha \in (0,1)$.

Therefore $\frac{1+\alpha}{2}\left[\frac{1-\alpha}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)}+1\right] > 1$ always holds. Since we assume $\frac{u(w_2)-u(w_3)}{u(w_3)} > 1$, $-\frac{1+\alpha}{2}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)+\frac{u(w_3)}{u(w_1)-u(w_2)} < 0$ always holds. Thus $f(\lambda_1=\hat{\lambda}) > 0$ always holds.

$$
\begin{aligned}
f'(\lambda_1) = &-\lambda_1^{-2}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)+\lambda_1^{-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1} \\
&-\frac{1+\alpha}{2}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\alpha} \\
&\left[(\alpha-2)\lambda_1^{\alpha-3}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{1-\alpha}\right. \\
&+\lambda_1^{\alpha-2}(1-\alpha)\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{-\alpha} \\
&\left.\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\right]-\frac{u(w_3)}{u(w_1)-u(w_2)}\lambda_1^{-2} \\
= &\lambda_1^{-2}\left[-\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)+\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right. \\
&-\frac{1+\alpha}{2}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\alpha} \\
&\left((\alpha-2)\lambda_1^{\alpha-1}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{1-\alpha}\right. \\
&\left.+(1-\alpha)\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{-\alpha}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{\alpha-\frac{\alpha+1}{2\alpha+1}-1}\right) \\
&\left.-\frac{u(w_3)}{u(w_1)-u(w_2)}\right]
\end{aligned}
$$

Let

$$
\begin{aligned}
g(\lambda_1) = &-\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)+\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \\
&-\frac{1+\alpha}{2}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\alpha} \\
&\left[(\alpha-2)\lambda_1^{\alpha-1}\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{1-\alpha}\right. \\
&\left.+(1-\alpha)\left(1+\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{-\alpha}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{\alpha-\frac{\alpha+1}{2\alpha+1}-1}\right] \\
&-\frac{u(w_3)}{u(w_1)-u(w_2)}
\end{aligned}
$$

$$
\begin{aligned}
g(\lambda_1 = 1) = &- \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} \\
&\Bigg[(\alpha - 2)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{1-\alpha} + (1-\alpha) \\
&\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha} \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\Bigg] - \frac{u(w_3)}{u(w_1) - u(w_2)} \\
= &- \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)(\alpha - 2) \\
&- \frac{1+\alpha}{2}(1-\alpha)\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} - \frac{u(w_3)}{u(w_1) - u(w_2)} \\
= &\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left[-1 - \frac{1+\alpha}{2}(\alpha - 2)\right] \\
&+ \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left[1 - \frac{1+\alpha}{2}(1-\alpha)\right] - \frac{u(w_3)}{u(w_1) - u(w_2)} \\
= &\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left[-1 + \frac{1+\alpha}{2}(2 - \alpha)\right] \\
&+ \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left(1 - \frac{1-\alpha^2}{2}\right) - \frac{u(w_3)}{u(w_1) - u(w_2)} \\
= &\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\frac{-2 + (-\alpha^2 + \alpha + 2)}{2} \\
&+ \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{2 - (1-\alpha^2)}{2} - \frac{u(w_3)}{u(w_1) - u(w_2)} \\
= &\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\frac{-\alpha^2 + \alpha}{2} \\
&+ \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2} - \frac{u(w_3)}{u(w_1) - u(w_2)}
\end{aligned}
$$

$$
\begin{aligned}
&f'(\lambda_1 = 1) \lesseqqgtr 0 \\
\iff &\left(\frac{1-\alpha}{2} + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)}\right)\frac{-\alpha^2 + \alpha}{2} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2} \lesseqqgtr \frac{u(w_3)}{u(w_1) - u(w_2)}
\end{aligned}
$$

We can easily get that $\left(\frac{1-\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)\frac{-\alpha^2+\alpha}{2} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2} > 0$ always holds. Thus

$$f'(\lambda_1 = 1) \lesseqqgtr 0$$

$$\Longleftrightarrow$$

$$\left[\left(\frac{1-\alpha}{2} + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)}\right)\frac{-\alpha^2+\alpha}{2} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2}\right]\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)}$$

$$\lesseqqgtr \frac{u(w_3)}{u(w_2)-u(w_3)}$$

$$\Longleftrightarrow$$

$$\frac{1-\alpha}{2}\frac{-\alpha^2+\alpha}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} + \frac{-\alpha^2+\alpha}{2}$$

$$\lesseqqgtr \frac{u(w_3)}{u(w_2)-u(w_3)}$$

We show that $\frac{1-\alpha}{2}\frac{-\alpha^2+\alpha}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} + \frac{-\alpha^2+\alpha}{2} > \frac{u(w_3)}{u(w_2)-u(w_3)}$ always holds:

Since $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$,

$$\begin{aligned}
&\frac{1-\alpha}{2}\frac{-\alpha^2+\alpha}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2}\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} + \frac{-\alpha^2+\alpha}{2}\\
&> \frac{1-\alpha}{2}\frac{-\alpha^2+\alpha}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)} + \frac{-\alpha^2+\alpha}{2}
\end{aligned}$$

Let $t(\alpha) = \frac{1-\alpha}{2}\frac{-\alpha^2+\alpha}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)} + \frac{-\alpha^2+\alpha}{2}$,

$$\begin{aligned}
t(\alpha) &= \frac{1-\alpha}{2}\frac{-\alpha^2+\alpha}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)} + \frac{-\alpha^2+\alpha}{2} + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha^2+1}{2}\frac{2(2\alpha+1)}{\alpha(3\alpha+1)}\\
&= \frac{1-\alpha}{2}(1-\alpha)\frac{2\alpha+1}{3\alpha+1} + \frac{-\alpha^2+\alpha}{2} + \frac{(1+\alpha)(1+\alpha)(1+\alpha^2)}{2\alpha(3\alpha+1)}\\
&= \frac{(1-\alpha)(1-\alpha)\alpha(1+2\alpha) + (-\alpha^2+\alpha)\alpha(3\alpha+1) + (1+2\alpha+\alpha^2)(1+\alpha^2)}{2\alpha(3\alpha+1)}\\
&= \frac{2\alpha^4 - 3\alpha^3 + \alpha - 3\alpha^4 + 2\alpha^3 + \alpha^2 + \alpha^4 + 2\alpha^3 + 2\alpha^2 + 2\alpha + 1}{2\alpha(3\alpha+1)}\\
&= \frac{\alpha^3 + 3\alpha^2 + 3\alpha + 1}{2\alpha(3\alpha+1)}
\end{aligned}$$

We have shown before that $\frac{\alpha^3+3\alpha^2+3\alpha+1}{2\alpha(3\alpha+1)} > 1$ always holds when $\alpha \in (0,1)$.

Thus we have proved that $f'(\lambda_1 = 1) > 0$ always holds.

Therefore, it is certain that under the conditions $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 < \hat{\lambda}$, there must exist some parameter configurations where $e_1^q > e_2^q$ is satisfied.

When $e_1^q > e_2^q{}^q$ is satisfied, $\lambda_1^{-1}\widetilde{v}_1^q > \overline{v}^q$. Thus

$$\begin{aligned} e_1^q &= \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1^q)^{1-\alpha}(v_2^q)^{\alpha} \\ &= \frac{\alpha}{2c}(\lambda_1^{-1}\widetilde{v}_1^q)^{1-\alpha}(v_2^q)^{\alpha} \\ &> \frac{\alpha}{2c}\overline{v}^q = \overline{e}^q \end{aligned}$$

$$\begin{aligned} e_2^q &= \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1^q)^{-\alpha}(v_2^q)^{1+\alpha} \\ &= \frac{\alpha}{2c}(\lambda_1^{-1}\widetilde{v}_1^q)^{-\alpha}(v_2^q)^{1+\alpha} \\ &< \frac{\alpha}{2c}\overline{v}^q = \overline{e}^q \end{aligned}$$

where $e_1^q > \overline{e}^q > e_2^q$.

The equilibrium winning probabilities are

$$p_{21}^q = \frac{1}{2}\left(\frac{e_2^q}{e_1^q}\right)^{\alpha} = \frac{1}{2}\left(\frac{\lambda_1 v_2^q}{\widetilde{v}_1^q}\right)^{\alpha} = \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1^q\right)^{-\alpha}\left(v_2^q\right)^{\alpha}$$

$$p_{12}^q = 1 - p_{21}^q = 1 - \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1^q\right)^{-\alpha}\left(v_2^q\right)^{\alpha}$$

$$\widetilde{p}_{12}^q = 1 - \frac{1}{2}\frac{(e_2^q)^{\alpha}}{\lambda_1\left(e_1^q\right)^{\alpha}} = 1 - \frac{1}{2}\lambda_1^{-1}\left(\lambda_1\left(\widetilde{v}_1^q\right)^{-1}\left(v_2^q\right)\right)^{\alpha} = 1 - \frac{1}{2}\lambda_1^{\alpha-1}\left(\widetilde{v}_1^q\right)^{-\alpha}\left(v_2^q\right)^{\alpha}$$

where $\widetilde{p}_{12}^q > p_{12}^q > \frac{1}{2} > p_{21}^q$.

(2) case 2: $\lambda_1 e_1^{\alpha} \geqslant e_2^{\alpha} \quad and \quad e_2 \geqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^{\alpha}}{\lambda_1 e_1^{\alpha}}\right)\widetilde{v}_1^q - ce_1$

Player 2 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^{\alpha}}{e_2^{\alpha}}\right)v_2^q - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^{\alpha}}{e_1^{\alpha+1}}\widetilde{v}_1^q - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^{\alpha}}{e_2^{\alpha+1}}v_2^q - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^{\alpha}}{e_1^{\alpha+2}}\widetilde{v}_1^q < 0$

$[e_2] \quad \frac{\alpha}{2}(-\alpha-1)\frac{e_1^{\alpha}}{e_2^{\alpha+2}}v_2^q < 0$

Solve F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1^q\right)^{\frac{\alpha+1}{2\alpha+1}}\left(v_2^q\right)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2 = \frac{\alpha}{2c} \lambda_1^{-\frac{\alpha}{2\alpha+1}} (\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}} (v_2^q)^{\frac{\alpha+1}{2\alpha+1}}$$

$$\frac{e_2}{e_1} = \lambda_1^{\frac{1}{2\alpha+1}} (\widetilde{v}_1^q)^{-\frac{1}{2\alpha+1}} (v_2^q)^{\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \geqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

$$\begin{aligned} \lambda_1 e_1^\alpha \geqslant e_2^\alpha &\Longleftrightarrow \frac{\lambda_1 e_1^\alpha}{e_2^\alpha} \geqslant 1 \\ &\Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}} (\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}} (v_2^q)^{-\frac{\alpha}{2\alpha+1}} \geqslant 1 \end{aligned}$$

always holds.

② $e_2 \geqslant e_1$

$$\begin{aligned} \frac{e_2}{e_1} \geqslant 1 &\Longleftrightarrow \lambda_1^{\frac{1}{2\alpha+1}} (\widetilde{v}_1^q)^{-\frac{1}{2\alpha+1}} (v_2^q)^{\frac{1}{2\alpha+1}} \geqslant 1 \\ &\Longleftrightarrow \frac{v_2^q}{\lambda_1^{-1} \widetilde{v}_1^q} \geqslant 1 \\ &\Longleftrightarrow \overline{v}^q \geqslant \lambda_1^{-1} \widetilde{v}_1^q \end{aligned}$$

According to the results we got from (1), we do not know if there exist some parameter configurations where $e_2 \geqslant e_1$ is satisfied.

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad \textit{and} \quad e_2 \geqslant e_1$

Player 1 $\quad max \quad \frac{1}{2} \frac{\lambda_1 e_1^\alpha}{e_2^\alpha} \widetilde{v}_1^q - c e_1$

Player 2 $\quad max \quad \left(1 - \frac{1}{2} \frac{e_1^\alpha}{e_2^\alpha}\right) v_2^q - c e_2$

F.o.c

$[e_1] \quad \frac{\alpha \lambda_1}{2} \frac{e_1^{\alpha-1}}{e_2^\alpha} \widetilde{v}_1^q - c = 0$

$[e_2] \quad \frac{\alpha}{2} \frac{e_1^\alpha}{e_2^{\alpha+1}} v_2^q - c = 0$

divide the two F.O.C , we get

$$\frac{e_2}{e_1} = \frac{v_2^q}{\lambda_1 \widetilde{v}_1^q} < 1 \quad (\widetilde{v}_1^q > v_2^q)$$

which contradicts the condition that $e_2 \geqslant e_1$

To conclude, when $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 < \hat{\lambda}$, we know that case 1 equilibrium, where $e_1^q > \overline{e}^q > e_2^q$, certainly exists but we are not sure about case 2 equilibrium. It could be the case that when the overconfident player exerts higher effort than the rational opponent in the semifinal, his perceived expected utility of reaching the semifinal is so high that the encouraging effect always prevails in the quarterfinal and he always exerts higher effort than the rational opponent in the quarterfinal.

3. The equilibrium when $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 \geqslant \hat{\lambda}$, which corresponds to (i)

Player 1 $\quad max \quad \widetilde{E}^q(U_{12}) = \widetilde{p}_{12}^q \widetilde{v}_1^q - ce_1$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1^q - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_2^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1^q - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_2^\alpha \end{cases}$$

Player 2 $\quad max \quad E^q(U_{21}) = p_{21}^q v_2^q - ce_2$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right) v_2^q - ce_2 & \text{if}\, e_2 \geqslant e_1 \\ \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha} v_2^q - ce_2 & \text{if}\, e_2 \leqslant e_1 \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \end{cases}$$

Since $\lambda_1 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1^q - ce_1$

Player 2 $\quad max \quad \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha} v_2^q - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1^q - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_2^{\alpha-1}}{e_1^\alpha} v_2^q - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1^q < 0$

$[e_2] \quad \frac{\alpha}{2}(\alpha-1)\frac{e_2^{\alpha-2}}{e_1^\alpha} v_2^q < 0$

Solve the two F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1^q)^{1-\alpha}(v_2^q)^\alpha$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1^q)^{-\alpha}(v_2^q)^{1+\alpha}$$

$$\frac{e_2}{e_1} = \lambda_1 \frac{v_2^q}{\widetilde{v}_1^q}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \leqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

As long as $e_1 \geqslant e_2$ is satisfied, $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ is satisfied.

② $e_2 \leqslant e_1$

$$\frac{e_2}{e_1} = \lambda_1 \frac{v_2^q}{\widetilde{v}_1^q} = \frac{\overline{v}^q}{\lambda_1^{-1}\widetilde{v}_1^q}$$

Let

$$f(\lambda_1) = \frac{\lambda_1^{-1}\widetilde{v}_1^q - \overline{v}^q}{u(w_1) - u(w_2)}$$

$$\begin{aligned} f(\lambda_1) = \lambda_1^{-1}\Bigg[&\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) \\ &- \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}} \\ &\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} + \frac{u(w_3)}{u(w_1) - u(w_2)}\Bigg] \\ &- \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right] \end{aligned}$$

Similar to before, we can get that $f(\lambda_1 = \hat{\lambda}) > 0$.

We can easily get that $f(\lambda_1 \to \infty) < 0$.

Therefore there must exist some parameter configurations and domains of overconfidence level where $e_1^q > e_2^q$ is satisfied. When $e_1^q > e_2^q$ is satisfied, we have $e_1^q > \overline{e}^q > e_2^q$.

The equilibrium winning probabilities are

$$p_{21}^q = \frac{1}{2}\left(\frac{e_2^q}{e_1^q}\right)^\alpha = \frac{1}{2}\left(\frac{\lambda_1 v_2^q}{\widetilde{v}_1^q}\right)^\alpha = \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1^q\right)^{-\alpha}\left(v_2^q\right)^\alpha$$

$$p_{12}^q = 1 - p_{21}^q = 1 - \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1^q\right)^{-\alpha}\left(v_2^q\right)^\alpha$$

$$\widetilde{p}_{12}^q = 1 - \frac{1}{2}\frac{\left(e_2^q\right)^\alpha}{\lambda_1\left(e_1^q\right)^\alpha} = 1 - \frac{1}{2}\lambda_1^{-1}\left(\lambda_1\left(\widetilde{v}_1^q\right)^{-1}\left(v_2^q\right)\right)^\alpha = 1 - \frac{1}{2}\lambda_1^{\alpha-1}\left(\widetilde{v}_1^q\right)^{-\alpha}\left(v_2^q\right)^\alpha$$

where $\widetilde{p}_{12}^q > p_{12}^q > \frac{1}{2} > p_{21}^q$.

(2) case 2: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1^q - ce_1$

Player 2 $\quad max \quad \left(1-\frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right)v_2^q - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1^q - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_2^{\alpha+1}}v_2^q - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1^q < 0$

$[e_2] \quad \frac{\alpha}{2}(-\alpha-1)\frac{e_1^\alpha}{e_2^{\alpha+2}}v_2^q < 0$

Solve F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha+1}{2\alpha+1}}(v_2^q)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{\frac{\alpha+1}{2\alpha+1}}$$

$$\frac{e_2}{e_1} = \lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1^q)^{-\frac{1}{2\alpha+1}}(v_2^q)^{\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \geqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

$$\begin{aligned}\lambda_1 e_1^\alpha \geqslant e_2^\alpha &\Longleftrightarrow \frac{\lambda_1 e_1^\alpha}{e_2^\alpha} \geqslant 1\\ &\Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{-\frac{\alpha}{2\alpha+1}} \geqslant 1\end{aligned}$$

always holds.

② $e_2 \geqslant e_1$

$$\begin{aligned}\frac{e_2}{e_1} \geqslant 1 &\Longleftrightarrow \lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1^q)^{-\frac{1}{2\alpha+1}}(v_2^q)^{\frac{1}{2\alpha+1}} \geqslant 1\\ &\Longleftrightarrow \frac{v_2^q}{\lambda_1^{-1}\widetilde{v}_1^q} \geqslant 1\\ &\Longleftrightarrow \overline{v}^q \geqslant \lambda_1^{-1}\widetilde{v}_1^q\end{aligned}$$

According to the results in (1), there must exist some parameter configurations where $e_2^q \geqslant e_1^q$ is satisfied.

When $e_2^q \geqslant e_1^q$ is satisfied, $\overline{v}^q \geqslant \lambda_1^{-1}\widetilde{v}_1^q$. The equilibrium efforts are

$$e_1^q = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha+1}{2\alpha+1}}(v_2^q)^{\frac{\alpha}{2\alpha+1}}$$

$$
\begin{aligned}
e_2^q &= \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \frac{\alpha}{2c}(\lambda_1^{-1}\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{\frac{\alpha+1}{2\alpha+1}} \\
&\leqslant \frac{\alpha}{2c}\overline{v}^q = \overline{e}^q
\end{aligned}
$$

where $e_1^q \leqslant e_2^q \leqslant \overline{e}^q$.

The equilibrium winning probabilities are

$$
p_{12}^q = \frac{1}{2}\left(\frac{e_1^q}{e_2^q}\right)^\alpha = \frac{1}{2}\left(\frac{(\lambda_1^{-1}\widetilde{v}_1^q)^{\frac{1}{2\alpha+1}}}{(v_2^q)^{\frac{1}{2\alpha+1}}}\right)^\alpha = \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{-\frac{\alpha}{2\alpha+1}}
$$

$$
p_{21}^q = 1 - p_{12}^q = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{-\frac{\alpha}{2\alpha+1}}
$$

$$
\widetilde{p}_{12}^q = 1 - \frac{1}{2}\frac{(e_2^q)^\alpha}{\lambda_1 (e_1^q)^\alpha} = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1^q)^{-\frac{\alpha}{2\alpha+1}}(v_2^q)^{\frac{\alpha}{2\alpha+1}}
$$

Thus we have $\widetilde{p}_{12}^q > p_{21}^q \geqslant \frac{1}{2} \geqslant p_{12}^q$.

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1^q - ce_1$

Player 2 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right)v_2^q - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^\alpha}\widetilde{v}_1^q - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_2^{\alpha+1}}v_2^q - c = 0$

divide the two F.O.C , we get

$$
\frac{e_2}{e_1} = \frac{v_2^q}{\lambda_1\widetilde{v}_1^q} < 1 \quad (\widetilde{v}_1^q > v_2^q)
$$

which contradicts the condition that $e_2 \geqslant e_1$

To conclude, under the condition that $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 \geqslant \hat{\lambda}$, depending on the utility spread and overconfidence level, in equilibrium the overconfident player can exert either lower or higher effort than the rational player. Both situations exist with certainty. We know for sure that $e_1^q > \overline{e}^q > e_2^q$ is satisfied at $\lambda_1 = \hat{\lambda}$.

4. The equilibrium when $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$, which corresponds to (ii)

Player 1 $\quad max \quad \widetilde{E}^q(U_{12}) = \widetilde{p}_{12}^q\widetilde{v}_1^q - ce_1$

$$
= \begin{cases} \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1^q - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_2^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1^q - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_2^\alpha \end{cases}
$$

Player 2 $\quad max \quad E^q(U_{21}) = p_{21}^q v_2^q - ce_2$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right) v_2^q - ce_2 & \text{if } e_2 \geqslant e_1 \\ \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha} v_2^q - ce_2 & \text{if } e_2 \leqslant e_1 \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \end{cases}$$

Since $\lambda_1 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1$

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1^q)^{1-\alpha}(v_2^q)^\alpha$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1^q)^{-\alpha}(v_2^q)^{1+\alpha}$$

$$\frac{e_2}{e_1} = \lambda_1 \frac{v_2^q}{\widetilde{v}_1^q}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \leqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

As long as $e_1 \geqslant e_2$ is satisfied, $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ is satisfied.

② $e_2 \leqslant e_1$

$$\frac{e_2}{e_1} = \lambda_1 \frac{v_2^q}{\widetilde{v}_1^q} = \frac{\overline{v}^q}{\lambda_1^{-1}\widetilde{v}_1^q}$$

Let

$$f(\lambda_1) = \frac{\lambda_1^{-1}\widetilde{v}_1^q - \overline{v}^q}{u(w_1) - u(w_2)}$$

$$\begin{aligned} f(\lambda_1) = \lambda_1^{-1} \Bigg[& \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) \\ & - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}} \\ & \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} + \frac{u(w_3)}{u(w_1) - u(w_2)} \Bigg] \\ & - \left[\left(1 - \frac{1+\alpha}{2}\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right) + \frac{u(w_3)}{u(w_1) - u(w_2)}\right] \end{aligned}$$

We can easily get

$$f(\lambda_1 = 1) = 0 \quad \text{and} \quad f(\lambda_1 \to \infty) < 0.$$

$$\begin{aligned}
f'(\lambda_1) = &-\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{u(w_3)}{u(w_1)-u(w_2)}\right)\lambda_1^{-2} \\
&+ \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1}+1\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-2} \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\Bigg[-\left(\frac{\alpha+1}{2\alpha+1}+1\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-2}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&+ \lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\frac{\alpha+1}{2\alpha+1}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}} \\
&\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\Bigg] \\
= \lambda_1^{-2}&\Bigg[-\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{u(w_3)}{u(w_1)-u(w_2)}\right) \\
&+ \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1}+1\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\Bigg(-\left(\frac{\alpha+1}{2\alpha+1}+1\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&+ \lambda_1^{-\frac{2\alpha+2}{2\alpha+1}}\frac{1+\alpha}{2}\frac{(\alpha+1)^2}{(2\alpha+1)^2}\left(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{1+\alpha}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}\Bigg)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
f'(\lambda_1 = 1) = &- \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} + \frac{u(w_3)}{u(w_1) - u(w_2)}\right) + \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\Bigg[- \left(\frac{\alpha+1}{2\alpha+1} + 1\right)\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&+ \frac{\alpha+1}{2\alpha+1}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{-\frac{\alpha}{2\alpha+1}} \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\Bigg] \\
= &- \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} + \frac{u(w_3)}{u(w_1) - u(w_2)}\right) + \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}\left(-\left(\frac{\alpha+1}{2\alpha+1} + 1\right)\right) \\
&\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&- \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} \frac{\alpha+1}{2\alpha+1} \\
&\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{-\frac{\alpha}{2\alpha+1}} \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
= &- \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} + \frac{u(w_3)}{u(w_1) - u(w_2)}\right) + \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&+ \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&- \frac{1+\alpha}{2}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha+1}{2\alpha+1}
\end{aligned}
$$

$$
\begin{aligned}
&f'(\lambda_1 = 1) \lesseqqgtr 0 \\
&\iff - \left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} + \frac{u(w_3)}{u(w_1) - u(w_2)}\right) + \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&+ \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&- \frac{1+\alpha}{2}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha+1}{2\alpha+1} \lesseqqgtr 0 \\
&\iff \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1} + 1\right) + \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&- \frac{1+\alpha}{2}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha+1}{2\alpha+1} \lesseqqgtr 1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} + \frac{u(w_3)}{u(w_1) - u(w_2)} \\
&\iff \frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1} + 1\right) + \frac{1+\alpha}{2}\left(1 + \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\left(\frac{\alpha+1}{2\alpha+1} + 1\right) \\
&- \frac{1+\alpha}{2}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\frac{\alpha+1}{2\alpha+1} - 1 - \frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} \lesseqqgtr \frac{u(w_3)}{u(w_1) - u(w_2)} \\
&\iff \frac{\alpha(3\alpha+1)}{2(2\alpha+1)}\frac{u(w_2) - u(w_3)}{u(w_1) - u(w_2)} + \frac{-7\alpha^4 + \alpha^3 + 8\alpha^2 + 5\alpha + 1}{4(2\alpha+1)^2} \lesseqqgtr \frac{u(w_3)}{u(w_1) - u(w_2)}
\end{aligned}
$$

If $\frac{u(w_3)}{u(w_1)-u(w_2)} < \frac{\alpha(3\alpha+1)}{2(2\alpha+1)}\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{-7\alpha^4+\alpha^3+8\alpha^2+5\alpha+1}{4(2\alpha+1)^2}$, then $f'(\lambda_1 = 1) > 0$ and thus $f(\lambda_1) > 0$ exists. This means $e_1 \geqslant e_2$ must be satisfied under some domains of overconfidence level given that $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} < \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$. In this situation we are certain that case 1 equilibrium exists under some parameter configurations.

If $\frac{u(w_3)}{u(w_1)-u(w_2)} > \frac{\alpha(3\alpha+1)}{2(2\alpha+1)}\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{-7\alpha^4+\alpha^3+8\alpha^2+5\alpha+1}{4(2\alpha+1)^2}$, then $f'(\lambda_1 = 1) < 0$ and thus $f(\lambda_1) < 0$ exists. In this situation we do not know if $e_1 \geqslant e_2$ will be satisfied given that $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} < \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$. It is not clear whether case 1 equilibrium exists.

(2) case 2: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1$

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha+1}{2\alpha+1}}(v_2^q)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{\frac{\alpha+1}{2\alpha+1}}$$

$$\frac{e_2}{e_1} = \lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1^q)^{-\frac{1}{2\alpha+1}}(v_2^q)^{\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \geqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

$$\lambda_1 e_1^\alpha \geqslant e_2^\alpha \Longleftrightarrow \frac{\lambda_1 e_1^\alpha}{e_2^\alpha} \geqslant 1$$
$$\Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{-\frac{\alpha}{2\alpha+1}} \geqslant 1$$

always holds.

② $e_2 \geqslant e_1$

$$\frac{e_2}{e_1} \geqslant 1 \Longleftrightarrow \lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1^q)^{-\frac{1}{2\alpha+1}}(v_2^q)^{\frac{1}{2\alpha+1}} \geqslant 1$$
$$\Longleftrightarrow \frac{v_2^q}{\lambda_1^{-1}\widetilde{v}_1^q} \geqslant 1$$
$$\Longleftrightarrow \overline{v}^q \geqslant \lambda_1^{-1}\widetilde{v}_1^q$$

Similar to the results in (1):
If $\frac{u(w_3)}{u(w_1)-u(w_2)} < \frac{\alpha(3\alpha+1)}{2(2\alpha+1)}\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{-7\alpha^4+\alpha^3+8\alpha^2+5\alpha+1}{4(2\alpha+1)^2}$, then $f'(\lambda_1 = 1) > 0$. This means $f(\lambda_1) < 0$ and thus $e_1 \leqslant e_2$ are satisfied when λ_1 is extremely large, given that $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$. In this situation we are certain that case 2 equilibrium exists under some parameter configurations.

If $\frac{u(w_3)}{u(w_1)-u(w_2)} > \frac{\alpha(3\alpha+1)}{2(2\alpha+1)}\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{-7\alpha^4+\alpha^3+8\alpha^2+5\alpha+1}{4(2\alpha+1)^2}$, then $f'(\lambda_1 = 1) < 0$ and thus $f(\lambda_1) < 0$ exists with certainty when λ_1 is close to 1 or extremely large. In this situation we are also certain that case 2 equilibrium exists under some parameter configurations.

When $e_2^q \geqslant e_1^q$ is satisfied, $\overline{v}^q \geqslant \lambda_1^{-1}\widetilde{v}_1^q$. The equilibrium efforts are

$$e_1^q = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha+1}{2\alpha+1}}(v_2^q)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2^q = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{\frac{\alpha+1}{2\alpha+1}}$$

where $e_1^q \leqslant e_2^q \leqslant \overline{e}^q$.

$$p_{12}^q = \frac{1}{2}\left(\frac{e_1^q}{e_2^q}\right)^{\alpha} = \frac{1}{2}\left(\frac{\left(\lambda_1^{-1}\widetilde{v}_1^q\right)^{\frac{1}{2\alpha+1}}}{(v_2^q)^{\frac{1}{2\alpha+1}}}\right)^{\alpha} = \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{-\frac{\alpha}{2\alpha+1}}$$

$$p_{21}^q = 1 - p_{12}^q = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1^q)^{\frac{\alpha}{2\alpha+1}}(v_2^q)^{-\frac{\alpha}{2\alpha+1}}$$

$$\widetilde{p}_{12}^q = 1 - \frac{1}{2}\frac{(e_2^q)^{\alpha}}{\lambda_1(e_1^q)^{\alpha}} = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1^q)^{-\frac{\alpha}{2\alpha+1}}(v_2^q)^{\frac{\alpha}{2\alpha+1}}$$

where $\widetilde{p}_{12}^q > p_{21}^q \geqslant \frac{1}{2} \geqslant p_{12}^q$.

(3) case 3: $\lambda_1 e_1^{\alpha} \leqslant e_2^{\alpha}$ *and* $e_2 \geqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^{\alpha}}{e_2^{\alpha}}\widetilde{v}_1^q - ce_1$

Player 2 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^{\alpha}}{e_2^{\alpha}}\right)v_2^q - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^{\alpha}}\widetilde{v}_1^q - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^{\alpha}}{e_2^{\alpha+1}}v_2^q - c = 0$

divide the two F.O.C , we get

$$\frac{e_2}{e_1} = \frac{v_2^q}{\lambda_1\widetilde{v}_1^q} < 1 \quad (\widetilde{v}_1^q > v_2^q)$$

which contradicts the condition that $e_2 \geqslant e_1$

To conclude, under the condition $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$, case 2 equilibrium exists with certainty. Whether case 1 equilibrium depends on the relationship between the utility spread in each stage. When $\frac{u(w_3)}{u(w_1)-u(w_2)} < \frac{\alpha(3\alpha+1)}{2(2\alpha+1)}\frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} + \frac{-7\alpha^4+\alpha^3+8\alpha^2+5\alpha+1}{4(2\alpha+1)^2}$ and λ_1 is close to 1, $e_1^q > \overline{e}^q > e_2^q$ is satisfied and the overconfident player exerts higher effort at equilibrium than the rational player and the benchmark.

5. Participation constraints

(1) When $e_1^q > e_2^q$

$$
\begin{aligned}
\widetilde{E}^q(U_{12}) &= \widetilde{p}_{12}^q \widetilde{v}_1^q - ce_1^q \\
&> p_{12}^q \widetilde{v}_1^q - ce_1^q \\
&= \left(1 - \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1^q\right)^{-\alpha} \left(v_2^q\right)^\alpha\right) \widetilde{v}_1^q - c\frac{\alpha}{2c}\lambda_1^{\alpha-1} \left(\widetilde{v}_1^q\right)^{1-\alpha} \left(v_2^q\right)^\alpha \\
&= \widetilde{v}_1^q - \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1^q\right)^{1-\alpha} \left(v_2^q\right)^\alpha - \frac{\alpha}{2}\lambda_1^{\alpha-1} \left(\widetilde{v}_1^q\right)^{1-\alpha} \left(v_2^q\right)^\alpha \\
&> \widetilde{v}_1^q - \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1^q\right)^{1-\alpha} \left(v_2^q\right)^\alpha - \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1^q\right)^{1-\alpha} \left(v_2^q\right)^\alpha \\
&= \widetilde{v}_1^q - \lambda_1^\alpha \left(\widetilde{v}_1^q\right)^{1-\alpha} \left(v_2^q\right)^\alpha \\
&= \left(\widetilde{v}_1^q\right)^{1-\alpha} \left[\left(\widetilde{v}_1^q\right)^\alpha - \lambda_1^\alpha \left(v_2^q\right)^\alpha\right] \\
&> 0
\end{aligned}
$$

$$
\begin{aligned}
E^q(U_{21}) &= p_{21}^q v_2^q - ce_2^q \\
&= \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1^q\right)^{-\alpha} \left(v_2^q\right)^\alpha v_2^q - c\frac{\alpha}{2c}\lambda_1^\alpha \left(\widetilde{v}_1^q\right)^{-\alpha} \left(v_2^q\right)^{\alpha+1} \\
&= \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1^q\right)^{-\alpha} \left(v_2^q\right)^{1+\alpha} - \frac{\alpha}{2}\lambda_1^\alpha \left(\widetilde{v}_1^q\right)^{-\alpha} \left(v_2^q\right)^{\alpha+1} \\
&= \frac{1-\alpha}{2}\lambda_1^\alpha \left(\widetilde{v}_1^q\right)^{-\alpha} \left(v_2^q\right)^{1+\alpha} \\
&\geqslant 0
\end{aligned}
$$

(2) When $e_1^q \leqslant e_2^q$

$$
\widetilde{E}^q(U_{12}) = \widetilde{p}_{12}^q \widetilde{v}_1^q - ce_1^q
$$

Since $\widetilde{p}_{12}^q > \frac{1}{2}$, $\widetilde{v}_1^q > \overline{v}^q$ and $e_1^q \leqslant \overline{e}^q$, we can get that $\widetilde{E}^q(U_{12}) > \overline{E}^q(U) \geqslant 0$.

$$
\begin{aligned}
E^q(U_{21}) &= p_{21}^q v_2^q - ce_2^q \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2^q\right)^{-\frac{\alpha}{2\alpha+1}}\right) v_2^q - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2^q\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2^q\right)^{-\frac{\alpha}{2\alpha+1}}\right) v_2^q - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2^q\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= v_2^q - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2^q\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \left(v_2^q\right)^{\frac{\alpha+1}{2\alpha+1}} \left(\left(v_2^q\right)^{\frac{\alpha}{2\alpha+1}} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1^q\right)^{\frac{\alpha}{2\alpha+1}}\right) \\
&\geqslant 0
\end{aligned}
$$

Next we characterize the equilibrium of the quarterfinal between rational players 3 and 4.

Proposition A10 *In a three-stage elimination contest where player 1 is overconfident and the other seven players are rational, consider the quarterfinal between two rational*

players who have the chance of meeting the overconfident player in the semifinal.
(i) If $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ *and* $\lambda_1 < \hat{\lambda}$ *where* $\hat{\lambda} > 1$ *solves* $1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}} = \hat{\lambda}(1 + \frac{u(w_2)-u(w_3)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2})$, *then the equilibrium efforts and winning probabilities satisfy* $e_3^q = e_4^q < \overline{e}^q$ *and* $p_{34}^q = p_{43}^q = 1/2$.
(ii) If either $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ *or* $\lambda_1 \geqslant \hat{\lambda}$, *then the equilibrium efforts and winning probabilities satisfy* $e_3^q = e_4^q \geqslant \overline{e}^q$ *and* $p_{34}^q = p_{43}^q = 1/2$.

Proposition A10 shows that, for the rational players who have the chance to meet the overconfident player in the semifinal, their expected utility of reaching the semifinal depend on the equilibrium in the semifinal. When the parameter configurations are such that the overconfident player exerts lower (higher) effort than his rational rival in the equilibrium of the semifinal, the rational players' expected utility of reaching the semifinal increase (decrease) and thus they both exert more (less) effort.

Proof of Proposition A10

1. Expected utilities of reaching the semifinal

Player 3's expected utility of reaching the semifinal is given by

$$v_3^q = p_{12}^q E^s(U_{31}) + p_{21}^q E^s(U_{32})$$

(1) When $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 < \hat{\lambda}$

$$\begin{aligned}
E^s(U_{31}) &= p_{31}^s\left(v_3^s - u(w_3)\right) - ce_3^s + u(w_3) \\
&= \frac{1}{2}\lambda_1^\alpha\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(v_3^s - u(w_3)\right)^\alpha\left(v_3^s - u(w_3)\right) \\
&\quad - c\frac{\alpha}{2c}\lambda_1^\alpha\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(v_3^s - u(w_3)\right)^{\alpha+1} + u(w_3) \\
&= \frac{1}{2}\lambda_1^\alpha\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(v_3^s - u(w_3)\right)^{1+\alpha} \\
&\quad - \frac{\alpha}{2}\lambda_1^\alpha\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(v_3^s - u(w_3)\right)^{\alpha+1} + u(w_3) \\
&= \frac{1-\alpha}{2}\lambda_1^\alpha\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(v_3^s - u(w_3)\right)^{1+\alpha} + u(w_3) \\
&= \frac{1-\alpha}{2}\lambda_1^\alpha\left(\widetilde{v}_1^s - u(w_3)\right)^{-\alpha}\left(\overline{v}^s - u(w_3)\right)^{1+\alpha} + u(w_3) \\
&= \frac{1-\alpha}{2}\left(\frac{\overline{v}^s - u(w_3)}{\lambda_1^{-1}\left(\widetilde{v}_1^s - u(w_3)\right)}\right)^\alpha\left(\overline{v}^s - u(w_3)\right) + u(w_3) \\
&< \overline{E}^s(U)
\end{aligned}$$

$$E^s(U_{32}) = \overline{E}^s(U)$$

Thus we have

$$v_3^q < \overline{v}^q$$

(2) When either $\frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ or $\lambda_1 \geqslant \hat{\lambda}$

$$
\begin{aligned}
E^s(U_{31}) &= p_{31}^s\left(v_3^s - u(w_3)\right) - ce_3^s + u(w_3) \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{-\frac{\alpha}{2\alpha+1}}\right)\left(v_3^s - u(w_3)\right) \\
&\quad - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} + u(w_3) \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{-\frac{\alpha}{2\alpha+1}}\right)\left(v_3^s - u(w_3)\right) \\
&\quad - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} + u(w_3) \\
&= \left(v_3^s - u(w_3)\right) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(v_3^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} + u(w_3) \\
&= \left(\overline{v}^s - u(w_3)\right) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1^s - u(w_3)\right)^{\frac{\alpha}{2\alpha+1}}\left(\overline{v}^s - u(w_3)\right)^{\frac{\alpha+1}{2\alpha+1}} + u(w_3) \\
&= \left(\overline{v}^s - u(w_3)\right)\left[1 - \frac{1+\alpha}{2}\left(\frac{\lambda_1^{-1}\left(\widetilde{v}_1^s - u(w_3)\right)}{\overline{v}^s - u(w_3)}\right)^{\frac{\alpha}{2\alpha+1}}\right] + u(w_3) \\
&\geqslant \overline{E}^s(U)
\end{aligned}
$$

$$E^s(U_{32}) = \overline{E}^s(U)$$

$$v_3^q \geqslant \overline{v}^q$$

Since player 3 and player 4 are identical, $v_4^q = v_3^q$.

2. Equilibrium efforts

$$e_3^q = e_4^q = \frac{\alpha}{2c}v_3^q$$

$$e_3^q = e_4^q \begin{cases} < \overline{e}^q & \text{when} \quad \frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)} \quad \text{and} \quad \lambda_1 < \hat{\lambda} \\ \geqslant \overline{e}^q & \text{when} \quad \text{either} \quad \frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)} \quad \text{or} \quad \lambda_1 \geqslant \hat{\lambda} \end{cases}$$

3. Participation constraints

$$
\begin{aligned}
E^q(U_{34}) = E^q(U_{43}) = p_{34}^q v_3^q - ce_3 &= \frac{1}{2}v_3^q - c\frac{\alpha}{2c}v_3^q \\
&= \frac{1-\alpha}{2}v_3^q \\
&\geqslant 0
\end{aligned}
$$

Last we characterize the equilibrium of the quarterfinals between rational players 5 and 6, and 7 and 8.

Proposition A11 *In a three-stage elimination contest where player 1 is overconfident and the other seven players are rational, consider the quarterfinals between the rational players who have the chance of meeting the overconfident player in the final. The equilibrium efforts and winning probabilities satisfy* $e_5^q = e_6^q = e_7^q = e_8^q > \overline{e}^q$, $p_{56}^q = p_{65}^q = p_{78}^q = p_{87}^q = 1/2$.

Proposition A11 shows that, for the rational players who only have the chance to meet the overconfident player in the final, their expected utilities of reaching the semifinal are higher and thus their equilibrium efforts in the quarterfinal increase.

Proof of Proposition A11

1. Expected utilities of reaching the semifinal

$$\begin{aligned} v_5^q &= E^s(U_{57}) \\ &= p_{57}^s\left(v_5^s - u(w_3)\right) - ce_5^s + u(w_3) \end{aligned}$$

   where

$$\begin{aligned} v_5^s &= p_{12}^q p_{13}^s E^f(U_{51}) + \left(1 - p_{12}^q p_{13}^s\right) E^f(U_{53}) \\ &= p_{12}^q p_{13}^s E^f(U_{51}) + \left(1 - p_{12}^q p_{13}^s\right) \overline{E}^f(U) \\ &> \overline{E}^f(U) = \overline{v}^s. \end{aligned}$$

   Therefore

$$v_5^q = v_6^q = v_7^q = v_8^q > \overline{v}^q$$

2. Equilibrium efforts

$$e_5^q = e_6^q = e_7^q = e_8^q = \frac{\alpha}{2c} v_5^q > \overline{e}^q$$

3. Participation constraints

$$\begin{aligned} E^q(U_{56}) = E^q(U_{65}) = E^q(U_{78}) = E^q(U_{87}) = p_{56}^q v_5^q - ce_5 &= \frac{1}{2} v_5^q - c\frac{\alpha}{2c} v_5^q \\ &= \frac{1-\alpha}{2} v_5^q \\ &\geqslant 0 \end{aligned}$$

5 Elimination Contest with One Underconfident and Three Rational Players

This section characterizes the equilibrium of a two-stage elimination contest with one underconfident player and three rational players. Throughout we assume player 1 is underconfident with $0 < \lambda_1 < 1$ and players 2, 3, and 4 are rational with $\lambda_2 = \lambda_3 = \lambda_4 = 1$. Players 1 and 2 are paired in one semifinal and players 3 and 4 are paired in the other semifinal.

5.1 Final

We start by analyzing the impact of underconfidence on the final. Since players 3 and 4 are identical, we consider a final with an underconfident player 1 and a rational player 3 without loss of generality.

Proposition A12 *In a final between an underconfident player and a rational player, the equilibrium effort of the underconfident player is*

$$e_1^f = \frac{\alpha}{2c} \lambda_1^{1+\alpha} \Delta u,$$

and the equilibrium effort of the rational player is

$$e_3^f = \frac{\alpha}{2c} \lambda_1^{\alpha} \Delta u.$$

with $e_1^f < e_3^f < \bar{e}^f$. The perceived equilibrium winning probability of the underconfident player is

$$\widetilde{p}_{13}^f = \frac{1}{2}\lambda_1^{1+\alpha},$$

and the true equilibrium winning probabilities are

$$p_{13}^f = \frac{1}{2}\lambda_1^\alpha$$

$$p_{31}^f = 1 - \frac{1}{2}\lambda_1^\alpha$$

with $p_{31}^f > 1/2 > p_{13}^f > \widetilde{p}_{13}^f$. The perceived equilibrium expected utility of the underconfident player is

$$\widetilde{E}^f(U_{13}) = u(w_2) + \frac{1-\alpha}{2}\lambda_1^{\alpha+1}\Delta u,$$

and the equilibrium expected utility of the rational player is

$$E^f(U_{31}) = u(w_1) - \frac{1+\alpha}{2}\lambda_1^\alpha \Delta u,$$

with $E^f(U_{31}) > \overline{E}^f(U) > \widetilde{E}^f(U_{13})$.

Proposition A12 shows that an underconfident player exerts less effort than a rational rival in the final and that both players exert less effort than if both were rational. It also shows that the underconfident player's perceived and true probabilities of winning the final are decreasing in his bias whereas the rational player's true probability of winning the final is increasing with the bias of the underconfident player. Finally, Proposition A12 shows that the underconfident player's perceived expected utility of the final is decreasing in his bias whereas the rational player's expected utility of the final is increasing in the bias of the underconfident player. Hence, underconfidence makes reaching the final less attractive for an underconfident player and more attractive for a rational rival.

Proof of Proposition A12

The perceived winning probabilities of the players are:

$$\widetilde{p}_{13}^f = \begin{cases} 1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha} & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha} & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

$$p_{31}^f = \begin{cases} 1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha} & \text{if} \quad e_3 \geqslant e_1 \\ \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha} & \text{if} \quad e_3 \leqslant e_1 \end{cases}$$

Underconfident player 1 $\quad max \quad \widetilde{E}^f(U_{13}) = \widetilde{p}_{13}^f \Delta u - ce_1 + u(w_2)$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)\Delta u - ce_1 + u(w_2) & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}\Delta u - ce_1 + u(w_2) & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

Rational player 3 $\quad max \quad E^f(U_{31}) = p_{31}^f \Delta u - ce_3 + u(w_2)$

$$=\begin{cases} \left(1-\frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)\Delta u - ce_3 + u(w_2) & \text{if} \quad e_3 \geqslant e_1 \\ \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\Delta u - ce_3 + u(w_2) & \text{if} \quad e_3 \leqslant e_1 \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \leqslant e_3^\alpha & and \quad e_3 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_3^\alpha & and \quad e_3 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_3^\alpha & and \quad e_3 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_3^\alpha & and \quad e_3 \geqslant e_1 \end{cases}$$

Since $\lambda_1 < 1$, the fourth case is impossible.

1. Equilibrium efforts

   (1) case 1 $\lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1$

   Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}\Delta u - ce_1 + u(w_2)$

   Player 3 $\quad max \quad \left(1-\frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)\Delta u - ce_3 + u(w_2)$

   F.o.c

   $[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_3^\alpha}\Delta u - c = 0$

   $[e_3] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_3^{\alpha+1}}\Delta u - c = 0$

   S.o.c

   $[e_1] \quad \frac{\alpha\lambda_1}{2}(\alpha-1)\frac{e_1^{\alpha-2}}{e_3^\alpha}\Delta u < 0$

   $[e_2] \quad \frac{\alpha}{2}(-\alpha-1)\frac{e_1^\alpha}{e_3^{\alpha+2}}\Delta u < 0$

   Solve F.O.C , we get

   $$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha+1}\Delta u$$

   $$e_3 = \frac{\alpha}{2c}\lambda_1^{\alpha}\Delta u$$

   Check the conditions $\lambda_1 e_1^\alpha \leqslant e_3^\alpha$ and $e_3 \geqslant e_1$:

   ① $\lambda_1 e_1^\alpha \leqslant e_3^\alpha$

   $$\begin{aligned} \lambda_1 e_1^\alpha \leqslant e_3^\alpha &\Longleftrightarrow \lambda_1 \left(\frac{e_1}{e_3}\right)^\alpha \leqslant 1 \\ &\Longleftrightarrow \lambda_1^{1+\alpha} \leqslant 1 \end{aligned}$$

   which always holds.

   ② $e_3 \geqslant e_1$

   $$\frac{e_3}{e_1} = \frac{1}{\lambda_1} > 1$$

$e_3 \geqslant e_1$ always holds.

(2) case 2 $\lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}\Delta u - ce_1 + u(w_2)$

Player 3 $\quad max \quad \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\Delta u - ce_3 + u(w_2)$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_3^\alpha}\Delta u - c = 0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_3^{\alpha-1}}{e_1^\alpha}\Delta u - c = 0$

divide the two F.O.C , we get

$$\lambda_1 \left(\frac{e_1}{e_3}\right)^{2\alpha-1} = 1$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$ and $e_3 \geqslant e_1$:

$$\frac{e_3}{e_1} = \lambda_1^{\frac{1}{2\alpha-1}}$$

$$\lambda_1 \left(\frac{e_1}{e_3}\right)^{\alpha} = \lambda_1^{\frac{\alpha-1}{2\alpha-1}}$$

Since $\alpha - 1 < 0$, $\lambda_1^{\frac{1}{2\alpha-1}}$ and $\lambda_1^{\frac{\alpha-1}{2\alpha-1}}$ have different signs, one of the conditions must be contradicted. Case 2 does not hold.

(3) case 3 $\lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)\Delta u - ce_1 + u(w_2)$

Player 3 $\quad max \quad \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\Delta u - ce_3 + u(w_2)$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_3^\alpha}{e_1^{\alpha+1}}\Delta u - c = 0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_3^{\alpha-1}}{e_1^\alpha}\Delta u - c = 0$

divide the two F.O.C , we get

$$\frac{e_3}{e_1} = \lambda_1 < 1$$

Thus we get

$$\lambda_1 \left(\frac{e_1}{e_3}\right)^{\alpha} = \lambda_1^{1-\alpha} < 1$$

which contradicts the condition that $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$. Case 3 does not hold.

Thus the unique equilibrium is

$$e_1^f = \frac{\alpha}{2c}\lambda_1^{1+\alpha}\Delta u < \bar{e}^f$$

$$e_3^f = \frac{\alpha}{2c}\lambda_1^\alpha \Delta u < \overline{e}^f$$

where $\lambda_1 e_1^\alpha < e_3^\alpha$ and $e_3 > e_1$.

2. Winning probabilities

$$p_{13}^f = \frac{1}{2}\left(\frac{e_1}{e_3}\right)^\alpha = \frac{1}{2}\lambda_1^\alpha$$

$$p_{31}^f = 1 - p_{13}^f = 1 - \frac{1}{2}\lambda_1^\alpha$$

$$\widetilde{p}_{13}^f = \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha} = \frac{1}{2}\lambda_1^{1+\alpha}$$

where $p_{31}^f > \frac{1}{2} > p_{13}^f > \widetilde{p}_{13}^f$.

3. Perceived expected utilities of the final

$$\begin{aligned}
\widetilde{E}^f(U_{13}) &= \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}\Delta u - ce_1 + u(w_2) \\
&= \frac{1}{2}\lambda_1^{1+\alpha}\Delta u - \frac{\alpha}{2}\lambda_1^{1+\alpha}\Delta u + u(w_2) \\
&= \left[\left(1 - \frac{1+\alpha}{2}\right)\lambda_1^{\alpha+1} + \frac{u(w_2)}{u(w_1) - u(w_2)}\right]\Delta u < \overline{E}^f(U)
\end{aligned}$$

$$\begin{aligned}
E^f(U_{31}) &= \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)\Delta u - ce_3 + u(w_2) \\
&= \left(1 - \frac{1}{2}\lambda_1^\alpha\right)\Delta u - \frac{\alpha}{2}\lambda_1^\alpha \Delta u + u(w_2) \\
&= \left(1 + \frac{u(w_2)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^\alpha\right)\Delta u > \overline{E}^f(U)
\end{aligned}$$

The participation constraints are satisfied.

5.2 Semifinals

We now analyze the impact of underconfidence on the two semifinals. We start with the semifinal with an underconfident and a rational player. Next, we consider the semifinal with two rational players.

Proposition A13 *In a semifinal between an underconfident player and a rational player of a two-stage elimination contest where player 1 is underconfident and players 2, 3, and 4 are rational, the equilibrium efforts and winning probabilities satisfy* $\overline{e}^s > e_2^s > e_1^s$ *and* $p_{21}^s > 1/2 > p_{12}^s$.

Proposition A13 shows that in the semifinal, unlike overconfidence who has opposite effects on overconfident players, underconfidence only has negative effects on the underconfident player since the perceived expected utility of reaching the final of the underconfident player is lower than the benchmark. The underconfident player lowers his effort relative to the benchmark, and the rational player reacts to this by reducing his effort but not as much.

Proof of Proposition A13

1. Perceived expected utilities of reaching the final

   Underconfident player 1:

$$\widetilde{v}_1 = p_{34}^s \widetilde{E}^f(U_{13}) + p_{43}^s \widetilde{E}^f(U_{14})$$

   Since players 3 and 4 are identical, $\widetilde{E}^f(U_{13}) = \widetilde{E}^f(U_{14})$

$$\widetilde{v}_1 = \widetilde{E}^f(U_{13}) = \left[\left(1 - \frac{1+\alpha}{2}\right)\lambda_1^{\alpha+1} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right]\Delta u < \overline{v}$$

   Rational player 2:

$$v_2 = p_{34}^s E^f(U_{23}) + p_{43}^s E^f(U_{24})$$

   Since players 3 and 4 are identical, $E^f(U_{23}) = E^f(U_{24})$

$$v_2 = \left(1 + \frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)\Delta u = \overline{v}$$

2. The equilibrium

   Player 1 $\quad max \quad \widetilde{E}^s(U_{12}) = \widetilde{p}_{12}^s \widetilde{v}_1 - ce_1$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_2^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_2^\alpha \end{cases}$$

   Player 2 $\quad max \quad E^s(U_{21}) = p_{21}^s v_2 - ce_2$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right) v_2 - ce_2 & \text{if}\, e_2 \geqslant e_1 \\ \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha} v_2 - ce_2 & \text{if}\, e_2 \leqslant e_1 \end{cases}$$

   There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad \textit{and} \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad \textit{and} \quad e_2 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad \textit{and} \quad e_2 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad \textit{and} \quad e_2 \geqslant e_1 \end{cases}$$

   Since $\lambda_1 < 1$, the fourth case is impossible.

   (1) case 1: $\lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad \textit{and} \quad e_2 \geqslant e_1$

   Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1$

   Player 2 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right) v_2 - ce_2$

F.o.c

$[e_1]$ $\quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^{\alpha}}\widetilde{v}_1 - c = 0$

$[e_2]$ $\quad \frac{\alpha}{2}\frac{e_1^{\alpha}}{e_2^{\alpha+1}}v_2 - c = 0$

S.o.c

$[e_1]$ $\quad \frac{\alpha\lambda_1}{2}(\alpha-1)\frac{e_1^{\alpha-2}}{e_2^{\alpha}}\widetilde{v}_1 < 0$

$[e_2]$ $\quad \frac{\alpha}{2}(-\alpha-1)\frac{e_1^{\alpha}}{e_2^{\alpha+2}}v_2 < 0$

Solve the two F.O.C , we get

$$\begin{aligned}
e_1 &= \frac{\alpha}{2c}\lambda_1^{\alpha+1}(\widetilde{v}_1)^{1+\alpha}(v_2)^{-\alpha} \\
&= \frac{\alpha}{2c}\lambda_1^{\alpha+1}\left(\left(1-\frac{1+\alpha}{2}\right)\lambda_1^{\alpha+1}+\frac{u(w_2)}{u(w_1)-u(w_2)}\right)^{1+\alpha} \\
&\quad \left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{-\alpha}\Delta u \\
&< \overline{e}^s
\end{aligned}$$

$$\begin{aligned}
e_2 &= \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1)^{\alpha}(v_2)^{1-\alpha} \\
&= \frac{\alpha}{2c}\lambda_1^{\alpha}\left(\left(1-\frac{1+\alpha}{2}\right)\lambda_1^{\alpha+1}+\frac{u(w_2)}{u(w_1)-u(w_2)}\right)^{\alpha} \\
&\quad \left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{1-\alpha}\Delta u \\
&< \overline{e}^s
\end{aligned}$$

Check the conditions $\lambda_1 e_1^{\alpha} \leqslant e_2^{\alpha}$ and $e_2 \geqslant e_1$:

① $\lambda_1 e_1^{\alpha} \leqslant e_2^{\alpha}$

$$\lambda_1 e_1^{\alpha} \leqslant e_2^{\alpha} \quad \text{is satisfied as long as } e_2 \geqslant e_1 \text{ holds.}$$

② $e_2 \geqslant e_1$

$$\frac{e_2}{e_1} = \frac{v_2}{\lambda_1\widetilde{v}_1} > 1$$

always holds.

(2) case 2: $\lambda_1 e_1^{\alpha} \leqslant e_2^{\alpha} \quad and \quad e_2 \leqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^{\alpha}}{e_2^{\alpha}}\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \frac{1}{2}\frac{e_2^{\alpha}}{e_1^{\alpha}}v_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^{\alpha}}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_2^{\alpha-1}}{e_1^{\alpha}}v_2 - c = 0$

Divide the two F.O.C , we get

$$\frac{e_1}{e_2} = \left(\frac{v_2}{\lambda_1\widetilde{v}_1}\right)^{\frac{1}{2\alpha-1}}$$

Since $v_2 > \lambda_1\widetilde{v}_1$, the condition $e_2 \leqslant e_1$ is satisfied if and only if $2\alpha - 1 \geqslant 0$. Now assume that $\alpha \geqslant \frac{1}{2}$ is satisfied.

$$\begin{aligned} e_1 &= \frac{\alpha}{2c}\lambda_1^{\frac{\alpha-1}{2\alpha-1}}(\widetilde{v}_1)^{\frac{\alpha-1}{2\alpha-1}}(v_2)^{\frac{\alpha}{2\alpha-1}} \\ &> \overline{e}^s \end{aligned}$$

$$\begin{aligned} e_2 &= \frac{\alpha}{2c}\lambda_1^{\frac{\alpha}{2\alpha-1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha-1}}(v_2)^{\frac{\alpha-1}{2\alpha-1}} \\ &< \overline{e}^s \end{aligned}$$

Check the condition $\lambda_1 e_1^{\alpha} \leqslant e_2^{\alpha}$:

$$\lambda_1\left(\frac{e_1}{e_2}\right)^{\alpha} = \lambda_1^{\frac{\alpha-1}{2\alpha-1}}(\widetilde{v}_1)^{-\frac{\alpha}{2\alpha-1}}(v_2)^{\frac{\alpha}{2\alpha-1}}$$

$$\begin{aligned} \lambda_1\left(\frac{e_1}{e_2}\right)^{\alpha} < 1 &\Longleftrightarrow \lambda_1^{\frac{\alpha-1}{2\alpha-1}}(\widetilde{v}_1)^{-\frac{\alpha}{2\alpha-1}}(v_2)^{\frac{\alpha}{2\alpha-1}} < 1 \\ &\Longleftrightarrow (v_2)^{\frac{\alpha}{2\alpha-1}} < \lambda_1^{\frac{1-\alpha}{2\alpha-1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha-1}} \end{aligned}$$

which is not satisfied under the assumption $\alpha \geqslant \frac{1}{2}$.

(3) case 3: $\lambda_1 e_1^{\alpha} \geqslant e_2^{\alpha} \quad and \quad e_2 \leqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^{\alpha}}{\lambda_1 e_1^{\alpha}}\right)\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \frac{1}{2}\frac{e_2^{\alpha}}{e_1^{\alpha}}v_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^{\alpha}}{e_1^{\alpha+1}}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_2^{\alpha-1}}{e_1^{\alpha}}v_2 - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^{\alpha}}{e_1^{\alpha+2}}\widetilde{v}_1 < 0$

$[e_2] \quad \frac{\alpha}{2}(\alpha-1)\frac{e_2^{\alpha-2}}{e_1^{\alpha}}v_2 < 0$

Divide the two F.O.C , we get

$$\frac{e_2}{e_1} = \lambda_1\frac{v_2}{\widetilde{v}_1}$$

Check the condition $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$:

$$\begin{aligned}\lambda_1 e_1^\alpha \geqslant e_2^\alpha &\iff \frac{\lambda_1 e_1^\alpha}{e_2^\alpha} \geqslant 1\\ &\iff \lambda_1^{1-\alpha}(\widetilde{v}_1)^\alpha (v_2)^{-\alpha} \geqslant 1\end{aligned}$$

which contradicts with $\lambda_1 < 1$ and $\widetilde{v}_1 < v_2$

Thus the unique equilibrium is

$$\begin{aligned}e_1^s = &\frac{\alpha}{2c}\lambda_1^{\alpha+1}\left(\frac{1-\alpha}{2}\lambda_1^{1+\alpha} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)^{1+\alpha}\\ &\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha}\Delta u\end{aligned}$$

$$\begin{aligned}e_2^s = &\frac{\alpha}{2c}\lambda_1^{\alpha}\left(\frac{1-\alpha}{2}\lambda_1^{1+\alpha} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)^{\alpha}\\ &\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{1-\alpha}\Delta u\end{aligned}$$

where $e_1^s < e_2^s < \overline{e}^s$.

$$\begin{aligned}p_{12}^s &= \frac{1}{2}\left(\frac{e_1^s}{e_2^s}\right)^\alpha\\ &= \frac{1}{2}\left(\frac{\lambda_1\widetilde{v}_1}{v_2}\right)^\alpha\\ &= \frac{1}{2}\lambda_1^\alpha\left(\frac{1-\alpha}{2}\lambda_1^{1+\alpha} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)^{\alpha}\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha}\end{aligned}$$

$$\begin{aligned}p_{21}^s &= 1 - p_{12}^s\\ &= 1-\frac{1}{2}\lambda_1^\alpha\left(\frac{1-\alpha}{2}\lambda_1^{1+\alpha} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)^{\alpha}\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha}\end{aligned}$$

$$\begin{aligned}\widetilde{p}_{12}^s &= \frac{1}{2}\lambda_1\left(\frac{e_1^s}{e_2^s}\right)^\alpha\\ &= \frac{1}{2}\lambda_1^{\alpha+1}\left(\frac{1-\alpha}{2}\lambda_1^{1+\alpha} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)^{\alpha}\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha}\end{aligned}$$

where $p_{21}^s > \frac{1}{2} > p_{12}^s > \widetilde{p}_{12}^s$.

3. Participation constraints

$$\begin{aligned}\widetilde{E}^s(U_{12}) &= \widetilde{p}_{12}^s\widetilde{v}_1 - ce_1^s\\ &= \frac{1}{2}\lambda_1^{\alpha+1}\left(\widetilde{v}_1\right)^\alpha\left(v_2\right)^{-\alpha}\widetilde{v}_1 - c\frac{\alpha}{2c}\lambda_1^{\alpha+1}(\widetilde{v}_1)^{1+\alpha}(v_2)^{-\alpha}\\ &= \frac{1-\alpha}{2}\lambda_1^{\alpha+1}(\widetilde{v}_1)^{1+\alpha}(v_2)^{-\alpha}\\ &\geqslant 0\end{aligned}$$

$$
\begin{aligned}
E^s(U_{21}) &= p_{21}^s v_2 - c e_2^s \\
&= \left(1 - \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^\alpha \left(v_2\right)^{-\alpha}\right) v_2 - c\frac{\alpha}{2c}\lambda_1^\alpha (\widetilde{v}_1)^\alpha (v_2)^{1-\alpha} \\
&= \left(1 - \frac{1+\alpha}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^\alpha \left(v_2\right)^{-\alpha}\right) v_2 \\
&> 0
\end{aligned}
$$

Next, we characterize the equilibrium of the semifinal between players 3 and 4.

Proposition A14 *In a semifinal between two rational players of a two-stage elimination contest where player 1 is underconfident and players 2, 3, and 4 are rational, the equilibrium efforts and winning probabilities satisfy* $e_3^s = e_4^s > \overline{e}^s$ *and* $p_{34}^s = p_{43}^s = 1/2$.

Proposition A14 shows that since playing against an underconfident player in the final raises a rational player's expected utility of reaching the final, the rational players 3 and 4 exert higher efforts in the semifinal.

Proof of Proposition A14

1. Expected utilities of reaching the final

   Rational player 3:

$$
\begin{aligned}
v_3 &= p_{12}^s E^f(U_{31}) + p_{21}^s E^f(U_{32}) \\
&= \left[p_{12}^s \left(1 + \frac{u(w_2)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^\alpha\right) + p_{21}^s \left(1 + \frac{u(w_2)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)\right] \\
&\quad \Delta u \\
&> \overline{v}
\end{aligned}
$$

   Rational player 4:

$$
v_4 = v_3 > \overline{v}
$$

2. The equilibrium

$$\begin{aligned}
e_3^s = e_4^s &= \frac{\alpha}{2c} v_3 \\
&= \frac{\alpha}{2c}\left[p_{12}^s\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^\alpha\right)+p_{21}^s\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)\right] \\
&\quad \Delta u \\
&= \frac{\alpha}{2c}\left[p_{12}^s\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^\alpha\right)\right. \\
&\quad \left. +\left(1-p_{12}^s\right)\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)\right] \\
&\quad \Delta u \\
&= \frac{\alpha}{2c}\left[p_{12}^s\left(\frac{1+\alpha}{2}-\frac{1+\alpha}{2}\lambda_1^\alpha\right)+\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)\right]\Delta u \\
&= \frac{\alpha}{2c}\left[\frac{1}{2}\frac{1+\alpha}{2}\lambda_1^\alpha\left(\frac{\left(\left(1-\frac{1+\alpha}{2}\right)\lambda_1^{1+\alpha}+\frac{u(w_2)}{u(w_1)-u(w_2)}\right)}{1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}}\right)^\alpha\left(1-\lambda_1^\alpha\right)\right. \\
&\quad \left. +\left(1+\frac{u(w_2)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)\right]\Delta u \\
&> \overline{e}^s
\end{aligned}$$

$$p_{34}^s = p_{43}^s = \frac{1}{2}$$

3. Participation constraints

$$\begin{aligned}
E^s(U_{34}) = p_{34}^s v_3 - c e_3 &= \frac{1}{2} v_3 - c\frac{\alpha}{2c} v_3 \\
&= \frac{1-\alpha}{2} v_3 \\
&\geqslant 0
\end{aligned}$$