Learning and Overconfidence in Elimination Contests

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Abstract

This paper studies how overconfidence shapes behavior in a two-stage elimination contest with incomplete information. An overconfident newcomer, uncertain about his ability, overestimates his ex-ante probability of being high ability. Conditional on a first stage victory, the newcomer and his rival update their beliefs about the newcomer's ability using Bayes' rule. We show that a first stage win amplifies the newcomer's overconfidence when his ex-ante probability of being high ability is low, and dampens it otherwise. Overconfidence can raise the newcomer's equilibrium effort in both stages and thus increase his chance of winning the contest. The model clarifies when success feeds further overconfidence and helps explain why overconfident individuals so frequently attain top position in organizations.

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1 Introduction

Organizations typically staff open positions by promoting from within or by drawing on the external labor market (Bidwell and Keller, 2014). In such cases, the promotion contest sets a newcomer whose ability is largely untested against incumbents whose ability is already known. Early victories in these contests not only advance the outsider but also reshape beliefs: both incumbents and the newcomer update their assessments of the latter's true talent.

Examples abound. Consider an external hire in a law firm who, after joining mid-career, must compete with long-tenured associates for a partnership slot in an up-or-out promotion contest. The partners can rely on years of information about their incumbent associates, whereas the newcomer enters with uncertain prospects (Rebitzer and Taylor, 2007). Promotions to CEO positions often display a similar dynamic: boards weigh candidates from inside the firm, whose past performance is well documented, against external recruits whose ability to lead the organization remains largely untested (Parrino, 1997; Zhang and Rajagopalan, 2004). In politics, two-round elections—where candidates first compete in their party for party leadership before facing off a leader from a rival party in the general election—often place a political newcomer, uncertain about how voters perceive her abilities, against seasoned opponents whose reputations are already well established (Crutzen et al., 2010; Andreottola, 2021). Similarly, in academic grant applications, junior faculty face funding tournaments alongside senior faculty whose publication records and expertise are broadly known (Azoulay et al., 2011). In each case, the eliminatory nature of the process creates settings where early successes affect expectations about an unknown entrant.

Crucially, this uncertainty about an newcomer's true ability opens the door to overconfidence. When individuals face incomplete information about how their skills compare to others', initial victories can inflate their beliefs about their own competence beyond what is objectively warranted. Indeed, overconfidence is most likely to emerge precisely when the contestant's type is uncertain (Benoît and Dubra, 2011).

Does earning a promotion increase a newcomer's overconfidence? How does overconfidence shape contestants' efforts across the successive stages of a promotion contest? Are overconfident newcomers more likely to win the contest than their rational rivals? These questions matter since, for instance, empirical studies show that approximately 40 percent of CEOs of companies listed in the Standard & Poor's 1500 index are overconfident (Malmendier and Tate, 2015).

To address these questions, we develop a formal model that embeds overconfidence—defined as an overestimation of the probability of being high ability—within a two-stage elimination contest with incomplete information. In the first stage (semifinal stage), four players are matched pairwise, and each pair competes in one semifinal. The semifinal winners go on to the second stage and compete in the final. In each pairwise interaction the players choose their efforts simultaneously and their winning probabilities are determined by their efforts and abilities via a Tullock contest success function. We consider a winner-take-all contest where players' utility of the prize is v and their constant marginal cost of effort is $c \in [1, v)$.

Player 1, the *newcomer*, can have either low, θ_L, or high, θ_H, ability, with $0 \leq \theta_L < 1 < \theta_H$. The ex-ante probability player 1 has high ability is $\pi \in (0, 1)$, and this is common knowledge. We sequentially analyze two scenarios. In the first scenario, the newcomer is rational, while in the second he is overconfident. A rational newcomer holds the correct prior belief that his ability is high with probability π. An overconfident newcomer holds the mistaken prior belief that his ability is high with probability $\tilde{\pi} = \pi + b^s$, where b^s is the newcomer's semifinal stage bias which satisfies $b^s \in (0, 1 - \pi]$. Players 2, 3, and 4, the *incumbents*, possess identical ability normalized to 1, and this value is common knowledge. The incumbents know the newcomer's ability is θ_H with ex-ante probability π and θ_L with ex-ante probability $1 - \pi$, and that an overconfident newcomer perceives his ability is θ_H with probability $\tilde{\pi}$ and θ_L with probability $1 - \tilde{\pi}$. Hence we are considering an incomplete information setup where players hold no private information.

A semifinal win prompts the newcomer to update his self-belief via Bayes' rule, while the incumbent who reaches the final also updates her belief about the newcomer's type via Bayes' rule. A rational newcomer's posterior belief is denoted by μ and that of an overconfident one by $\tilde{\mu}$. Posterior beliefs about the newcomer's ability are determined by the equilibrium semifinal efforts and prior beliefs.[1] Accordingly, an overconfident newcomer's final-stage bias b^f is determined endogenously and equals $b^f = \tilde{\mu} - \mu$. Given the resulting posterior beliefs, the semifinal winners choose their efforts in the final. Comparing b^s to b^f reveals whether a semifinal victory amplifies or attenuates the newcomer's overconfidence.

Our main findings are as follows. First, a semifinal victory *amplifies* the newcomer's overconfidence bias when his ex-ante probability of being high ability is low: $b^f > b^s$

[1] At equilibrium with incomplete but symmetric information about ability, semifinal efforts do not need to be observable for finalists to form posterior beliefs about ability as semifinal efforts are perfectly anticipated through the equilibrium strategy.

whenever π is small. Conversely, the same win *dampens* the newcomer's overconfidence bias when his ex-ante probability of being high ability is high: $b^f < b^s$ for large π. Thus an early victory in an elimination contest can either heighten or temper a player's overconfidence depending on the ex-ante likelihood of being high ability. The mechanism behind this unexpected result is straightforward once beliefs are traced. When π is close to zero, the incumbent's posterior hardly moves if the newcomer wins his semifinal. On the other hand, following a semifinal victory, an overconfident newcomer significantly upgrades his self-belief because $\tilde{\pi} = \pi + b^s$ shifts appreciably away from zero: this unexpected win is mistakenly attributed in the newcomer's mind to a significantly higher probability he is of high type. By contrast, when π is near one, both players already regard the newcomer as almost certainly high-ability, leaving little scope for further upward revision, so the bias naturally contracts. Overall, a semifinal victory amplifies overconfidence in players with lower expected ability, while attenuating it in those with higher expected ability.

Second, in a final between a rational newcomer and an incumbent, our first scenario, both players select the same effort at equilibrium regardless of how the newcomer's possible abilities compare to that of the incumbent.[2] In contrast, in our second scenario where the final involves an overconfident newcomer and an incumbent, the players' efforts at equilibrium are sensitive to the ability comparison: the newcomer's effort exceeds the incumbent's only when the product of the newcomer's possible abilities $\theta_L \theta_H$ lies below the incumbent's ability (normalized to 1). The intuition behind this result is as follows. When $\theta_L \theta_H \in (0, 1)$, the overconfident newcomer attributes a higher weight than the incumbent to a scenario where the gap between the players' abilities is not too large. This in turn incentivizes the overconfident newcomer to invest more effort than the incumbent. The opposite happens when $\theta_L \theta_H > 1$ and the overconfident newcomer attributes a higher weight than the incumbent to a scenario where the gap between the players' abilities is large. This in turn incentivizes him to invest less effort than the incumbent. Moreover, the overconfident newcomer's equilibrium effort as well as his true probability of winning the final rise monotonically with his bias in the final, b^f.

Third, in a semifinal between a rational newcomer and an incumbent, our first scenario, the identity of the higher-effort player hinges on the ex-ante probability π that the newcomer is high ability: when π is small the rational newcomer expends less effort

[2]This is a well known result in Tullock contests with complete information. With identical linear costs, heterogeneous abilities cancel out in the first-order conditions, forcing identical effort. In our model with a rational newcomer, where both players hold the same posterior beliefs, they can be seen as optimizing a weighted average of two complete information contests.

than the incumbent at equilibrium, but when π is large he expends more. The intuition for this result can be grasped by considering the two extreme cases where π is either close to 0 or to 1. If π is close to 0, the incentives for the newcomer to invest in the semifinal are very low, since both the likelihood of winning the semifinal (for any effort) and the expected utility of the final are low. The incumbent is equally incentivized to invest little effort in the semifinal, yet, she will invest more than the newcomer since both her winning probability and expected utility of the final are much higher. Combined, this implies that for low values of π the incumbent invests higher effort in the semifinal. When π is close to 1, the incentives of the newcomer and the incumbent are reversed since the newcomer holds both a high probability of winning the semifinal (for a given effort) and has a higher expected utility of the final compared to the incumbent. Accordingly, for high values of π the rational newcomer exerts higher effort in the semifinal.

Fourth, in our second scenario where the semifinal is played between an overconfident newcomer and an incumbent, an increase in the newcomer's semifinal stage bias b^s raises his prior belief $\tilde{\pi}$ which has similar effects as an increase in π in the rational newcomer case. In addition, an increase in b^s changes the wedge between the newcomer's posterior belief $\tilde{\mu}$ and the incumbent's posterior belief μ as previously described. This introduces a new linkage between the semifinal and the final whereby, the overconfident newcomer's choice of effort in the semifinal affects b^f which, in turn, affects the newcomer's rival choice of effort in the final. The complexity of the problem prevents us from deriving general results for any value of θ_L and θ_H. However, imposing $\theta_L \theta_H = 1$ we are able to characterize the equilibrium of the semifinal since the final stage equilibrium efforts are identical and unaffected by the newcomer's final stage bias b^f. We find that when π is low and the bias b^s is large enough, an overconfident newcomer exerts higher semifinal effort than the incumbent whereas, in our first scenario, for such low values of π a rational newcomer exerts lower semifinal effort than the incumbent. Indeed, when π is low, a rational newcomer anticipates a low expected utility from winning the semifinal. By contrast, an overconfident newcomer misattributes a semifinal victory to his own ability, thereby reinforcing his overconfidence and making advancement to the final appear more attractive. This in turn incentivizes the overconfident newcomer to exert higher effort than his rival in the semifinal.

Fifth, we demonstrate that overconfidence raises the newcomer's true equilibrium probability of winning the elimination contest when $\theta_L \theta_H = 1$. An increase in the overconfident newcomer's semifinal stage bias b^s raises the newcomer's semifinal equilibrium relative effort a_1^s/a_3^s. This raises the newcomer's true equilibrium probability of winning the semifinal. However, the increase in the newcomer's semifinal relative effort also lowers μ which, in turn, reduces the newcomer's true equilibrium probability of wining the final. Still, we are able to show that the increase in the probability of winning the semifinal dominates the drop in the probability of winning the final.

The rest of the paper proceeds as follows. Section 2 discusses related literature. Section 3 sets-up the model. Section 4 studies the elimination contest with a rational newcomer. Section 5 considers the case of an overconfident newcomer. Section 6 concludes the paper. All proofs can be found in the Appendix.

2 Related Literature

Our paper contributes to three strands of the literature: elimination contests, the dynamics of overconfidence, and the effect of overconfidence in contests and tournaments.

First, our contribution to elimination contests is most closely related to Rosen (1986) and Chen and Santos-Pinto (2025).[3] The main focus of Rosen (1986) is to explain why contest organizers set increasingly larger prizes as players advance in an elimination contest. Although most of the analysis is performed under complete information, Rosen (1986) also discusses the extension of his model to an incomplete information elimination contest. Unlike Rosen, we focus on the effect of overconfidence on effort provision and winning probabilities in elimination contests. Rosen (1986) finds that uncertainty about ability is a force that dampens incentives to perform in the early stages since it creates incentives to experiment to discover own strength. In our setup we have the same effect. However, we find that uncertainty about ability coupled with overconfidence can instead induce an overconfident player to invest more in the early stage of the elimination contest, thereby revealing that overconfidence can totally flip the results.

Chen and Santos-Pinto (2025) provide the first analysis of overconfidence in elimination contests. They show that in the second stage the overconfident player always exerts less effort than a rational rival, and if the prize spread is large and confidence moderate, may exert more effort in the first stage. In their model there is no uncertainty and so the overconfidence bias remains fixed across stages. In our framework instead, there is uncertainty about the overconfident player's ability, and as a consequence the

[3]Related analyses of elimination contests examine aspects of optimal design and organization, such as rent-seeking structures (Gradstein and Konrad, 1999), information revelation (Zhang and Wang, 2009), optimal seeding (Groh et al., 2012), multi-stage design (Fu and Lu, 2012), and sabotage (Klunover, 2021).

overconfidence bias changes endogenously from the first to the second stage.[4] We show that winning the first stage amplifies the bias when the player's true prior probability of being high-ability is low and dampens it otherwise. This novel mechanism yields new predictions for efforts and outcomes. In the second stage, when $\theta_L\theta_H \in (0,1)$, the equilibrium effort of the overconfident newcomer is larger than his rational rival, and both his effort and true winning probability increase with his bias. The opposite result obtains if $\theta_L\theta_H \geq 1$. In addition, we show that the overconfident player's true probability of winning the contest, measured as the product of the first and second stages true winning probabilities, can increase with his first stage bias.

Second, we also contribute to the literature on the emergence and evolution of overconfidence. Gervais and Odean (2001) show that initial success increases posterior assessments of one's ability because agents apply a self-serving weighting to outcomes—successes receive greater subjective weight as evidence of skill than failures do as evidence of low ability—producing upward-biased belief updating.

In Compte and Postlewaite (2004), confidence—shaped by recalled past successes and failures—affects the actual probability of success, and overconfidence arises from a biased recollection of past events. They find that this type of overconfidence can have positive welfare implications and that overconfidence decreases with experience. In contrast, our model assumes that while overconfident players begin with biased priors about their abilities, they perfectly recall past performance and update beliefs in accordance to Bayes' rule. Our findings are also more nuanced, since we find that an overconfident newcomer who is successful early on (i.e. wins in the semifinal) will either dampen or amplify his overconfidence bias depending on the ex-ante true probability of being high ability. Interestingly, when the newcomer is initially highly overconfident, an early success will induce him to attribute a higher probability he is of a high type, thereby reducing his overconfidence bias.

In Bénabou and Tirole (2002) a time-inconsistent agent, with imperfect knowledge about his ability, may choose to remain ignorant about his ability to induce a future self to work harder. Likewise in our model, that is admittedly very different, the overconfident newcomer can influence both knowledge about his ability and his final stage overconfidence via the choice of semifinal stage effort. We show that players may contain their efforts at the semifinal stage to increase everyone's belief that they are of

[4] Another fundamental difference between the two studies is that in Chen and Santos-Pinto (2025) the players' winning probabilities are determined by Alcalde and Dahm's (2007) contest success function while here we consider a Tullock contest success function. This is important given the prevalence and wide use of the Tullock CSF in multistage contests.

a high ability, thereby increasing their expected payoff from the final. We also find that depending on the newcomer's ex-ante true probability of being high ability, victory increases the true likelihood the player is of high ability, and it can either amplify or dampen his overconfidence bias. Our results do not rely on time-inconsistency.

Last, Zábojník (2004) and Benoît and Dubra (2011) develop Bayesian decision-theoretic learning models where agents might end up rationally overestimating their abilities. We instead assume that some agents are overconfident, and study how confidence biases evolve endogenously through agents' choice of effort in a dynamic elimination contest.

Third, we contribute to the literature that explores the role of overconfidence in contests and tournaments. Santos-Pinto (2010) studies how a principal chooses prizes in static tournaments featuring players overestimating their abilities. Ludwig et al. (2011), model overconfidence in a static contest as an underestimation of the cost of effort, and conclude that it leads to higher equilibrium efforts. Santos-Pinto and Sekeris (2023, 2025) demonstrate that the results are totally reversed when players overestimate their abilities in contests, while the results are shown to be more nuanced in tournaments (Santos-Pinto and Sekeris 2025). This paper shows that overconfidence in the presence of uncertainty can result in the overconfident player exerting lower or high effort than a rational rival even in a static contest.

Denter et al. (2022) explore how one-sided asymmetric information about the marginal cost of effort affects effort levels in a contest. They assume that prior to competing in the contest, a newcomer–the informed player–can signal his type to an incumbent–the uninformed player–through a costly signal. They find that only newcomers who have a very low marginal cost of effort, relative to the incumbent, benefit from disclosing their type in equilibrium. In addition, they allow the newcomer to be overconfident as we do. Our study differs from Denter et al. (2022) in at least two main dimensions. First, we assume information is symmetric whereas they assume it is asymmetric. Second, we study a two-stage elimination contest whereas they allow for costly signaling prior to competing in a one shot contest. We show that the newcomer must take into account that, everything else constant, higher effort in the semifinal lowers his posterior belief of having high ability in final.

Last, our paper is related to the literature on learning in dynamic contests and tournaments. In Altmann et al. (2012), Kubitz (2023), Barbieri and Serena (2025), and Catepillán et al. (2025), each player has private information about either his ability, cost of effort, valuation of the prize, or more generally objective function. We instead assume that nobody (including the newcomer himself) knows the newcomer's ability. This distinction is essential, since unlike private information setups where players attempt signaling or concealing their known identity, in our setup the newcomer's first stage effort influences everyone's beliefs about his ability at the start of the second stage. Indeed, an early victory achieved with little effort lead to a sharper updating of beliefs, since such success is more likely attributed to the winner's unknown — and potentially high — ability than it would be if the effort had been greater. Such behaviour therefore potentially trumps everyone, the newcomer included, and may confer a strategic advantage to the newcomer if he reaches the subsequent round. This mechanism is highlighted in Krähmer (2007) in a repeated contest with symmetric incomplete information about players' abilities. In contrast to Krähmer (2007), we focus on the role of overconfidence in an elimination contest.

3 Set-up

Consider a two-stage elimination contest where players 1 and 3 compete in one semifinal and 2 and 4 in the other semifinal. The semifinal winners move on to the final. Player 1, the newcomer, can have either low, θ_L, or high, θ_H, ability, with $0 \leq \theta_L < 1 < \theta_H$. The ex-ante probability player 1 has high ability is $\pi \in (0, 1)$. The abilities of players 2, 3, and 4 are common knowledge, identical, and normalized to 1. Players 2, 3 and 4 know that the ex-ante probability player 1 has high ability is π.

The utility of the winning prize is v and the utility of the losers' prize is normalized to 0. Player i's cost of exerting effort a_i is equal to $C(a_i) = ca_i$, with $c \in [1, v)$. We assume that the ratio v/c is large enough to ensure that in both stages of the elimination contest the game admits interior pure-strategy equilibria.

The players' winning probabilities in any pairwise interaction are determined according to a Tullock contest success function. Moreover, the probability player i's wins against player j when player i has ability θ_i and player j has ability θ_j is as follows:

$$P_i(a_i, a_j; \theta_i, \theta_j) = \frac{\theta_i a_i}{\theta_i a_i + \theta_j a_j}, \tag{1}$$

where $\theta_1 \in \{\theta_L, \theta_H\}$, $\theta_2 = \theta_3 = \theta_4 = 1$, and $j \neq i$.

Player 1 is rational when his prior belief of having high ability is equal to π. To model overconfidence, we assume player 1 has a (subjective) prior belief of having high ability equal to $\tilde{\pi} = \pi + b^s$, where $b^s \in (0, 1 - \pi]$ is the overconfidence bias in the semifinal.

Player 1's rivals know that player 1 is overconfident, that is they know b^s, but think, correctly, player 1 is mistaken.

We work with the perfect Bayesian equilibrium concept (PBE) and solve the game by backwards induction. A semifinal victory prompts the newcomer to update his belief about his own type via Bayes' rule, while an incumbent who also reaches the final also updates her belief about the newcomer's type via Bayes' rule. The players' posterior beliefs are a function of their prior beliefs and the semifinal efforts. Given the posterior beliefs, we derive the Bayesian-Nash equilibrium (BNE) of the final and compute the corresponding equilibrium payoffs (continuation values). Finally, given the continuation values, we then solve for the BNE of each semifinal. The resulting strategy profile and belief system jointly satisfy the requirements of a PBE for the two-stage elimination contest.

To be able to compute the equilibrium taking into account that players can hold mistaken beliefs we assume: (i) a player who faces a biased opponent is aware that the latter's perception is mistaken, (ii) each player thinks that his own perception is correct, and (iii) both players have a common understanding of each other's beliefs, despite their disagreement on the accuracy of their opponent's beliefs. Hence, players agree to disagree about their perceptions. This approach follows Squintani (2006).[5]

4 Rational Newcomer

In this section we analyze the model under the assumption that the newcomer is rational. We proceed backward in time. First, we derive the equilibrium efforts in a final between a rational newcomer and an incumbent. Second, using the final stage solution, we characterize the equilibrium efforts in the semifinal between a rational newcomer and an incumbent.

4.1 Final

We study the final between the rational newcomer, player 1, and the incumbent player 2, without loss of generality. The expected utility of player $i = 1, 2$ in the final is

$$E[U_i^f(a_i, a_j)] = P_i^f(a_i, a_j)v - ca_i.$$

[5]These assumptions are consistent with the psychology literature on the "Blind Spot Bias" according to which individuals believe that others are more susceptible to behavioral biases than themselves (Pronin et al. 2002, Pronin and Kugler 2007).

Player 1's expected utility in the final is therefore

$$E[U_1^f(a_1,a_2)] = \left[\mu\frac{\theta_H a_1}{\theta_H a_1 + a_2} + (1-\mu)\frac{\theta_L a_1}{\theta_L a_1 + a_2}\right] v - c a_1, \tag{2}$$

and player 2's expected utility in the final is

$$E[U_2^f(a_1,a_2)] = \left[\mu\frac{a_2}{\theta_H a_1 + a_2} + (1-\mu)\frac{a_2}{\theta_L a_1 + a_2}\right] v - c a_2, \tag{3}$$

where μ is the players' common posterior belief that player 1 has high ability, given by

$$\begin{aligned}
\mu &= \frac{\pi P_1^s(a_1^s,a_3^s;\theta_H)}{\pi P_1^s(a_1^s,a_3^s;\theta_H) + (1-\pi)P_1^s(a_1^s,a_3^s;\theta_L)} \\
&= \frac{\pi \dfrac{\theta_H a_1^s}{\theta_H a_1^s + a_3^s}}{\pi\dfrac{\theta_H a_1^s}{\theta_H a_1^s + a_3^s} + (1-\pi)\dfrac{\theta_L a_1^s}{\theta_L a_1^s + a_3^s}} \\
&= \frac{\pi\theta_H\left(\theta_L a_1^s + a_3^s\right)}{\theta_L\theta_H a_1^s + \left[\pi\theta_H + (1-\pi)\theta_L\right]a_3^s}.
\end{aligned} \tag{4}$$

Note that μ is decreasing with a_1^s since

$$\frac{\partial\mu}{\partial a_1^s} = -\frac{\pi(1-\pi)\theta_L\theta_H(\theta_H-\theta_L)a_3^s}{\left[\theta_L\theta_H a_1^s + \left[\pi\theta_H + (1-\pi)\theta_L\right]a_3^s\right]^2} < 0, \tag{5}$$

and μ is increasing in a_3^s since

$$\frac{\partial\mu}{\partial a_3^s} = \frac{\pi(1-\pi)\theta_L\theta_H(\theta_H-\theta_L)a_1^s}{\left[\theta_L\theta_H a_1^s + \left[\pi\theta_H + (1-\pi)\theta_L\right]a_3^s\right]^2} > 0. \tag{6}$$

The first-order conditions of players 1 and 2 in the final are given by

$$\left[\mu\frac{\theta_H}{(\theta_H a_1 + a_2)^2} + (1-\mu)\frac{\theta_L}{(\theta_L a_1 + a_2)^2}\right] a_2 v = c, \tag{7}$$

and

$$\left[\mu\frac{\theta_H}{(\theta_H a_1 + a_2)^2} + (1-\mu)\frac{\theta_L}{(\theta_L a_1 + a_2)^2}\right] a_1 v = c. \tag{8}$$

It follows from the first-order conditions that the final has a unique pure-strategy equilibrium. Our first result characterizes the equilibrium of the final.

Proposition 1. *In a final between a rational newcomer and an incumbent, the equilib-*

rium efforts are symmetric and given by:

$$a_1^f = a_2^f = a^f = \left[\mu \frac{\theta_H}{(\theta_H+1)^2} + (1-\mu)\frac{\theta_L}{(\theta_L+1)^2}\right]\frac{v}{c}. \tag{9}$$

The proof of Proposition 1 follows directly from the combination of first-order conditions (7) and (8). Note that the equilibrium effort (i) is unaffected by μ when $\theta_L\theta_H \in \{0,1\}$, (ii) increases in μ for $\theta_L\theta_H \in (0,1)$, and (iii) decreases in μ when $\theta_L\theta_H > 1$. The intuition of this result is the following. In any complete information Tullock contest with linear costs and asymmetric abilities, equilibrium efforts are symmetric across players. Moreover, the higher the asymmetry in abilities, the lower the equilibrium efforts (see e.g. Corchón 2000). In our setup where there is incomplete information about the newcomer's ability, players can be seen as optimizing a weighted average of two complete information Tullock contests with linear costs, where the weights are given by the posterior beliefs μ and $(1-\mu)$. Accordingly, the players will invest equal efforts at equilibrium.

Second, observe that with probability $(1-\mu)$, the newcomer has an ability of θ_L and the incumbent an ability of 1. Alternatively, by dividing the numerator and the denominator of the Tullock contest success function by θ_L, one can re-interpret this as the newcomer having an ability of 1 and the incumbent having an ability of $1/\theta_L$. Therefore if $\theta_H = 1/\theta_L$, this implies that the players invest the same equilibrium efforts in the two "degenerate" cases where $\mu = 0$ and $\mu = 1$. Consequently, for any probability μ the players invest these same equilibrium efforts as explained above in point (i).

Extending this logic, we deduce that $\theta_H < 1/\theta_L$ implies that when $\mu = 1$ equilibrium efforts are higher than when $\mu = 0$. Hence, equilibrium efforts are monotonically increasing in μ when $\theta_L\theta_H \in (0,1)$, as stated above in point (ii). Likewise we deduce observation (iii) according to which equilibrium efforts are monotonically decreasing in μ when $\theta_L\theta_H > 1$.

It follows from (9) that player 1's equilibrium probability of winning the final is

$$P_1^f(a_1^f, a_2^f) = \mu\frac{\theta_H}{\theta_H+1} + (1-\mu)\frac{\theta_L}{\theta_L+1}.$$

Hence, player 1 has a higher chance of winning the final at equilibrium when

$$\mu > \frac{1}{2} + \frac{1-\theta_L\theta_H}{2(\theta_H-\theta_L)}.$$

It also follows from (9) that player 1's equilibrium expected utility of the final is

$$E[U_1^f(a_1^f, a_2^f)] = \left[\left(\frac{\theta_L}{\theta_L+1}\right)^2 + \mu\chi\right] v, \tag{10}$$

where $\chi = \left(\frac{\theta_H}{\theta_H+1}\right)^2 - \left(\frac{\theta_L}{\theta_L+1}\right)^2$.

From equation (10) we can determine how a change in μ affects payer 1's equilibrium expected utility of the final. We have

$$\frac{\partial E[U_1^f(a_1^f, a_2^f)]}{\partial \mu} = \chi v > 0$$

Hence, $E[U_1^f(a_1^f, a_2^f)]$ is increasing with μ. It follows from (9) that player 2's equilibrium expected utility of the final is

$$E[U_2^f(a_1^f, a_2^f)] = \left[\left(\frac{1}{\theta_L+1}\right)^2 - \mu\left[\left(\frac{1}{\theta_L+1}\right)^2 - \left(\frac{1}{\theta_H+1}\right)^2\right]\right] v \tag{11}$$

From equation (11) we can determine how a change in μ affects payer 2's equilibrium expected utility of the final. We have

$$\frac{\partial E[U_2^f(a_1^f, a_2^f)]}{\partial \mu} = -\left[\left(\frac{1}{\theta_L+1}\right)^2 - \left(\frac{1}{\theta_H+1}\right)^2\right] v < 0$$

Hence, $E[U_2^f(a_1^f, a_2^f)]$ is decreasing with μ. We see that the rational newcomer has a strategic incentive to lower his semifinal effort in a way that boosts the common posterior belief μ. This result is intuitive. If the rational newcomer wins the semifinal without putting a lot of effort, this persuades incumbent 2 that the rational newcomer has high ability with high probability. This is beneficial to the rational newcomer as it gives him a strategic advantage in the final.

4.2 Semifinal

We now consider the semifinal between the rational newcomer and the incumbent player 3. Observe that in the semifinal opposing incumbents 2 and 4, the players' expected utility of reaching the final is a weighted average of their expected utility when facing either player 1 or player 3. Yet, given the symmetry of incumbents 2 and 4, their continuation value is identical, and their equilibrium semifinal effort as well. Although their equilibrium efforts eventually depend on their expectation of whom they will meet in the final, the identity of the winner is irrelevant, and we can then focus on the semifinal opposing players 1 and 3.

Player 1's expected utility of the semifinal is

$$\begin{aligned} E[U_1^s(a_1,a_3)] &= P_1^s(a_1,a_3)E[U_1^f(a_1^f,a_2^f)] - ca_1 \\ &= \left[\pi\frac{\theta_H a_1}{\theta_H a_1 + a_3} + (1-\pi)\frac{\theta_L a_1}{\theta_L a_1 + a_3}\right]\left[\left(\frac{\theta_L}{\theta_L+1}\right)^2 + \mu\chi\right] v - ca_1, \end{aligned}$$

where μ is the posterior belief that player 1 has high ability and is given by:

$$\mu = \frac{\pi\theta_H\big(\theta_L a_1 + a_3\big)}{\theta_L\theta_H a_1 + \big[\pi\theta_H + (1-\pi)\theta_L\big]a_3}.$$

Player 3's expected utility of the semifinal is

$$\begin{aligned} E[U_3^s(a_1,a_3)] &= P_3^s(a_1,a_3)E[U_3^f(a_3^f,a_2^f)] - ca_3 \\ &= \left[\pi\frac{a_3}{\theta_H a_1 + a_3} + (1-\pi)\frac{a_3}{\theta_L a_1 + a_3}\right]\frac{v}{4} - ca_3. \end{aligned}$$

If player 3 reaches the final and the opponent is player 2, then both players exert effort $a^f = v/4c$ and player 3's equilibrium expected utility of the final is $E[U_3^f(a_3^f,a_2^f)] = v/4$.

The first-order conditions of players 1 and 3 in the semifinal are

$$\frac{\partial P_1^s}{\partial a_1}E\big[U_1^f(a_1^f,a_2^f)\big] + P_1^s(a_1,a_3)\frac{dE[U_1^f(a_1^f,a_2^f)]}{d\mu}\frac{\partial\mu}{\partial a_1} = c, \tag{12}$$

and

$$\frac{\partial P_3^s}{\partial a_3}E[U_3^f(a_3^f,a_2^f)] = c, \tag{13}$$

respectively. The first term in the left-hand side of equation (12) captures the effect of an increase in the rational newcomer's semifinal effort on his probability of winning the semifinal, which is positive. The second term in the left-hand side of equation (12) captures the effect of an increase in the rational newcomer's semifinal effort on the posterior belief μ. This second effect is negative since, everything else constant, a higher effort by the rational newcomer is interpreted as a negative signal of ability ("a truly strong player would not need to try so hard"), thereby lowering the posterior belief μ which, in turn, reduces the rational newcomer's expected utility of the final. Hence, the rational newcomer's semifinal effort choice has a public-signaling component that affects both his and incumbent 2' posterior belief μ, that, in our setting, is negative—greater effort is interpreted as bad news about ability.

To show existence of a pure-strategy equilibrium we demonstrate in the Appendix that the second-order conditions are verified whenever the first-order conditions are satisfied. The assumption that v/c is sufficiently large ensures that both players attain strictly positive expected utilities for these strategy profile.

Proposition 2. *In the semifinal between a rational newcomer and an incumbent of a two-stage elimination contest, there exists a unique value* $\bar{\pi} \in (0,1)$ *for the ex-ante probability* π *that the newcomer has high ability such that when* $\pi \lesseqgtr \bar{\pi}$*, the players' equilibrium efforts satisfy* $a_1^s \lesseqgtr a_3^s$*.*

Four distinct forces govern how π shapes a rational newcomer's equilibrium effort in the semifinal. The first one, the *contest-sensitivity effect*, captures how an increase in π changes the rational newcomer's marginal probability of winning the semifinal. It's sign is determined by

$$\frac{\partial^2 P_1^s}{\partial a_1 \partial \pi} = \frac{(\theta_H - \theta_L)a_3}{(\theta_H a_1 + a_3)^2(\theta_L a_1 + a_3)^2}(a_3^2 - \theta_L \theta_H a_1^2).$$

Hence, the newcomer's marginal probability of winning the semifinal rises with π as long as $a_1 < a_3/\sqrt{\theta_L \theta_H}$, is zero at $a_1 = a_3/\sqrt{\theta_L \theta_H}$ and falls with π when $a_1 > a_3/\sqrt{\theta_L \theta_H}$. Everything else equal, this effect produces a inverted U-shaped response of $a_1^s(\pi)$ to π.

The second one, the *encouragement effect*, captures how an increase in π changes the rational newcomer's expected utility of the final $E[U_1^f(a_1^f, a_2^f)]$. This effect is positive since an increase in π raises the posterior belief μ ($\partial\mu/\partial\pi > 0$) and an increase in the posterior belief, in turn, raises the rational newcomer's expected utility of the final ($\partial E[U_1^f(a_1^f, a_2^f)]/\partial\mu > 0$). This effect pushes $a_1^s(\pi)$ upward with π.

The third one, the *public signal effect*, captures how an increase in π raises the probability the rational newcomer reaches the final and makes him more exposed to the negative signal of ability. This effect is negative since an increase in π raises the rational newcomer's probability of advancing to the final as

$$\frac{\partial P_1^s}{\partial \pi} = \frac{(\theta_H - \theta_L)a_1 a_3}{(\theta_H a_1 + a_3)(\theta_L a_1 + a_3)} > 0.$$

and higher semifinal effort leads to a lower posterior belief, $\partial\mu/\partial a_1 < 0$. This effect pushes $a_1^s(\pi)$ downward with π.

The fourth one, the *posterior-sensitivity effect*, captures how an increase in π changes the marginal posterior belief $\partial\mu/\partial a_1$. It's sign is determined by

$$\frac{\partial^2\mu}{\partial a_1\partial\pi} = \frac{\pi\left[2\theta_L(\theta_H a_1 + a_3) + (\theta_H - \theta_L)a_3\right] - \theta_L(\theta_H a_1 + a_3)}{\left[\theta_L\theta_H a_1 + \left[\pi\theta_H + (1-\pi)\theta_L\right]a_3\right]^3}.$$

Hence, $\partial\mu/\partial a_1$ falls with π as long as $\pi < \pi^*$, is zero at $\pi = \pi^*$, and rises with π when $\pi > \pi^*$, where $\pi^* = \theta_L(\theta_H a_1 + a_3)/[2\theta_L(\theta_H a_1 + a_3) + (\theta_H - \theta_L)a_3]$. Everything else equal, this effect produces a U-shaped response of $a_1^s(\pi)$ to π. The intuition behind this effect is rooted in the fact that both with very low or very high π, the posterior belief μ will not be very sensitive to the newcomer's effort a_1^s. Indeed, with a very low ex-ante probability of being a high type π, whether the newcomer wins by exerting low or high effort, the posterior probability μ will be high, hence μ is not very sensitive to effort a_1^s. Likewise, if π is high, then there is little room for improvement in beliefs, thereby again implying that μ is not very sensitive to effort a_1^s. For intermediate cases, however, a victory with higher effort a_1^s tends to significantly reduce μ.

Let us now turn to the incumbent. Her ability and the value of reaching the final are fixed and hence a change in π only affects the incumbent's semifinal effort through a *contests-sensitivity effect* which has the opposite sign than the one of the newcomer.

To then better grasp Proposition 2, we first consider the two extreme cases where π is either close to 0 or to 1. If π is close to 0, the incentives for the newcomer to invest in the semifinal are very low, since both the likelihood of winning the semifinal (for any effort) and the expected utility of the final are low. The incumbent is equally incentivized to invest little effort in the semifinal, yet, she will invest more than the newcomer since both her winning probability and expected utility of the final are much higher. Combined, this implies that for low values of π the incumbent invests higher effort in the semifinal.

Consider next a situation where π is close to 1. In such instances the incentives of the newcomer and the incumbent are reversed since the newcomer holds both a high probability of winning the semifinal (for a given effort) and has a higher expected utility of the final compared to the incumbent. Accordingly, for high values of π the newcomer exerts higher effort in the semifinal.

The four effects described above for the newcomer together with the effect of a change in π on the incumbent's incentives to invest effort in the semifinal allow us to better comprehend how equilibrium relative efforts in the semifinal depend on π. For low values of π we know that, since $a_1^s/a_3^s < 1$, the incumbent's incentives to invest in effort in the semifinal will drop with π. On the other hand, the two first effects will be positive for the newcomer. Our findings therefore suggest that the two negative effects (public signal and posterior-sensitivity) are not strong enough to overturn the positive contest-sensitivity and encouragement effects. Moreover, although for values of $\pi > \bar{\pi}$ the newcomer invests higher effort in the semifinal, potentially flipping the contest-sensitivity effect for both the incumbent and the newcomer, we demonstrate that the encouragement effect is strong enough to secure that at equilibrium the newcomer invest more effort than the incumbent.

5 Overconfident Newcomer

This section analyzes the model with an overconfident newcomer. First, we characterize the equilibrium efforts in a final between the newcomer and an incumbent. Second, we analyze the first-order conditions that determine the equilibrium efforts in the semifinal between the newcomer and an incumbent. Third, we specialize the model to the case $\theta_L\theta_H = 1$ which allows us study how overconfidence affects winning probabilities.

5.1 Final

The perceived expected utilities of players 1 and 2 in the final are given

$$\tilde{E}[U_1^f(a_1,a_2)] = \tilde{P}_1^f(a_1,a_2)v - ca_1 = \left[\tilde{\mu}\frac{\theta_H a_1}{\theta_H a_1 + a_2} + (1-\tilde{\mu})\frac{\theta_L a_1}{\theta_L a_1 + a_2}\right]v - ca_1, \quad (14)$$

and

$$E[U_2^f(a_1,a_2)] = P_2^f(a_1,a_2)v - ca_2 = \left[\mu\frac{a_2}{\theta_H a_1 + a_2} + (1-\mu)\frac{a_2}{\theta_L a_1 + a_2}\right]v - ca_2. \quad (15)$$

In equation (14), $\tilde{\mu}$ is player 1's perceived posterior belief of having high ability given by

$$\tilde{\mu} = \frac{\tilde{\pi}\theta_H\big(\theta_L a_1 + a_3\big)}{\theta_L\theta_H a_1 + \big[\tilde{\pi}\theta_H + (1-\tilde{\pi})\theta_L\big]a_3}. \quad (16)$$

Likewise, in equation (15), μ designates player 2's posterior belief that player 1 has high ability and is given by:

$$\mu = \frac{\pi\theta_H\big(\theta_L a_1 + a_3\big)}{\theta_L\theta_H a_1 + \big[\pi\theta_H + (1-\pi)\theta_L\big]a_3}. \quad (17)$$

Define player 1's final stage overconfidence bias as $b^f = \tilde{\mu} - \mu$. Note that whereas the semifinal stage bias b^s is exogenous, the final stage bias b^f is endogenous because it depends on the posterior beliefs $\tilde{\mu}$ and μ which are determined by the equilibrium efforts exerted in the semifinal. Furthermore, since $\tilde{\mu}$ is a function of $\tilde{\pi}$, which itself is influenced by the semifinal stage bias b^s, and the semifinal equilibrium efforts depend on b^s, the final stage bias b^f depends on b^s as well. Comparing player 1's overconfidence biases in the two stages, b^s and b^f, we can make the following observation.

Result 1. *There exists a unique value $\hat{\pi} \in (0,1)$ for the ex-ante probability π the newcomer has high ability such that when $\pi \lesseqqgtr \hat{\pi}$ then $b^f \gtreqqless b^s$.*

This results says that in an elimination contest, winning the semifinal amplifies the overconfidence bias of a player whose ex-ante probability of being high ability is low, and dampens it otherwise. First observe that when the overconfident player 1 wins the semifinal, both player 1 and his rival in the final, player 2, revise upwards their beliefs about player 1 having high ability, i.e., $\tilde{\mu} > \tilde{\pi}$ and $\mu > \pi$. Given that both players revise their beliefs upwards, we wish to understand what makes either player revise his beliefs the most. When π is close to 0—meaning the rational player is almost certain that his opponent has low ability—a win by player 1 in the semifinal does little to change player 2's belief. The strong prior prevents the rational player to significantly revise his expectations. For intermediate values of π,—meaning the rational player is very uncertain about his opponent's ability—a win by player 1 in the semifinal leads to a large upwards revision in the beliefs of player 2. Last, when π is close to 1, there is little room for further increasing players' beliefs. The overconfident player mistakenly assigns a higher probability to having high ability. This, in turn, leads the overconfident player to update upwards his beliefs by a large amount when π is low. Consequently, for low values of π the update from the overconfident player will be larger that the one from the rational player. For intermediate values of π, the overconfident player believes his probability of being a high type is very high, and the update of beliefs will therefore be small. Accordingly, it will be the rational player who will update his beliefs the most for such priors.

Optimizing (14) and (15), we obtain the two following first-order conditions:

$$\left[\tilde{\mu}\frac{\theta_H}{(\theta_H a_1 + a_2)^2} + (1-\tilde{\mu})\frac{\theta_L}{(\theta_L a_1 + a_2)^2}\right] a_2 v = c, \tag{18}$$

and

$$\left[\mu\frac{\theta_H}{(\theta_H a_1 + a_2)^2} + (1-\mu)\frac{\theta_L}{(\theta_L a_1 + a_2)^2}\right] a_1 v = c. \tag{19}$$

From these expressions, we can derive the next two Lemmas:

Lemma 1. *The players' best response functions in the final are quasi-concave.*

Lemma 2. *The final admits a unique pure strategy equilibrium.*

Having shown that the final admits a unique pure strategy equilibrium, we can establish the next proposition.

Proposition 3. *In a final between an overconfident newcomer and an incumbent, the equilibrium efforts depend on the product of the newcomer's possible abilities as follows:*
(i) If $\theta_L\theta_H \in \{0,1\}$*, then* $a_1^f = a_2^f$*;*
(ii) If $\theta_L\theta_H \in (0,1)$*, then* $a_1^f > a_2^f$*;*
(iii) If $\theta_L\theta_H > 1$*, then* $a_1^f < a_2^f$*.*

Proposition 3 shows that whether the overconfident newcomer invests more than the incumbent in the final depends on how the incumbent's fixed ability, which equals 1, compares with the product of the newcomer's possible abilities, $\theta_L\theta_H$. Interestingly, the posterior beliefs that the newcomer is of high ability, μ and $\tilde{\mu}$, are irrelevant in determining which player exerts higher effort at equilibrium. Indeed, following the reasoning underlying Proposition 1, we know that when $\theta_H = 1/\theta_L$, the players can be seen as optimizing a weighted average of two Tullock contests where the most able player is either the newcomer with an ability θ_H, or the incumbent with an ability $1/\theta_L = \theta_H$. Hence, the highest ability player will always have the same ability for any weights μ or $\tilde{\mu}$. Accordingly, for any weighing (μ and $\tilde{\mu}$) the equilibrium efforts will be the same for both players, and equal to the efforts they would invest if information was complete.

When $\theta_L\theta_H = 0$, the newcomer can only reach the final is he has high ability, consequently the posterior beliefs are $\tilde{\mu} = \mu = 1$ (there is certainty in the final), and both players choose the same effort.

When $\theta_H < 1/\theta_L$, since $\tilde{\mu} > \mu$, the newcomer puts more weight than the incumbent on the scenario where his relative ability is θ_H, while the incumbent puts more weight than the newcomer where his own relative ability is $1/\theta_L$. Yet, we know that the more unequal the abilities of the player, the lower their equilibrium efforts in a complete information setup. Consequently, when $\theta_H < 1/\theta_L$, the newcomer puts more weight on the scenario where players invest higher equilibrium efforts, whereas the incumbent puts more weight on the scenario where the players invest lower equilibrium efforts. Accordingly, at equilibrium the newcomer will invest more effort than the incumbent.

When $\theta_L\theta_H > 1$ the opposite holds true. In such instances, the newcomer puts more weight on the scenario where the players invest lower equilibrium efforts, since θ_H is indeed higher to $1/\theta_L$. Therefore, at equilibrium the newcomer will invest less effort than the incumbent.

Proposition 3 uncovers a striking result: in a one-shot Tullock contest, an overconfident player may expend more effort than his rational rival. This finding differs from earlier work on overconfidence on Tullock contests that predicts lower effort by the overconfident player in one-shot Tullock contests with complete information (Santos-Pinto and Sekeris, 2025; Chen and Santos-Pinto, 2025). The divergence arises from how the bias is specified: whereas Santos-Pinto and Sekeris (2025) and Chen and Santos-Pinto (2025) assume an overconfident player overestimates his deterministic ability, we treat the overconfident player as uncertain about his ability and prone to exaggerating the chance that it is high.

The next result describes how a change in the newcomer's final stage bias b^f affects his equilibrium effort, a_1^f, his true equilibrium probability of winning the final

$$P_1^f = \mu \frac{\theta_H a_1^f}{\theta_H a_1^f + a_2^f} + (1-\mu)\frac{\theta_L a_1^f}{\theta_L a_1^f + a_2^f},$$

and his perceived equilibrium probability of winning the final

$$\tilde{P}_1^f = \tilde{\mu} \frac{\theta_H a_1^f}{\theta_H a_1^f + a_2^f} + (1-\tilde{\mu})\frac{\theta_L a_1^f}{\theta_L a_1^f + a_2^f}.$$

Proposition 4. *In a final between an overconfident newcomer and an incumbent, the newcomer's equilibrium effort increases with his bias b^f if and only if $\theta_L\theta_H \in (0,1)$. Therefore, the newcomer's true equilibrium probability of winning the final, P_1^f, increases with his bias b^f if and only if $\theta_L\theta_H \in (0,1)$. Moreover the newcomer's perceived equilibrium probability of winning the final, $\tilde{P}_1^f$, increases with his bias b^f if $\theta_L\theta_H \in (0,1]$, while the effect is undetermined otherwise.*

When the newcomer's downside prospect of being low ability looms large, $\theta_L\theta_H \in (0,1)$, an increase in the newcomer's final stage bias b^f raises his equilibrium effort and this, in turn, increases his true and perceived equilibrium probabilities of winning the final. In contrast, when the newcomer's upside potential of being high ability is pronounced, $\theta_L\theta_H > 1$, an increase in b^f, lowers the newcomer's equilibrium effort which, in turn, reduces his true equilibrium probability of winning the final. Note that in this case the impact of an increase in b^f on the newcomer's perceived probability of winning the final is undetermined since $\tilde{\mu}$ goes up but the newcomer's effort goes down.

Lemma 3. *An increase in the newcomer's posterior belief $\tilde{\mu}$ leads to a contraction of* *his best response function in the final, $\partial R_1^f(a_2)/\partial\tilde{\mu} < 0$, for $a_2 < a_1\sqrt{\theta_L\theta_H}$, and to an expansion of his best response function in the final otherwise.*

We next explore the effect of the newcomer's posterior belief $\tilde{\mu}$, taking the incumbent's posterior belief μ fixed, on the incumbent's equilibrium effort and on the newcomer's equilibrium perceived expected utility of the final denoted by $\tilde{E}[U_1^f(a_1^f, a_2^f)]$.

Proposition 5. *Consider a final between an overconfident newcomer and an incumbent and fix the incumbent's posterior belief, μ. The incumbent's equilibrium effort, and the newcomer's equilibrium perceived expected utility vary with the newcomer's posterior belief $\tilde{\mu}$, as follows:*
(i) If $\theta_L\theta_H = 1$, then $da_2^f/d\tilde{\mu} = 0$, and $d\tilde{E}[U_1^f(a_1^f, a_2^f)]/d\tilde{\mu} > 0$.
(ii) If $\theta_L\theta_H \in (0,1)$, and $\mu > \bar{\mu}$, then $da_2^f/d\tilde{\mu} < 0$, and $d\tilde{E}[U_1^f(a_1^f, a_2^f)]/d\tilde{\mu} > 0$. Otherwise, if $\mu < \bar{\mu}$, then $da_2^f/d\tilde{\mu} > 0$ and $d\tilde{E}[U_1^f(a_1^f, a_2^f)]/d\tilde{\mu}$ is undetermined.
(iii) If $\theta_L\theta_H > 1$, and $\mu < \bar{\mu}$, then $da_2^f/d\tilde{\mu} < 0$, and $d\tilde{E}[U_1^f(a_1^f, a_2^f)]/d\tilde{\mu} > 0$. Otherwise, if $\mu > \bar{\mu}$, then $da_2^f/d\tilde{\mu} > 0$ and $d\tilde{E}[U_1^f(a_1^f, a_2^f)]/d\tilde{\mu}$ is undetermined.

A change in the newcomer's posterior belief, $\tilde{\mu}$, for fixed μ (a change in b^f) has two effects on his equilibrium perceived expected utility. First, there is a direct positive effect of the biased posterior belief on the equilibrium perceived expected winning probability. Second, there is a strategic effect of the biased posterior belief going through the effort provision of the rival player. If the strategic effect is positive, then the overall effect is unambiguously positive as well. However, if player 2 increases his equilibrium effort in response to a higher posterior belief of player 1, then the overall effect is undetermined.

Observe first that when $\theta_L\theta_H = 1$, the equilibrium efforts are equal and unaffected by the newcomer's posterior belief. In such cases, only the direct positive effect of the biased posterior belief on the equilibrium perceived expected winning probability matters. When $\theta_L\theta_H \in (0,1)$, i.e. the newcomer's downside prospect of being low ability looms large, and the incumbent's posterior belief is high, $\mu > \bar{\mu}$, an increase in the final stage bias b^f makes the final more attractive to the newcomer. In such instances, higher levels of overconfidence incentivize player 1 to increase his equilibrium effort, while also pushing player 2 to reduce his equilibrium effort, i.e. the strategic effect is positive. In contrast, if the incumbent's posterior belief is low, $\mu < \bar{\mu}$, the strategic effect is negative, and the overall effect is therefore undetermined. Finally, when $\theta_L\theta_H > 1$, i.e. the newcomer's upside potential of being high ability is pronounced, and the incumbent's posterior belief is low, $\mu < \bar{\mu}$, an increase in final stage bias b^f makes the final more attractive to the newcomer.

We now study the effect of the incumbent's posterior belief μ, taking the newcomer's posterior belief $\tilde{\mu}$ fixed, on the incumbent's equilibrium effort and on the newcomer's equilibrium perceived expected utility.

Proposition 6. *Consider a final between an overconfident newcomer and an incumbent and fix the newcomer's posterior belief, $\tilde{\mu}$. The incumbent's equilibrium effort, and the newcomer's equilibrium perceived expected utility vary with the incumbent's posterior belief μ, as follows:*
(i) If $\theta_L\theta_H = 1$, then $da_2^f/d\mu = 0$, and $d\tilde{E}[U_1^f(a_1^f, a_2^f)]/d\mu = 0$.
(ii) If $\theta_L\theta_H \in (0,1)$, then $da_2^f/d\mu > 0$, and $d\tilde{E}[U_1^f(a_1^f, a_2^f)]/d\mu < 0$.
(iii) If $\theta_L\theta_H > 1$, then $da_2^f/d\mu < 0$, and $d\tilde{E}[U_1^f(a_1^f, a_2^f)]/d\mu > 0$.

In Proposition 6 we isolate the effect of a change in the posterior belief of an incumbent in the final, μ, on her equilibrium effort a_2^f, as well as on the overconfident newcomer's perceived expected utility of the final. When $\theta_L\theta_H = 1$, only the direct positive effect of the posterior belief matters, for the same reason as in Proposition 5(i). When $\theta_L\theta_H \in (0,1)$, an increase in the posterior belief μ incentivizes the incumbent to invest lower effort at equilibrium, thence resulting in a higher equilibrium perceived expected utility for the overconfident newcomer. In contrast, when $\theta_L\theta_H > 1$, the incumbent is incentivized to increase his equilibrium effort, thereby leading to a reduction of the newcomer's payoff.

5.2 Semifinal

Player 1's perceived expected utility of the semifinal is

$$\begin{aligned}\tilde{E}[U_1^s(a_1,a_3)] &= \tilde{P}_1^s(a_1,a_3)\tilde{E}[U_1^f(a_1^f,a_2^f;\tilde{\mu})] - ca_1\\ &= \left[\tilde{\pi}\frac{\theta_H a_1}{\theta_H a_1 + a_3} + (1-\tilde{\pi})\frac{\theta_L a_1}{\theta_L a_1 + a_3}\right]\tilde{E}[U_1^f(a_1^f,a_2^f;\tilde{\mu})] - ca_1,\end{aligned}$$

where $a_1^f = h_1(\tilde{\mu},\mu)$, $a_2^f = h_2(\tilde{\mu},\mu)$, $\tilde{\mu}$ is given by equation (16) and μ by equation (17).

The first-order condition of player 1 in the semifinal is

$$\frac{\partial \tilde{P}_1^s}{\partial a_1}\tilde{E}[U_1^f] + \tilde{P}_1^s(a_1,a_3)\left[\frac{d\tilde{E}[U_1^f]}{d\mu}\frac{\partial\mu}{\partial a_1} + \frac{d\tilde{E}[U_1^f]}{d\tilde{\mu}}\frac{\partial\tilde{\mu}}{\partial a_1}\right] = c. \tag{20}$$

To show existence of a pure-strategy equilibrium we demonstrate in the Appendix that the second-order conditions are verified whenever the first-order conditions are satisfied for the specific case $\theta_L\theta_H = 1$. By continuity of the players' perceived expected utilities in θ_L and θ_H it follows that there is an interval $[\underline{\Theta}, \bar{\Theta}]$, with $\underline{\Theta} < 1$ and $\bar{\Theta} > 1$, such that for any $\Theta = \theta_L\theta_H$ in that interval the second-order conditions are verified. The assumption that v/c is sufficiently large ensures that both players attain strictly positive expected utilities for these strategy profile.

We can re-express the equilibrium perceived utility of player 1 in the final $\tilde{E}[U_1^f]$, as a function of the updated beliefs in the final μ and the bias b^f, rather than as a function of μ and $\tilde{\mu}$. Accordingly, define $\tilde{E}[\hat{U}_1^f(b^f, \mu)] \equiv \tilde{E}[U_1^f(\tilde{\mu}, \mu)]$ and re-write (20) as:

$$\frac{\partial \tilde{P}_1^s}{\partial a_1}\tilde{E}\big[\hat{U}_1^f\big] + \tilde{P}_1^s(a_1, a_3)\left[\frac{d\tilde{E}[\hat{U}_1^f]}{d\mu}\frac{\partial \mu}{\partial a_1} + \frac{d\tilde{E}[\hat{U}_1^f]}{db^f}\frac{\partial b^f}{\partial a_1}\right] = c. \tag{21}$$

The above first-order condition equates the perceived marginal benefit of effort in the semifinal to the constant marginal cost c. The perceived marginal benefit of effort is itself composed of three terms. The first two terms are directly comparable to the ones obtained in the previous section with a rational newcomer. The third term is new and is related to the newcomer's overconfidence bias. Combined, these three effects shape the overconfident newcomer's behavior in the semifinal.

The first term captures the fact that increasing effort in the semifinal raises the newcomer's perceived probability of winning the semifinal. The sign of the first term in (21) is positive since an increase in semifinal effort raises the perceived probability of winning.

The second term in (21) describes how a change in the newcomer's semifinal effort affects the incumbent's posterior belief μ, which in turn affects the efforts of the newcomer and the incumbent in the final, thereby modifying the newcomer's perceived expected utility of the final. Bearing in mind that $\partial\mu/\partial a_1 < 0$ (Equation (5)), the sign of the second term is directly determined by Proposition 6. Indeed, we know that higher semifinal effort reduces the rival's posterior belief. Moreover, for a fixed posterior belief of the newcomer, we know from Proposition 5 that an increase in the rival's posterior belief μ raises the rival's effort in the final when $\theta_L\theta_H \in (0, 1)$, in turn leading to a lower perceived expected utility for the newcomer in final. Conversely, for $\theta_L\theta_H > 1$ an increase in μ will generate the opposite effects. Combined, these effects imply that for $\theta_L\theta_H < 1$, the second term is negative, thence providing the newcomer with lower incentives to invest in the semifinal effort. If, on the other hand, $\theta_L\theta_H > 1$, the effect is reversed.

Lastly, the third effect in (21) captures how a change in semifinal effort impacts the (endogenous) final stage overconfidence bias for a given posterior belief μ, which in turn affects the efforts of the newcomer and the incumbent in the final, thereby modifying the newcomer's perceived expected utility of reaching the final. This effect is the product of two terms. The sign of the first multiplicative term, $dE[\hat{U}_1^f]/db^f$, is described in Proposition 5. The sign of the second multiplicative term, $\partial b^f/\partial a_1$, the effect of semifinal effort on the newcomer's final stage bias b^f, is described in the next lemma.

Lemma 4. *There exists a value $\bar{\gamma}(a_1, a_3) \in (0,1)$ such that for any $\pi > \bar{\gamma}(a_1, a_3)$, $\partial b^f/\partial a_1 > 0$, and for any $\pi < \bar{x}(a_1, a_3)$, $\partial b^f/\partial a_1 > 0$ if $b^s > \bar{b}^s \in (0, 1-\pi)$, and $\partial b^f/\partial a_1 < 0$ otherwise.*

This lemma states that an increase in semifinal effort raises the newcomer's final stage bias b^f if the ex-ante probability the newcomer is of high ability, π is large enough. Moreover, this is also true for a small π provided the newcomer's semifinal stage bias, b^s, is sufficiently high. For low π and b^s, an increase in semifinal effort lowers the newcomer's final stage bias b^f.

We see that the overconfident newcomer's semifinal effort choice has two informational consequences: a public-signaling component that affects incumbent 2's posterior belief μ, and a self-signaling component that affects the newcomer's posterior belief $\tilde{\mu}$.[6] In both cases greater semifinal effort is interpreted as bad news about ability.

Observe that if the newcomer's final stage bias were exogenous and independent of the semifinal stage bias, we would be able to gauge the effect of b^f on the equilibrium semifinal efforts and winning probabilities. Yet, since b^f is endogenous to players' semifinal efforts, which in turn depend on b^s, we are not in a position to derive general results for any value of θ_L and θ_H. However, by imposing $\theta_L\theta_H = 1$, we are able to derive closed-form solutions for the final stage efforts, and this in turn allows us to derive additional results in the next section.

5.3 The Model with $\theta_L\theta_H = 1$

When $\theta_L\theta_H = 1$, we know that $a_1^f = a_2^f$, which using player 1's first-order condition is shown to equal:

$$a^f = \left[\tilde{\mu}\frac{\theta_H}{(\theta_H+1)^2} + (1-\tilde{\mu})\frac{\theta_L}{(\theta_L+1)^2}\right]\frac{v}{c}$$

Substituting for $\theta_L = 1/\theta_H$, this expression reads as:

$$a^f = \frac{\theta_H}{(\theta_H+1)^2}\frac{v}{c}$$

[6] We use the term self-signaling in a similar way as Bodner and Prelec (2003).

The equilibrium perceived expected utility of the final to player 1 is then equal to:

$$\begin{aligned}\tilde{E}[U_1^f(a_1^f, a_2^f; \tilde{\mu})] &= \left[\tilde{\mu}\frac{\theta_H}{\theta_H+1} + (1-\tilde{\mu})\frac{1}{\theta_H+1}\right]v - ca \\ &= \left[\left(\frac{1}{\theta_H+1}\right)^2 + \frac{\theta_H-1}{\theta_H+1}\tilde{\mu}\right]v,\end{aligned}$$

where, imposing $\theta_L = 1/\theta_H$, we have

$$\tilde{\mu} = \frac{\tilde{\pi}\theta_H\big(a_1 + \theta_H a_3\big)}{\theta_H a_1 + \big[\tilde{\pi}\theta_H^2 + (1-\tilde{\pi})\big]a_3}.$$

In the semifinal, the first-order condition of the overconfident newcomer is then:

$$\frac{\partial \tilde{P}_1^s}{\partial a_1}\tilde{E}\big[U_1^f\big] + \tilde{P}_1^s(a_1,a_3)\left[\frac{\partial \tilde{E}\big[U_1^f\big]}{\partial a_2}\Big(\underbrace{\frac{\partial a_2^f}{\partial \tilde{\mu}}}_{=0}\frac{\partial \tilde{\mu}}{\partial a_1} + \underbrace{\frac{\partial a_2^f}{\partial \mu}}_{=0}\frac{\partial \mu}{\partial a_1}\Big) + \frac{\partial \tilde{E}\big[U_1^f\big]}{\partial \tilde{\mu}}\frac{\partial \tilde{\mu}}{\partial a_1}\right] = c. \quad (22)$$

Observe that by focusing on the specific case where $\theta_L\theta_H = 1$, the players' equilibrium efforts in the final are independent of μ. Consequently, when modifying his effort in the semifinal, the overconfident newcomer is aware that his rival in the final will not subsequently adapt her effort, as shown in Propositions 5(i) and 6(i). This result explains why the two terms in (22) are equal to 0. Consequently, when deciding his semifinal effort, the overconfident newcomer accounts for the effect of his effort on the expected utility of the final only through the effect of $\tilde{\mu}$ on the perceived probability of winning the final. Equation (22) then simplifies to

$$\frac{\partial \tilde{P}_1^s}{\partial a_1}\left[\left(\frac{1}{\theta_H+1}\right)^2 + \frac{\theta_H-1}{\theta_H+1}\tilde{\mu}\right]v + \tilde{P}_1^s(a_1,a_3)\frac{\theta_H-1}{\theta_H+1}v\frac{\partial \tilde{\mu}}{\partial a_1} = c. \quad (23)$$

Using the incumbent's first-order condition in equation (13), alongside the overconfident newcomer's first-order condition in equation (23), we can state the next result.

Proposition 7. *In the semifinal between an overconfident newcomer and an incumbent of a two-stage elimination contest where $\theta_L\theta_H = 1$, there exists a unique value*

$$\breve{\pi} = \frac{\theta_H + 3}{4(\theta_H+1)}$$

for the ex-ante probability π the newcomer has high ability such that:
(i) For $\pi < \tilde{\pi} < \breve{\pi}$, both a rational and an overconfident newcomer exerts less effort than the incumbent at equilibrium in the semifinal.

(ii) For $\pi < \breve{\pi} < \tilde{\pi}$, an overconfident newcomer exerts more effort than the incumbent at equilibrium in the semifinal, while a rational newcomer exerts less effort.
(iii) For $\breve{\pi} < \pi < \tilde{\pi}$, both a rational and an overconfident newcomer exerts more effort than the incumbent at equilibrium in the semifinal.

When $\theta_L \theta_H = 1$ we have the following three scenarios. First, when the newcomer's ex-ante probability of having high ability is low and his biased prior belief $\tilde{\pi}$ is smaller than the threshold value, $\pi < \tilde{\pi} < \breve{\pi}$, both a rational and an overconfident newcomer exert less effort at equilibrium in the semifinal than the incumbent. This is intuitive, the small bias leads to a small change in effort provision and both a rational and an overconfident newcomer exert less effort than the incumbent for the same reasons highlighted after Proposition 2 in Section 4.2.

Second, when the newcomer's ex-ante probability of having high ability is low and his biased prior belief $\tilde{\pi}$ is greater than the threshold value, $\pi < \breve{\pi} < \tilde{\pi}$, an overconfident newcomer exerts more effort at equilibrium in the semifinal than the incumbent whereas a rational newcomer exerts less effort. In this case the higher bias leads to a large effort provision since the newcomer overestimates his probability of winning the semifinal as well at his expected utility of the final. Consequently the overconfident newcomer exerts more effort than the incumbent when a rational newcomer would have exerted less effort than the incumbent.

Third, when the newcomer's ex-ante probability of having high ability is greater than the threshold value, $\breve{\pi} < \pi < \tilde{\pi}$, both a rational and an overconfident newcomer exert more effort at equilibrium in the semifinal than the incumbent.

Last we analyze the impact of the newcomer's semifinal bias b^s on his true equilibrium probabilities of winning the final and semifinal, P_1^f and P_1^s, respectively. This allows us to determine how the semifinal bias b^s changes the newcomer's true equilibrium probability of winning the elimination contest $P_1^s P_1^f$.

Proposition 8. *In a two-stage elimination contest where $\theta_L \theta_H = 1$, the overconfident newcomer's true equilibrium probability of winning the final, P_1^f, decreases in his overconfidence bias b^s, and his true equilibrium probability of winning the semifinal, P_1^s, increases in b^s. His true equilibrium probability of winning the contest, $P_1^s P_1^f$, increases in his overconfidence bias b^s.*

When $\theta_L \theta_H = 1$, the newcomer's true equilibrium probability of winning the final is given by

$$P_1^f = \frac{1 + \mu(\theta_H - 1)}{\theta_H + 1}, \tag{24}$$

where

$$\mu = \pi \frac{\frac{a_1^s}{a_3^s} + \theta_H}{\frac{a_1^s}{a_3^s} + \pi\theta_H + (1-\pi)/\theta_H}.$$

Since

$$\partial\mu/\partial(a_1^s/a_3^s) = -\frac{(1-\pi)(\theta_H - 1/\theta_H)}{(a_1^s/a_3^s + \pi\theta_H + (1-\pi)/\theta_H)^2} < 0,$$

and since P_1^f is increasing in μ, it follows that the newcomer's true equilibrium probability of winning the final is decreasing in a_1^s/a_3^s.

The newcomer's true equilibrium probability of winning the semifinal is given by

$$P_1^s = \pi\frac{\theta_H a_1^s}{\theta_H a_1^s + a_3^s} + (1-\pi)\frac{a_1^s/\theta_H}{a_1^s/\theta_H + a_3^s}, \tag{25}$$

and we immediately observe that P_1^s is increasing with a_1^s/a_3^s. Consequently, a change in the equilibrium semifinal relative effort induces P_1^f and P_1^s to move in opposite directions. We show, in the proof of Proposition 8, that an increase in the newcomer's semifinal bias b^s raises the semifinal equilibrium relative effort a_1^s/a_3^s. Hence, an increase in the newcomer's semifinal bias b^s lowers his true equilibrium probability of winning the final while it raises his true equilibrium probability of winning the semifinal. We demonstrate in Proposition 8 that the net effect of the increase in P_1^s dominates.

Figure 1 illustrates Proposition 8. It depicts P_1^s, P_1^f, and their product as a function of b^s when $\pi = 1/4$, $\theta_L = 1/2$, $\theta_H = 2$, $v = 10$, and $c = 1$.

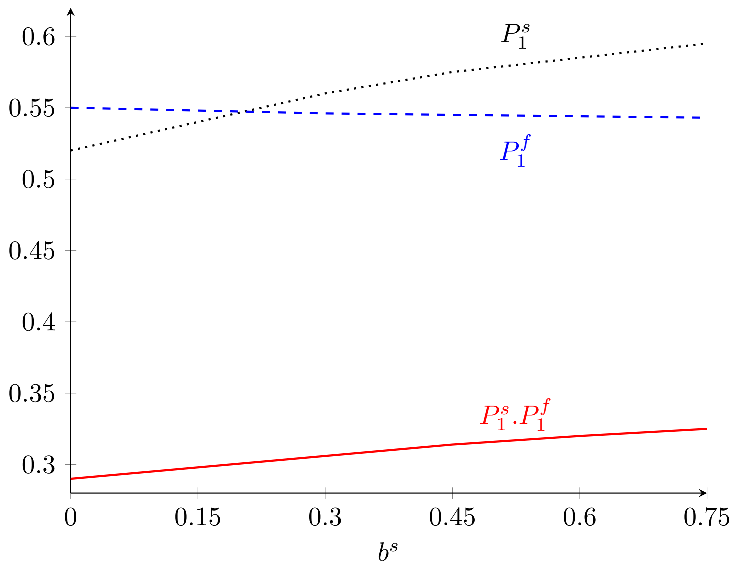

Figure 1: Newcomer's true winning probabilities in the elimination contest.

6 Conclusion

This paper analyzes how overconfidence shapes behavior and winning probabilities in a two-stage elimination contest with incomplete information. We start by showing that following a first stage victory the newcomer's overconfidence bias is boosted when his ex-ante probability of having high ability is low, otherwise the bias is dampened. Next, we demonstrate that an overconfident newcomer can exert high effort in the final of the elimination contest than his rival when the downside prospect of having low ability looms large. We also show that an increase in the semifinal stage bias can lead the overconfident newcomer to increase his effort in the semifinal of the elimination contest. Finally, we demonstrate that an overconfident newcomer's chances of winning the elimination contest can increase withe the size of the his bias. Our results clarify under which conditions success breeds further overconfidence or tames it. They also provide an explanation for why overconfident individuals can attain the upper levels of hierarchical promotion contests

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7 Appendix

Second-order conditions of the semifinal with a rational newcomer: The first derivative of the newcomer's expected utility in the semifinal is:

$$\frac{\partial E[U_1^s]}{\partial a_1} = \frac{\partial P_1^s}{\partial a_1}\left[\left(\frac{\theta_L}{\theta_L+1}\right)^2 + \mu\chi\right] v + P_1^s \chi v \frac{\partial \mu}{\partial a_1} - c.$$

The second derivative is therefore given by:

$$\frac{\partial^2 E[U_1^s]}{\partial a_1^2} = \frac{\partial^2 P_1^s}{\partial a_1^2}\left[\left(\frac{\theta_L}{\theta_L+1}\right)^2 + \mu\chi\right] v + 2\frac{\partial P_1^s}{\partial a_1}\chi v \frac{\partial \mu}{\partial a_1} + P_1^s \chi v \frac{\partial^2 \mu}{\partial a_1^2}.$$

Since $\frac{\partial^2 P_1^s}{\partial a_1^2} < 0$, this expression is a fortiori negative if:

$$\left[\frac{\partial^2 P_1^s}{\partial a_1^2}\mu + 2\frac{\partial P_1^s}{\partial a_1}\frac{\partial \mu}{\partial a_1} + P_1^s \frac{\partial^2 \mu}{\partial a_1^2}\right] \underbrace{\chi v}_{>0} < 0.$$

We thus need to show that the term inside squared brackets is negative. Substituting for the appropriate terms, and simplifying, the term inside squared brackets is given by:

$$-\frac{2\pi\theta_H^2 a_3}{(\theta_H a_1 + a_3)^3} < 0.$$

Hence, the second-order condition for the newcomer is satisfied. It is immediate to verify that the second-order condition for the incumbent is also satisfied.

Proof of Proposition 2: To prove this result, we consider an effort level of a player that maximizes his semifinal payoff. Accordingly, that effort level must satisfy his first-order condition. We next fix the other player's effort at the same level, i.e. $a_1 = a_3$, and deduce that $a_1^s \lesseqgtr a_3^s \Leftrightarrow \partial E[U_1^s(a_1,a_3)]/\partial a_1 \lesseqgtr \partial E[U_3^s(a_1,a_3)]/\partial a_3$, for $a_1 = a_3 = a > 0$. More specifically, we define the difference of the players' first order derivatives when evaluated at $a_1 = a_3$ as $\Psi(\pi)$, and we show that $\Psi(0) < 0$, $\Psi(1) > 0$, and $\Psi'(\pi) > 0$ on $\pi \in [0,1]$, thence implying that there is a unique ex-ante probability the newcomer has high ability $\bar{\pi}$ for which players 1 and 3 exert the same equilibrium effort in the semifinal.

We start by simplifying the first-order condition of player 1 given by equation (12):

$$\frac{\partial P_1^s}{\partial a_1} E\big[U_1^f(a_1^f, a_2^f)\big] + P_1^s(a_1, a_3)\chi v \frac{\partial \mu}{\partial a_1} = c.$$

We know that

$$\frac{\partial P_1^s}{\partial a_1} = \frac{\partial P_3^s}{\partial a_3}\frac{a_3}{a_1}.$$

Using this last equation, the first-order condition of player 3 given by equation (13) becomes

$$\frac{\partial P_1^s}{\partial a_1}\frac{a_1}{a_3}\frac{v}{4} = c.$$

Using these first-order conditions, we can next express $\Psi(\pi)$ as:

$$\begin{aligned}
\Psi(\pi) &= \left.\frac{\partial E[U_1^s(a_1, a_3)]}{\partial a_1}\right|_{a_1=a_3} - \left.\frac{\partial E[U_3^s(a_1, a_3)]}{\partial a_3}\right|_{a_1=a_3} \\
&= \left.\left\{\frac{\partial P_1^s}{\partial a_1} E\big[U_1^f(a_1^f, a_2^f)\big] + P_1^s(a_1, a_3)\chi v \frac{\partial \mu}{\partial a_1} - \frac{\partial P_1^s}{\partial a_1}\frac{a_1}{a_3}\frac{v}{4}\right\}\right|_{a_1=a_3} \\
&= \left.\left\{\frac{\partial P_1^s}{\partial a_1}\left[E\big[U_1^f(a_1^f, a_2^f)\big] - \frac{a_1}{a_3}\frac{v}{4}\right] + P_1^s(a_1, a_3)\chi v \frac{\partial \mu}{\partial a_1}\right\}\right|_{a_1=a_3}.
\end{aligned}$$

We have

$$P_1^s(a_1, a_3) = \pi \frac{\theta_H a_1}{\theta_H a_1 + a_3} + (1-\pi)\frac{\theta_L a_1}{\theta_L a_1 + a_3}.$$

So, we have

$$\frac{\partial P_1^s}{\partial a_1} = \pi \frac{\theta_H a_3}{(\theta_H a_1 + a_3)^2} + (1-\pi)\frac{\theta_L a_3}{(\theta_L a_1 + a_3)^2}$$

Observe that when imposing $a_1 = a_3 = a$, we have:

$$\begin{aligned}
\left.P_1^s(a_1, a_3)\right|_{a_1=a_3=a} &= \frac{\theta_L}{1+\theta_L} + \pi\left(\frac{\theta_H}{1+\theta_H} - \frac{\theta_L}{1+\theta_L}\right), \\
\left.\frac{\partial P_1^s}{\partial a_1}\right|_{a_1=a_3=a} &= \left[\frac{\theta_L}{(\theta_L+1)^2} + \pi\left(\frac{\theta_H}{(\theta_H+1)^2} - \frac{\theta_L}{(\theta_L+1)^2}\right)\right]\frac{1}{a}, \\
\left.E[U_1^f(a_1^f, a_2^f)]\right|_{a_1=a_3=a} &= \left[\left(\frac{\theta_L}{\theta_L+1}\right)^2 + \frac{\pi\theta_H\big(\theta_L+1\big)}{\theta_L\theta_H + \pi\theta_H + (1-\pi)\theta_L}\chi\right] v,
\end{aligned}$$

and

$$\left.\frac{\partial \mu}{\partial a_1}\right|_{a_1=a_3=a} = -\frac{\pi(1-\pi)\theta_L\theta_H(\theta_H-\theta_L)}{\big[\theta_L\theta_H + \pi\theta_H + (1-\pi)\theta_L\big]^2 a}$$

.

Substituting in Ψ for the appropriate terms we then obtain:

$$\begin{aligned}\Psi(\pi) &= \left[\frac{\theta_L}{(\theta_L+1)^2} + \pi\left(\frac{\theta_H}{(\theta_H+1)^2} - \frac{\theta_L}{(\theta_L+1)^2}\right)\right]\frac{1}{a}\left[\left[\left(\frac{\theta_L}{\theta_L+1}\right)^2 + \frac{\pi\theta_H(\theta_L+1)}{\theta_L\theta_H+\pi\theta_H+(1-\pi)\theta_L}\chi\right] - \frac{1}{4}\right]v \\ &- \left[\frac{\theta_L}{1+\theta_L} + \pi\left(\frac{\theta_H}{1+\theta_H} - \frac{\theta_L}{1+\theta_L}\right)\right]\chi v\frac{\pi(1-\pi)\theta_L\theta_H(\theta_H-\theta_L)}{\left[\theta_L\theta_H+\pi\theta_H+(1-\pi)\theta_L\right]^2 a}\end{aligned}$$

After some manipulations we obtain

$$\Psi(\pi) = \frac{-\theta_L(\theta_H+1)^4(1+2\theta_L-3\theta_L^2) + \left[\theta_H(\theta_L+1)^4(3\theta_H^2-2\theta_H-1) + \theta_L(\theta_H+1)^4(1+2\theta_L-3\theta_L^2)\right]\pi}{(\theta_H+1)^4(\theta_L+1)^4}\frac{v}{a}$$

Setting $\Psi(\pi) = 0$, and solving for π, we obtain:

$$\bar{\pi} = \frac{\theta_L(\theta_H+1)^4(1+2\theta_L-3\theta_L^2)}{\theta_H(\theta_L+1)^4(3\theta_H^2-2\theta_H-1) + \theta_L(\theta_H+1)^4(1+2\theta_L-3\theta_L^2)}.$$

Proof of Result 1: There exists a unique prior belief $\hat{\pi} \in [0,1]$ which is such that when $\pi \lesseqgtr \hat{\pi}$ then $b^f \gtreqless b^s$.

$$\begin{aligned}b^f - b^s &= (\tilde{\mu} - \mu) - (\tilde{\pi} - \pi) \\ &= (\tilde{\mu} - \tilde{\pi}) - (\mu - \pi) \\ &= \tilde{\pi}\left[\frac{P_1^s(a_1^s,a_3^s;\theta_H)}{\tilde{\pi}P_1^s(a_1^s,a_3^s;\theta_H) + (1-\tilde{\pi})P_1^s(a_1^s,a_3^s;\theta_L)} - 1\right] - \pi\left[\frac{P_1^s(a_1^s,a_3^s;\theta_H)}{\pi P_1^s(a_1^s,a_3^s;\theta_H) + (1-\pi)P_1^s(a_1^s,a_3^s;\theta_L)} - 1\right] \\ &= \tilde{\pi}(1-\tilde{\pi})\frac{P_1^s(a_1^s,a_3^s;\theta_H) - P_1^s(a_1^s,a_3^s;\theta_L)}{\tilde{\pi}P_1^s(a_1^s,a_3^s;\theta_H) + (1-\tilde{\pi})P_1^s(a_1^s,a_3^s;\theta_L)} - \pi(1-\pi)\frac{P_1^s(a_1^s,a_3^s;\theta_H) - P_1^s(a_1^s,a_3^s;\theta_L)}{\pi P_1^s(a_1^s,a_3^s;\theta_H) + (1-\pi)P_1^s(a_1^s,a_3^s;\theta_L)}\end{aligned}$$

Thence, the sign of $b^f - b^s$ is given by:

$$\begin{aligned}\mathrm{sgn}\{b^f - b^s\} &= \mathrm{sgn}\{(1-\tilde{\pi})(1-\pi)\,P_1^s(a_1^s,a_3^s;\theta_L) - \pi\tilde{\pi}\,P_1^s(a_1^s,a_3^s;\theta_H)\} \\ &= \mathrm{sgn}\{(1-\pi-b^s)(1-\pi)\,P_1^s(a_1^s,a_3^s;\theta_L) - \pi(\pi+b^s)\,P_1^s(a_1^s,a_3^s;\theta_H)\} \\ &= \mathrm{sgn}\Big\{-\left[P_1^s(a_1^s,a_3^s;\theta_H) - P_1^s(a_1^s,a_3^s;\theta_L)\right]\pi^2 \\ &\quad - \left[2 + (P_1(a_1,a_3;\theta_H) - P_1^s(a_1^s,a_3^s;\theta_L))\right]b^s\,\pi \\ &\quad + (1-b^s)\,P_1^s(a_1^s,a_3^s;\theta_L)\Big\}.\end{aligned}$$

Since for $\pi = 0$ the above expression is always positive, for $\pi = 1$ it is always negative, and given the quadratic nature of the expression there exists a unique threshold $\hat{\pi} \in (0,1)$ such that $\pi \lesseqgtr \hat{\pi} \Leftrightarrow b^f \gtreqless b^s$.

Proof of Lemma 1: To show that the best response functions of players in the final, $R_i^f(a_{-i})$, $i \in \{1,2\}$, are quasi-concave, we focus on the best response of the overconfident player, player 1, and the reasoning extends to the rival player 2. We show that (i) the slope of player 1's best response function is strictly positive for $a_2 = 0$, i.e. $(\partial R_1^f/\partial a_2)\big|_{a_2=0} > 0$, (ii) that it it strictly negative for $a_2 \to \infty$, i.e. $(\partial R_1^f/\partial a_2)\big|_{a_2\to\infty} < 0$, and (iii) that whenever $\partial R_1^f/\partial a_2 = 0$, then $\partial^2 R_1^f/\partial a_2^2 < 0$.

By implicit differentiation of the first-order condition of player 1 as given by equation (18), we deduce that the sign of the slope of player 1's best response is given by:

$$\text{sgn}\left\{\frac{\partial R_1^f}{\partial a_2}\right\} = \text{sgn}\left\{\frac{\partial^2 E[U_1^f(a_1,a_2)]}{\partial a_1 a_2}\right\} = \text{sgn}\left\{\frac{\tilde{\mu}\theta_H(\theta_H a_1 - a_2)}{(\theta_H a_1 + a_2)^3} + \frac{(1-\tilde{\mu})\theta_L(\theta_L a_1 - a_2)}{(\theta_L a_1 + a_2)^3}\right\}. \tag{26}$$

Points (i) and (ii) are immediately deduced upon observing the above expression. Turning next to (iii), define first $\phi(a_2) = \frac{\partial^2 E[U(a_1,a_2)]}{\partial a_1 a_2}$. To establish (iii) it is then sufficient to show that when $\phi(a_2) = 0$, then $\phi'(a_2) < 0$. We thus compute $\phi'(a_2)$ which is given by:

$$\phi'(a_2) = -\frac{\tilde{\mu}\theta_H}{(\theta_H a_1 + a_2)^3} - \frac{3\tilde{\mu}\theta_H(\theta_H a_1 - a_2)}{(\theta_H a_1 + a_2)^4} - \frac{(1-\tilde{\mu})\theta_L}{(\theta_L a_1 + a_2)^3} - \frac{3(1-\tilde{\mu})\theta_L(\theta_L a_1 - a_2)}{(\theta_L a_1 + a_2)^4}.$$

Substituting for $\phi(a_2) = 0$, we can show that the above expression can be re-expressed as:

$$\phi'(a_2) = \frac{\tilde{\mu}\theta_H}{(\theta_H a_1 + a_2)^3}\left(\frac{(\theta_H - \theta_L)a_1}{\theta_L a_1 - a_2} - 3(\theta_H a_1 - a_2)\left(\frac{1}{\theta_H a_1 + a_2} - \frac{1}{\theta_L a_1 + a_2}\right)\right),$$

or,

$$\phi'(a_2) = \frac{\tilde{\mu}\theta_H 4(\theta_H - \theta_L)a_1}{(\theta_H a_1 + a_2)^3(\theta_L a_1 - a_2)} < 0,$$

with the sign following from the observation that to have $\phi(a_2) = 0$, it is necessary that $\theta_L a_1 - a_2 < 0$.

Proof of Lemma 2: To prove uniqueness, we show that the contraction mapping $\partial R_1^f(a_2)/\partial a_2 . \partial R_2^f(a_1)/\partial a_1$ is smaller to 1. The slopes of $R_1^f(a_2)$ and $R_2^f(a_1)$ are respectively given by:

$$\frac{\partial R_1^f(a_2)}{\partial a_2} = \frac{\frac{\tilde{\mu}\theta_H(\theta_H a_1 - a_2)}{(\theta_H a_1 + a_2)^3} + \frac{(1-\tilde{\mu})\theta_L(\theta_L a_1 - a_2)}{(\theta_L a_1 + a_2)^3}}{2a_2\left[\frac{\tilde{\mu}\theta_H^2}{(\theta_H a_1 + a_2)^3} + \frac{(1-\tilde{\mu})\theta_L^2}{(\theta_L a_1 + a_2)^3}\right]},$$

and,

$$\frac{\partial R_2^f(a_1)}{\partial a_1} = -\frac{\frac{\mu\theta_H(\theta_H a_1 - a_2)}{(\theta_H a_1 + a_2)^3} + \frac{(1-\mu)\theta_L(\theta_L a_1 - a_2)}{(\theta_L a_1 + a_2)^3}}{2a_1\left[\frac{\mu\theta_H}{(\theta_H a_1 + a_2)^3} + \frac{(1-\mu)\theta_L}{(\theta_L a_1 + a_2)^3}\right]}.$$

Observe first that if $\theta_L a_1 \geq a_2$, then $\frac{\partial R_1^f(a_2)}{\partial a_2} > 0$ and $\frac{\partial R_2^f(a_1)}{\partial a_1} < 0$. Next, observe that if $\theta_H a_1 \leq a_2$, then $\frac{\partial R_1^f(a_2)}{\partial a_2} < 0$ and $\frac{\partial R_2^f(a_1)}{\partial a_1} > 0$. In both cases the product of the slopes of the best responses is negative and the contraction mapping is smaller to 1. A necessary condition for the slopes of the best responses at equilibrium to be of equal sign is that

$$\theta_H a_1 - a_2 > 0 > \theta_L a_1 - a_2. \tag{27}$$

We now demonstrate that at equilibrium it is impossible for both best responses to be negatively sloped. Denote $\Psi_K = \frac{\theta_K(\theta_K a_1 - a_2)}{(\theta_K a_1 + a_2)^3}$, $K = \{H, L\}$. Since at equilibrium a_1 and a_2 cannot be negative, when condition (27) holds we have $\Psi_H > 0$ and $\Psi_L < 0$. Accordingly, the sign of the slope of $R_1^f(a_2)$ is given by the sign of $\tilde{\mu}\Psi_K + (1-\tilde{\mu})\Psi_L$, and the sign of the slope of $R_2^f(a_1)$ is given by the sign of $\mu\Psi_K + (1-\mu)\Psi_L$. Assume then that the sign of the slope of $R_2^f(a_1)$ is negative. For this to be the case when $\Psi_H > 0$ and $\Psi_L < 0$ we must have $-\frac{\Psi_H}{\Psi_L} > \frac{1-\mu}{\mu}$. Thence, for $R_1^f(a_2)$ to also be negative when $\Psi_H > 0$ and $\Psi_L < 0$, we need that $-\frac{\Psi_H}{\Psi_L} < \frac{1-\tilde{\mu}}{\tilde{\mu}}$. For both these inequalities to hold, we need that $\frac{1-\tilde{\mu}}{\tilde{\mu}} > \frac{1-\mu}{\mu}$, which contradicts $\tilde{\mu} > \mu$. Hence, at equilibrium it is impossible for both best responses to be negatively sloped.

We are left with the case where at equilibrium both best responses are positively sloped. To next show that the contraction mapping is smaller to 1, this is equivalent to inquiring whether the following inequality is true:

$$\left(\frac{\tilde{\mu}\theta_H(\theta_H a_1 - a_2)}{(\theta_H a_1 + a_2)^3} + \frac{(1-\tilde{\mu})\theta_L(\theta_L a_1 - a_2)}{(\theta_L a_1 + a_2)^3}\right)\left(\frac{\mu\theta_H(\theta_H a_1 - a_2)}{(\theta_H a_1 + a_2)^3} + \frac{(1-\mu)\theta_L(\theta_L a_1 - a_2)}{(\theta_L a_1 + a_2)^3}\right)$$
$$+4a_1a_2\left(\frac{\tilde{\mu}\theta_H^2}{(\theta_H a_1 + a_2)^3} + \frac{(1-\tilde{\mu})\theta_L^2}{(\theta_L a_1 + a_2)^3}\right)\left(\frac{\mu\theta_H}{(\theta_H a_1 + a_2)^3} + \frac{(1-\mu)\theta_L}{(\theta_L a_1 + a_2)^3}\right) > 0.$$

We then drop from the above expression the following two positive terms:

$$\left(\frac{\tilde{\mu}\,\theta_H(\theta_H a_1 - a_2)}{(\theta_H a_1 + a_2)^3} + \frac{(1-\tilde{\mu})\,\theta_L(\theta_L a_1 - a_2)}{(\theta_L a_1 + a_2)^3}\right)\frac{\mu\,\theta_H(\theta_H a_1 - a_2)}{(\theta_H a_1 + a_2)^3}$$
$$+\ 4a_1a_2\left(\frac{\tilde{\mu}\,\theta_H^2}{(\theta_H a_1 + a_2)^3} + \frac{(1-\tilde{\mu})\,\theta_L^2}{(\theta_L a_1 + a_2)^3}\right)\frac{\mu\,\theta_H}{(\theta_H a_1 + a_2)^3}.$$

The original expression is then necessarily true if:

$$\left(\frac{\tilde{\mu}\theta_H(\theta_H a_1 - a_2)}{(\theta_H a_1 + a_2)^3} + \frac{(1-\tilde{\mu})\theta_L(\theta_L a_1 - a_2)}{(\theta_L a_1 + a_2)^3}\right)(\theta_L a_1 - a_2) + 4a_1 a_2 \left(\frac{\tilde{\mu}\theta_H^2}{(\theta_H a_1 + a_2)^3} + \frac{(1-\tilde{\mu})\theta_L^2}{(\theta_L a_1 + a_2)^3}\right) > 0.$$

or,

$$\frac{\tilde{\mu}\theta_H}{(\theta_H a_1 + a_2)^3}\left((\theta_H a_1 - a_2)(\theta_L a_1 - a_2) + 4a_1 a_2 \theta_H\right) + \frac{(1-\tilde{\mu})\theta_L}{(\theta_L a_1 + a_2)^3}\left((\theta_L a_1 - a_2)^2 + 4a_1 a_2 \theta_L\right) > 0.$$

It is then sufficient to prove that the first expression is true, or:

$$\theta_L \theta_H a_1^2 + a_2^2 - \theta_L a_1 a_2 + 3a_1 a_2 \theta_H > 0,$$

and this expression is always true since $\theta_H > \theta_L$.

Proof of Proposition 3: Let $\phi_K = \frac{\theta_K}{(\theta_K a_1 + a_2)^2}$, $K = \{L, H\}$. The first-order conditions become

$$[\tilde{\mu}\phi_H + (1-\tilde{\mu})\phi_L]\, a_2 v = c,$$

and

$$[\mu\phi_H + (1-\mu)\phi_L]\, a_1 v = c.$$

Let us study the sign of

$$\begin{aligned}
\Xi &= [\tilde{\mu}\phi_H + (1-\tilde{\mu})\phi_L] - [\mu\phi_H + (1-\mu)\phi_L] \\
&= \tilde{\mu}(\phi_H - \phi_L) - \mu(\phi_H - \phi_L) \\
&= (\tilde{\mu} - \mu)(\phi_H - \phi_L) \\
&= (\tilde{\mu} - \mu)(\theta_H - \theta_L)\frac{a_2^2 - \theta_L\theta_H a_1^2}{(\theta_H a_1 + a_2)^2(\theta_L a_1 + a_2)^2}
\end{aligned}$$

If we know the sign of Ξ, then we know which player exerts higher effort as $\Xi = 0$ implies $a_1^f = a_2^f$, $\Xi > 0$ implies $a_1^f > a_2^f$, and $\Xi < 0$ implies $a_1^f < a_2^f$. Note that $\tilde{\pi} > \pi$ implies $\tilde{\mu} - \mu > 0$. Hence, to determine the sign of Ξ we only need to consider the sign of the numerator of the third term. To do that we consider three possible situations: (i) $\theta_L\theta_H \in \{0, 1\}$; (ii) $\theta_L\theta_H \in (0, 1)$; and (iii) $\theta_L\theta_H > 1$.

Consider case (i): $\theta_L\theta_H \in \{0, 1\}$. When $\theta_L\theta_H = 0$, then we must have $\theta_L = 0$. Substituting $\theta_L = 0$ in the expressions for the posterior beliefs we have $\tilde{\mu} = \mu = 1$, and hence $\Xi = 0$ and $a_1^f = a_2^f = v\theta_H / c(\theta_H + 1)^2$. When $\theta_L\theta_H = 1$, the sign of Ξ is given by

$a_2^2 - a_1^2$. If $a_2 = a_1$, then $\Xi = 0$ and both first-order conditions are satisfied. If $a_2 > a_1$, then $\Xi > 0$, and the first-order conditions are violated. If $a_2 < a_1$, then $\Xi < 0$, and the first-order conditions are violated. Hence, when $\theta_L \theta_H = 1$ the equilibrium satisfies $a_1^f = a_2^f = v\theta_H / c(\theta_H + 1)^2$.

Consider case (ii): $\theta_L \theta_H \in (0, 1)$. The sign of Ξ is given by $a_2^2 - \theta_L \theta_H a_1^2$. If $a_2 = a_1$, then $\Xi > 0$, and the first-order conditions are violated. If $a_2 > a_1$, then $\Xi > 0$, and the first-order conditions are violated. For $a_2 < a_1$ there are three subcases: (a) if $\theta_L \theta_H a_1^2 < a_2^2 < a_1^2$, then $\Xi > 0$, and there will exist values of a_1 and a_2 that satisfy the first-order conditions; (b) if $a_2^2 = \theta_L \theta_H a_1^2$, then $\Xi = 0$, and the first-order conditions are violated; (c) if $a_2^2 < \theta_L \theta_H a_1^2$, then $\Xi < 0$, and the first-order conditions are violated. Hence, when $\theta_L \theta_H \in (0, 1)$ the equilibrium satisfies $a_1^f > a_2^f$.

Consider case (iii): $\theta_L \theta_H > 1$. The sign of Ξ is given by $a_2^2 - \theta_L \theta_H a_1^2$. If $a_2 = a_1$, then $\Xi < 0$, and the first-order conditions are violated. If $a_2 < a_1$, then $\Xi < 0$, and the first-order conditions are violated. For $a_2 > a_1$ there are three subcases: (a) if $a_2^2 < \theta_L \theta_H a_1^2$, then $\Xi < 0$, and there will exist values of a_1 and a_2 that satisfy the first-order conditions; (b) if $a_2^2 = \theta_L \theta_H a_1^2$, then $\Xi = 0$, and the first-order conditions are violated; (c) if $a_2^2 > \theta_L \theta_H a_1^2$, then $\Xi > 0$, and the first-order conditions are violated. Hence, when $\theta_L \theta_H > 1$ the equilibrium satisfies $a_2^f > a_1^f$.

Proof of Proposition 4: Observe first that the bias b^f does not affect the best response of player 2 as implicitly defined in (19). Second, we can show by implicit differentiation that the best response function of player 1, as implicitly defined in (18), is shifting outwards with the bias b^f if and only if the sign of the following expression is positive:

$$\frac{\theta_H}{(\theta_H a_1^f + a_2^f)^2} - \frac{\theta_L}{(\theta_L a_1^f + a_2^f)^2}.$$

In the proof of Proposition 2 we demonstrate that the sign of this expression is positive if and only if $\theta_L \theta_H \in (0, 1)$. Consequently, if $\theta_L \theta_H \in (0, 1)$, then an increase in b^f shift outwards the best response of player 1, thence resulting in a higher equilibrium effort a_1^f, and, given the quasi-concavity of player 2's best response function, the true as well as the perceived equilibrium winning probability of player 1 increases. If $\theta_L \theta_H > 1$, the equilibrium effort of player 1 as well as his true winning probability will therefore decrease with the bias b^f. Yet, the effect on player 1's perceived equilibrium probability is undetermined in this case since, even though the bias raises the perceived equilibrium winning probability, the change in players' efforts pushes the perceived equilibrium winning probability in the other direction.

Proof of Lemma 3: The sign of $\partial R_1^f(a_2)/\partial\tilde{\mu}$ is given by the sign of $\partial^2\tilde{E}[U_1^f(a_1,a_2)]/\partial a_1\partial\tilde{\mu}$, which is given by:

$$\begin{aligned}\text{sgn}\{\theta_H(\theta_L a_1+a_2)^2-\theta_L(\theta_H a_1+a_2)^2\} &= \text{sgn}\{(\theta_H\theta_L^2-\theta_L\theta_H^2)a_1^2+(\theta_H-\theta_L)a_2^2\}\\ &= \text{sgn}\{(\theta_H-\theta_L)(a_2^2-\theta_L\theta_H a_1^2)\}.\end{aligned}$$

Hence, if $a_2 < a_1\sqrt{\theta_L\theta_H}$, then $\partial R_1^f(a_2)/\partial\tilde{\mu} < 0$. However, if $a_2 > a_1\sqrt{\theta_L\theta_H}$, then $\partial R_1^f(a_2)/\partial\tilde{\mu} > 0$.

Proof of Proposition 5: Making use of the Envelope theorem, we deduce that the effect of the newcomer's posterior belief $\tilde{\mu}$ on his equilibrium perceived expected utility is given by:

$$\frac{d\tilde{E}[U_1^f(a_1^f,a_2^f;\tilde{\mu})]}{d\tilde{\mu}}=\frac{\partial\tilde{E}[U_1^f(a_1^f,a_2^f;\tilde{\mu})]}{\partial\tilde{\mu}}+\frac{\partial\tilde{E}[U_1^f(a_1^f,a_2^f;\tilde{\mu})]}{\partial a_2}\frac{da_2^f}{d\tilde{\mu}}$$
$$=\underbrace{\left[\frac{\theta_H a_1^f}{\theta_H a_1^f+a_2^f}-\frac{\theta_L a_1^f}{\theta_L a_1^f+a_2^f}\right]}_{>0}v-\underbrace{\left[\tilde{\mu}\frac{\theta_H a_1^f}{(\theta_H a_1^f+a_2^f)^2}+(1-\tilde{\mu})\frac{\theta_L a_1^f}{(\theta_L a_1^f+a_2^f)^2}\right]}_{>0}v\underbrace{\frac{da_2^f}{d\tilde{\mu}}}_{?} \quad (28)$$

Case (i): $\theta_L\theta_H=1$. From Proposition 3 part (i), we know that if $\theta_L\theta_H=1$, then $a_1^f=a_2^f$. Substituting for $\theta_L=1/\theta_H$ and $a^f=a_1^f=a_2^f$ in the first-order condition (18), we obtain:

$$a^f=\frac{\theta_H v}{c(1+\theta_H)^2}.$$

We thus obtain that the players' equilibrium efforts are independent of their beliefs, which necessarily implies that $d\tilde{E}[U_1^f(a_1^f,a_2^f;\tilde{\mu})]/d\tilde{\mu}>0$.
Case (ii): $\theta_L\theta_H\in(0,1)$. Observe first that from Proposition 3 part (ii) we know that $\theta_L\theta_H\in(0,1)$ implies $a_2^f > a_1^f\sqrt{\theta_L\theta_H}$, which in turn, thanks to Lemma 3 implies $\partial R_1^f(a_2^f)/\partial\tilde{\mu}>0$.

Assume that $\tilde{\mu}=\mu$, which implies that $a_1^f=a_2^f$. Using Lemma 2, the sign of the slope of the best response of player 2 at equilibrium, $\partial R_2^f(a_1^f)/\partial a_1$, is given by:

$$\begin{aligned}\text{sgn}\left\{\frac{R_2^f(a_1^f)}{\partial a_1}\right\} &= -\,\text{sgn}\left\{\mu\frac{\theta_H(\theta_H a_1^f-a_2^f)}{(\theta_H a_1^f+a_2^f)^3}+(1-\mu)\frac{\theta_L(\theta_L a_1^f-a_2^f)}{(\theta_L a_1^f+a_2^f)^3}\right\}\\ &= -\,\text{sgn}\left\{\mu\frac{\theta_H(\theta_H-1)}{(\theta_H+1)^3}+(1-\mu)\frac{\theta_L(\theta_L-1)}{(\theta_L+1)^3}\right\}.\end{aligned}$$

Observe that the above expression is decreasing in μ since $\theta_H > 1 > \theta_L$. Moreover for $\mu = 0$, the expression is positive, and for $\mu = 1$ it is negative. We can then deduce that there exists a unique $\mu = \bar{\mu}$, such that for $\mu \lesseqgtr \bar{\mu}$, $\partial R_2^f(a_1^f)/\partial a_1 \gtreqless 0$. This $\bar{\mu}$ is defined as:

$$\bar{\mu} = \frac{\frac{\theta_L(1-\theta_L)}{(\theta_L+1)^3}}{\frac{\theta_L(1-\theta_L)}{(\theta_L+1)^3} + \frac{\theta_H(\theta_H-1)}{(\theta_H+1)^3}} = \frac{\theta_L(1-\theta_L)(\theta_H+1)^3}{\theta_L(1-\theta_L)(\theta_H+1)^3 + \theta_H(\theta_H-1)(\theta_L+1)^3}.$$

It follows that for $\mu = \bar{\mu}$, $d\tilde{E}[U_1^f(a_1^f, a_2^f; \tilde{\mu})]/d\tilde{\mu} > 0$.

Consider next any $\tilde{\mu} > \mu$. We know from above, that the slope of the best response of player 2 at the 45^o line is positive for $\mu < \bar{\mu}$, nil for $\mu = \bar{\mu}$, and negative for $\mu > \bar{\mu}$. Moreover, we know from Lemma 1 that the best response functions are quasi-concave. In addition, the best response of player 2 is independent of $\tilde{\mu}$. We can therefore deduce that for any $\tilde{\mu} > \mu$, and provided $\mu > \bar{\mu}$, the best response of player 1 intersects the best response of player 2 below the 45^o line where $a_1^f > a_2^f$ and where the best response of player 2 has a negative slope. Hence, for $\mu > \bar{\mu}$, $da_2^f/d\tilde{\mu} < 0$, and $d\tilde{E}[U_1^f(a_1^f, a_2^f; \tilde{\mu})]/d\tilde{\mu} > 0$.

For $\mu < \bar{\mu}$, the slope of the best response of player 2 is positive on the 45^o line, in which case an increase in $\tilde{\mu}$ leads to an increase in a_2^f when the slope of the best response of player 2 is positive at equilibrium. Consequently, we are unable to determine the effect of $\tilde{\mu}$ on the equilibrium perceived expected utility of player 1 in such instances.

Case (iii): $\theta_L\theta_H > 1$. Observe first that from Proposition 3 part (iii) we know that $\theta_L\theta_H > 1$ implies $a_2^f < a_1^f\sqrt{\theta_L\theta_H}$ which in turn, thanks to Lemma 3 implies $\partial R_1^f(a_2^f)/\partial\tilde{\mu} < 0$. If we then consider $\mu = \bar{\mu}$ as above, it follows that $E[U_1^f(a_1^f, a_2^f; \tilde{\mu})]/d\tilde{\mu} > 0$. For $\mu < \bar{\mu}$, the slope of the best response of player 2 is positive above the 45^o line, and any increase in $\tilde{\mu}$ will then result in reductions of a_2^f. Hence, for $\mu < \bar{\mu}$, $d\tilde{E}[U_1^f(a_1^f, a_2^f; \tilde{\mu})]/d\tilde{\mu} > 0$. Last for $\mu > \bar{\mu}$, for similar reasons to the ones in Case (ii), the effect of $\tilde{\mu}$ on the equilibrium expected utility of player 1 us undetermined.

Proof of Proposition 6: Observe that the perceived expected utility of the newcomer in the final is a function of $\tilde{\mu}$, and that it depends on μ only through its effect on the players' equilibrium efforts. By the Envelope theorem, we therefore deduce that $d\tilde{E}[U_1^f(a_1^f, a_2^f)]/d\mu \gtreqless 0$ if $da_2^f/d\mu \lesseqgtr 0$.

From the first-order condition (18) we know that $R_1^f(a_2)$ is independent of μ, for a fixed $\tilde{\mu}$. Therefore $da_2^f/d\mu \lesseqgtr 0 \Leftrightarrow \partial R_2^f(a_1)/\partial\mu \lesseqgtr 0$. Using the first-order condition of player 2 as given by (19), we deduce that the sign of $\partial R_2^f(a_1)/\partial\mu$ is given by:

$$\text{sgn}\left\{\frac{\partial R_2^f(a_1)}{\partial \mu}\right\} = \frac{\theta_H}{(\theta_H a_1 + a_2)^2} - \frac{\theta_L}{(\theta_L a_1 + a_2)^2},$$

which is given by:

$$\begin{aligned}\text{sgn}\{\theta_H(\theta_L a_1 + a_2)^2 - \theta_L(\theta_H a_1 + a_2)^2\} &= \text{sgn}\{(\theta_H \theta_L^2 - \theta_L \theta_H^2)a_1^2 + (\theta_H - \theta_L)a_2^2\} \\ &= \text{sgn}\{(\theta_H - \theta_L)(a_2^2 - \theta_L \theta_H a_1^2)\}.\end{aligned}$$

Hence, if $a_2 < a_1\sqrt{\theta_L\theta_H}$, then $\partial R_2^f(a_1)/\partial\mu < 0$. However, if $a_2 > a_1\sqrt{\theta_L\theta_H}$, then $\partial R_2^f(a_1)/\partial\mu > 0$. We can then consider the three cases (i)-(iii).

Case (i): If $\theta_L\theta_H = 1$, $a_1^f = a_2^f$, and therefore $\partial R_2^f(a_1)/\partial\mu = 0$.

Case (ii): $\theta_L\theta_H \in (0,1)$. Observe first that from Proposition 3 part (ii) we know that $\theta_L\theta_H \in (0,1)$ implies $a_2^f > a_1^f\sqrt{\theta_L\theta_H}$, which in turn implies $\partial R_2^f(a_1^f)/\partial\mu > 0$.

Case (iii): $\theta_L\theta_H > 1$. Observe that from Proposition 3 part (iii) we know that $\theta_L\theta_H > 1$ implies $a_2^f < a_1^f\sqrt{\theta_L\theta_H}$ which in turn implies $\partial R_2^f(a_1^f)/\partial\mu < 0$.

Second-order conditions of the semifinal with an overconfident newcomer when $\theta_L\theta_H = 1$: The first derivative of the newcomer's perceived expected utility in the semifinal is:

$$\frac{\partial \tilde{E}[U_1^s]}{\partial a_1} = \frac{\partial \tilde{P}_1^s}{\partial a_1}\left[\left(\frac{1}{\theta_H + 1}\right)^2 + \frac{\theta_H - 1}{\theta_H + 1}\tilde{\mu}\right] v + \tilde{P}_1^s(a_1, a_3)\frac{\theta_H - 1}{\theta_H + 1} v \frac{\partial \tilde{\mu}}{\partial a_1} - c.$$

The second derivative is therefore given by:

$$\frac{\partial^2 \tilde{E}[U_1^s]}{\partial a_1^2} = \frac{\partial^2 \tilde{P}_1^s}{\partial a_1^2}\left[\left(\frac{1}{\theta_H + 1}\right)^2 + \frac{\theta_H - 1}{\theta_H + 1}\tilde{\mu}\right] v + 2\frac{\partial \tilde{P}_1^s}{\partial a_1}\frac{\theta_H - 1}{\theta_H + 1} v \frac{\partial \tilde{\mu}}{\partial a_1} + \tilde{P}_1^s \frac{\theta_H - 1}{\theta_H + 1} v \frac{\partial^2 \tilde{\mu}}{\partial a_1^2}.$$

Since $\frac{\partial^2 \tilde{P}_1^s}{\partial a_1^2} < 0$, this expression is a fortiori negative if:

$$\left[\frac{\partial^2 \tilde{P}_1^s}{\partial a_1^2}\tilde{\mu} + 2\frac{\partial \tilde{P}_1^s}{\partial a_1}\frac{\partial \tilde{\mu}}{\partial a_1} + \tilde{P}_1^s \frac{\partial^2 \tilde{\mu}}{\partial a_1^2}\right] \underbrace{\frac{\theta_H - 1}{\theta_H + 1} v}_{>0} < 0.$$

We thus need to show that the term inside squared brackets is negative. Substituting for the appropriate terms, and simplifying, we can show that the term inside squared brackets is given by:

$$-\frac{2\tilde{\pi}\theta_H^2 a_3}{(\theta_H a_1 + a_3)^3} < 0.$$

Hence, the second-order condition for the newcomer is satisfied. It is immediate to verify that the second-order condition for the incumbent is also satisfied.

Proof of Lemma 4: Using the definitions of $\tilde{\mu}$ and μ as given in (16) and (17), respectively, we have:

$$\begin{aligned}\frac{\partial b^f}{\partial a_1} &= \frac{\partial(\tilde{\mu}-\mu)}{\partial a_1}\\ &= \left[\frac{\pi(1-\pi)}{\left[\theta_L\theta_H a_1+(\theta_L+\pi(\theta_H-\theta_L))\,a_3\right]^2}-\frac{\tilde{\pi}(1-\tilde{\pi})}{\left[\theta_L\theta_H a_1+(\theta_L+\tilde{\pi}(\theta_H-\theta_L))\,a_3\right]^2}\right]\theta_L\theta_H(\theta_H-\theta_L)a_3.\end{aligned}$$

Observe that the expression inside square brackets can be seen as a difference of the same function evaluated at two different arguments, π and $\tilde{\pi}$. Denote this function by $h(\gamma)$, i.e.

$$h(\gamma)=\frac{\gamma(1-\gamma)}{[\theta_L\theta_H a_1+(\theta_L+\gamma(\theta_H-\theta_L))a_3]^2}$$

Note that $h(\gamma)=0$ for $\gamma=\{0,1\}$. We then have:

$$h'(\gamma)=\frac{\theta_L(\theta_H a_1+a_3)-(2a_1\theta_H\theta_L+a_3(\theta_L+\theta_H)\gamma)}{[\theta_L\theta_H a_1+(\theta_L+\gamma(\theta_H-\theta_L))a_3]^3}.$$

We thus have that $h'(\gamma)=0$ for:

$$\gamma=\bar{\gamma}(a_1,a_3)=\frac{\theta_L(\theta_H a_1+a_3)}{2a_1\theta_L\theta_H+a_3(\theta_L+\theta_H)}.$$

Next one can easily show that $\bar{\gamma}(a_1,a_3)\in(0,1)$.

Moreover, observe that function $h(\gamma)$ is quasi-concave on $\gamma\in[0,1]$, since it is immediate to see that $h''(\gamma)<0$ when evaluated at $\gamma=\bar{\gamma}(a_1,a_3)$. Summarizing, $h(\gamma)$ is a function that starts in 0 is monotonically increasing in γ up to $\bar{\gamma}$ and then monotonically decreases in γ until is reaches 0 for $\gamma=1$.

We can then deduce that if $\pi\geq\bar{\gamma}$, then $h(\pi)>h(\tilde{\pi})$, and hence for any $\pi>\bar{\gamma}(a_1,a_3)$, $\partial b^f/\partial a_1>0$. Furthermore, for any $0<\pi<\bar{\gamma}(a_1,a_3)$, there exists a bias $\bar{b}^s$ such that $\forall b^s>\bar{b}^s$, $h(\pi)>h(\tilde{\pi})$ so that $\partial b^f/\partial a_1>0$. Likewise, for $\forall b^s<\bar{b}^s$, $h(\pi)<h(\tilde{\pi})$ so that $\partial b^f/\partial a_1<0$.

Proof of Proposition 7: Computing each term separately, we have:

$$\frac{\partial\tilde{P}_1^s}{\partial a_1}=\frac{\partial\tilde{P}_1^s(a_1,a_3;\tilde{\pi})}{\partial a_1}=\left[\frac{\theta_L a_3}{(\theta_L a_1+a_3)^2}+\tilde{\pi}\left(\frac{\theta_H a_3}{(\theta_H a_1+a_3)^2}-\frac{\theta_L a_3}{(\theta_L a_1+a_3)^2}\right)\right]$$

Imposing $\theta_L = 1/\theta_H$, this reads as:

$$\frac{\partial \tilde{P}_1^s}{\partial a_1} = \left[\frac{\theta_H a_3}{(a_1+\theta_H a_3)^2} + \tilde{\pi}\left(\frac{\theta_H a_3}{(\theta_H a_1 + a_3)^2} - \frac{\theta_H a_3}{(a_1+\theta_H a_3)^2}\right)\right]$$

Evaluating this expression at $a_1 = a_3$, this expression becomes:

$$\frac{\partial \tilde{P}_1^s}{\partial a_1} = \frac{\theta_H}{a(\theta_H+1)^2}.$$

Turning next to the squared-bracketed term of expression (23), since a_2^f is independent of posterior beliefs, the first multiplicative term is then nil. Focusing next on the last term of expression (23), we have that when evaluated at $a_1 = a_3$, then:

$$\begin{aligned}
\frac{\partial \tilde{E}\left[U_1^f\right]}{\partial \tilde{\mu}} \frac{\partial \tilde{\mu}}{\partial a_1} &= -\left(\frac{\theta_H}{\theta_H+1} - \frac{\theta_L}{\theta_L+1}\right) \frac{\tilde{\pi}(1-\tilde{\pi})\theta_L\theta_H(\theta_H-\theta_L)}{\left[\theta_L\theta_H + \tilde{\pi}\theta_H + (1-\tilde{\pi})\theta_L\right]^2 a} v \\
&= -\frac{\theta_H-1}{\theta_H+1} \frac{\tilde{\pi}(1-\tilde{\pi})\theta_H(\theta_H^2-1)}{\left[\theta_H + \tilde{\pi}\theta_H^2 + (1-\tilde{\pi})\right]^2 a} v \\
&= -\frac{\tilde{\pi}(1-\tilde{\pi})\theta_H(\theta_H-1)^2}{\left[\theta_H + \tilde{\pi}\theta_H^2 + (1-\tilde{\pi})\right]^2 a} v \\
&= -\frac{\tilde{\pi}(1-\tilde{\pi})\theta_H(\theta_H-1)^2}{(\theta_H+1)^2(1-\tilde{\pi}+\tilde{\pi}\theta_H)^2 a} v
\end{aligned}$$

The first-order condition of player 1 when evaluated at $a_1 = a_3 = a$ is then equal to:

$$\frac{\theta_H}{a(\theta_H+1)^2} \frac{1-\tilde{\pi}+\tilde{\pi}\,\theta_H^3}{(\theta_H+1)^2\ (1-\tilde{\pi}+\tilde{\pi}\,\theta_H)} v - \frac{1-\tilde{\pi}+\tilde{\pi}\theta_H}{\theta_H+1} \frac{\tilde{\pi}(1-\tilde{\pi})\theta_H(\theta_H-1)^2}{(\theta_H+1)^2(1-\tilde{\pi}+\tilde{\pi}\theta_H)^2 a} v = c,$$

or

$$\frac{\theta_H}{a(\theta_H+1)^2} \frac{1-\tilde{\pi}+\tilde{\pi}\,\theta_H^3}{(\theta_H+1)^2\ (1-\tilde{\pi}+\tilde{\pi}\,\theta_H)} v - \frac{\theta_H}{\theta_H+1} \frac{\tilde{\pi}(1-\tilde{\pi})(\theta_H-1)^2}{(\theta_H+1)^2(1-\tilde{\pi}+\tilde{\pi}\theta_H) a} v = c,$$

or

$$\frac{\theta_H\left(1-\tilde{\pi}+\tilde{\pi}\,\theta_H^2\right)}{(\theta_H+1)^4} \frac{v}{a} = c$$

We next explore the first-order condition of player 3 which, when evaluated at $a_1 = a_3$, is given by:

$$\frac{\theta_H}{4(\theta_H+1)^2} \frac{v}{a} = c.$$

Using the first-order derivatives, we can next express $\Psi(\tilde{\pi})$ evaluated at $a_1 = a_3$ as:

$$\begin{aligned}\Psi(\tilde{\pi}) &= \left.\frac{\partial \tilde{E}[U_1^s(a_1,a_3;\tilde{\pi})]}{\partial a_1}\right|_{a_1=a_3} - \left.\frac{\partial E[U_3^s(a_1,a_3)]}{\partial a_3}\right|_{a_1=a_3} \\ &= \left[\frac{\theta_H\left(1-\tilde{\pi}+\tilde{\pi}\,\theta_H^2\right)}{(\theta_H+1)^4} - \frac{\theta_H}{4(\theta_H+1)^2}\right]\frac{v}{a} \\ &= -\frac{\theta_H(\theta_H-1)\left[\theta_H+3-4\tilde{\pi}(\theta_H+1)\right]}{4(\theta_H+1)^4}\frac{v}{a},\end{aligned}$$

with $\Psi(\tilde{\pi}) \gtreqless 0 \Leftrightarrow a_1^s \gtreqless a_3^s$. It follows that $\Psi(\tilde{\pi}) = 0$ when

$$\breve{\pi} = \frac{\theta_H+3}{4(\theta_H+1)}$$

So, when the prior belief of the overconfident player 1, $\tilde{\pi}$, is higher than $\breve{\pi}$, he exerts more effort than the rational rival in the semifinal. Note that $\theta_H > 1$ implies $\breve{\pi} \in (1/4, 1/2)$ and that the higher is θ_H the closer is $\breve{\pi}$ to 1/4.

Proof of Proposition 8: To prove this result, we first show that $P_1^s P_1^f$ is monotonically increasing in the ratio a_1^s/a_3^s, and we then show that this ratio is monotonically increasing in b^s.

Making use of equations (24) and (25), the newcomer's ex-ante true probability of winning the elimination contest is given by:

$$P_1^s P_1^f = \left[\pi\frac{\theta_H a_1^s}{\theta_H a_1^s + a_3^s} + (1-\pi)\frac{a_1^s/\theta_H}{a_1^s/\theta_H + a_3^s}\right]\frac{1+\mu(\theta_H-1)}{\theta_H+1},$$

which after substituting for μ as given by Equation (17) becomes:

$$P_1^s P_1^f = \left[\pi\frac{\theta_H a_1^s}{\theta_H a_1^s + a_3^s} + (1-\pi)\frac{a_1^s/\theta_H}{a_1^s/\theta_H + a_3^s}\right]\frac{1+\frac{\pi\theta_H(a_1^s/\theta_H + a_3^s)}{a_1^s + (\pi\theta_H + (1-\pi)/\theta_H)a_3^s}(\theta_H-1)}{\theta_H+1}.$$

Rewritting the above expression as a function of $x = a_1^s/a_3^s$, we obtain:

$$\begin{aligned}P_1^s P_1^f &= \left[\frac{\pi\theta_H}{\theta_H x+1} + \frac{(1-\pi)}{x+\theta_H}\right]\left[1+\frac{\pi\theta_H(x/\theta_H+1)(\theta_H-1)}{x+(\pi\theta_H+(1-\pi)/\theta_H)}\right]\frac{x}{\theta_H+1} \\ &= \frac{\theta_H x + \pi\theta_H^2 + (1-\pi)}{(\theta_H x+1)(x+\theta_H)}\left[\frac{\theta_H x + \pi\theta_H^2 + (1-\pi) + \pi\theta_H(x+\theta_H)(\theta_H-1)}{\theta_H x + \pi\theta_H^2 + (1-\pi)}\right]\frac{x}{\theta_H+1} \\ &= \frac{(1-\pi)(\theta_H x+1) + \pi\theta_H^2(x+\theta_H)}{(\theta_H x+1)(x+\theta_H)}\frac{x}{\theta_H+1} \\ &= \left[(1-\pi)\frac{x}{x+\theta_H} + \pi\theta_H\frac{\theta_H x}{\theta_H x+1}\right]\frac{1}{\theta_H+1}.\end{aligned}$$

Since the two terms inside the squared brackets are increasing in x, we deduce that

$P_1^s P_1^f$ is equally increasing in x.

We next prove that $dx/db^s > 0$. At optimality, the first order conditions of both players 1 and 3 ought to be simultaneously satisfied. The first order condition of player 1 is given by:

$$\left[\tilde{\pi}\frac{\theta_H a_3^s}{(\theta_H a_1^s + a_3^s)^2} + (1-\tilde{\pi})\frac{a_3^s/\theta_H}{(a_1^s/\theta_H + a_3^s)^2}\right]\left[\left(\frac{1}{\theta_H+1}\right)^2 + \frac{\theta_H-1}{\theta_H+1}\frac{\tilde{\pi}\theta_H(a_1^s/\theta_H + a_3^s)}{a_1^s + (\tilde{\pi}\theta_H + (1-\tilde{\pi})/\theta_H)a_3^s}\right]v$$
$$-\left[\tilde{\pi}\frac{\theta_H a_1^s}{\theta_H a_1^s + a_3^s} + (1-\tilde{\pi})\frac{a_1^s/\theta_H}{a_1^s/\theta_H + a_3^s}\right]\frac{\theta_H-1}{\theta_H+1}\frac{\tilde{\pi}(1-\tilde{\pi})(\theta_H - 1/\theta_H)a_3^s}{\left[a_1^s + (\tilde{\pi}\theta_H + (1-\tilde{\pi})/\theta_H)a_3^s\right]^2}v = c$$

Rewritting this condition as a function of x, we obtain:

$$\left[\tilde{\pi}\frac{\theta_H}{(\theta_H x + 1)^2} + (1-\tilde{\pi})\frac{1/\theta_H}{(x/\theta_H + 1)^2}\right]\left[\left(\frac{1}{\theta_H+1}\right)^2 + \frac{\theta_H-1}{\theta_H+1}\frac{\tilde{\pi}\theta_H(x/\theta_H + 1)}{x + \tilde{\pi}\theta_H + (1-\tilde{\pi})/\theta_H}\right]\frac{v}{a_3^s}$$
$$-\left[\tilde{\pi}\frac{\theta_H x}{\theta_H x + 1} + (1-\tilde{\pi})\frac{x/\theta_H}{x/\theta_H + 1}\right]\frac{\theta_H-1}{\theta_H+1}\frac{\tilde{\pi}(1-\tilde{\pi})(\theta_H - 1/\theta_H)}{\left[x + \tilde{\pi}\theta_H + (1-\tilde{\pi})/\theta_H\right]^2}\frac{v}{a_3^s} = c.$$

Proceeding likewise for player 3, we obtain:

$$\left[\pi\frac{\theta_H a_1^s}{(\theta_H a_1^s + a_3^s)^2} + (1-\pi)\frac{a_1^s/\theta_H}{(a_1^s/\theta_H + a_3^s)^2}\right]\frac{v}{4} = c$$
$$\left[\pi\frac{\theta_H x}{(\theta_H x + 1)^2} + (1-\pi)\frac{x/\theta_H}{(x/\theta_H + 1)^2}\right]\frac{v}{4a_3^s} = c.$$

Combining these two first-order conditions, we obtain:

$$\begin{aligned}\mathcal{A}(x) \quad = \quad & \left[\tilde{\pi}\frac{\theta_H}{(\theta_H x + 1)^2} + (1-\tilde{\pi})\frac{1/\theta_H}{(x/\theta_H + 1)^2}\right]\left[\left(\frac{1}{\theta_H+1}\right)^2 + \frac{\theta_H-1}{\theta_H+1}\frac{\tilde{\pi}\theta_H(x/\theta_H + 1)}{x + \tilde{\pi}\theta_H + (1-\tilde{\pi})/\theta_H}\right] \\ - \quad & \left[\tilde{\pi}\frac{\theta_H x}{\theta_H x + 1} + (1-\tilde{\pi})\frac{x/\theta_H}{x/\theta_H + 1}\right]\frac{\theta_H-1}{\theta_H+1}\frac{\tilde{\pi}(1-\tilde{\pi})(\theta_H - 1/\theta_H)}{\left[x + \tilde{\pi}\theta_H + (1-\tilde{\pi})/\theta_H\right]^2} \\ - \quad & \left[\pi\frac{\theta_H x}{(\theta_H x + 1)^2} + (1-\pi)\frac{x/\theta_H}{(x/\theta_H + 1)^2}\right]\frac{1}{4} = 0.\end{aligned}$$

This expression can be re-written as:

$$\mathcal{A}(x) = -\frac{\theta_H}{4(\theta_H+1)^2(\theta_H x^2 + \theta_H^2 x + x + \theta_H)^2}\left[c_1 x^3 + c_2 x^2 + c_3 x + c_4\right] = 0,$$

where:

$$\begin{aligned}
c_1 &= \pi + 2\pi\theta_H + \theta_H^2 + 2(1-\pi)\theta_H^3 + (1-\pi)\theta_H^4 \\
c_2 &= 2\theta_H(\theta_H^2 + 1) \\
c_3 &= 8\tilde{\pi}\theta_H - 6\theta_H - 2\pi\theta_H - \pi + \theta_H^2 + 2\pi\theta_H^3 + \pi\theta_H^2 - 8\tilde{\pi}\theta_H^3 + 1 \\
c_4 &= -4(1 - \pi + \pi\theta_H^4)
\end{aligned}$$

Observe that the fraction in $\mathcal{A}(x)$ is always positive, thence implying that the first-order conditions are satisfied if $\mathcal{B}(x) = c_1x^3 + c_2x^2 + c_3x + c_4 = 0$, thence implying that this condition defines the equilibrium value of x. We are interested in the sign of dx/db^s which is the same to the sign of $dx/d\tilde{\pi}$. Applying implicit differentiation to $\mathcal{B}(x)$, we have:

$$\frac{dx}{d\tilde{\pi}} = -\frac{\frac{\partial \mathcal{B}(x)}{\partial \tilde{\pi}}}{\frac{\partial \mathcal{B}(x)}{\partial x}} = -\frac{4(2\theta_H x(1-\theta_H^2) + 1 - \theta_H^4)}{3c_1x^2 + 2c_2x + c_3}.$$

The sign of the numerator is negative, thence implying that the sign of the entire expression is given by the sign of the denominator. Exploiting the fact that $\mathcal{B}(x) = 0$ implies:

$$c_3 = -\frac{c_4 + c_1x^3 + c_2x^2}{x}.$$

Substituting for c_3 in the denominator of the above expression, we deduce that $dx/d\tilde{\pi} > 0$ if:

$$3c_1x^2 + 2c_2x - \frac{c_4}{x} - c_1x^2 - c_2x > 0.$$

Since $x > 0$, simplifying the above expression and multiplying by x, we deduce that $dx/d\tilde{\pi} > 0$ if:

$$2c_1x^3 + c_2x^2 - c_4 > 0,$$

which is necessarily true since $c_1 > 0$, $c_2 > 0$, and $c_4 < 0$.