PHILIPPE BACCHETTA
ERIC VAN WINCOOP

Higher Order Expectations in Asset Pricing

We examine formally Keynes' idea that higher order beliefs can drive a wedge between an asset price and its fundamental value based on expected future payoffs. We call this the higher order wedge, which depends on the difference between higher and first order expectations of future payoffs. We analyze the determinants of this wedge and its impact on the equilibrium price in the context of a dynamic noisy rational expectations model. We show that the wedge reduces asset price volatility and disconnects the price from the present value of future payoffs. The impact of the higher order wedge on the equilibrium price can be quantitatively large.

JEL codes: G0, G1, D8
Keywords: higher order beliefs, beauty contest, asset pricing.

In his General Theory, Keynes devotes significant attention to factors that can drive a wedge between an asset price and its fundamental value based on expected future payoffs (Keynes 1936, ch. 12, sect. 5). He emphasizes in particular two factors: mass psychology and higher order opinions. Although market psychology had largely been neglected for decades, it is now receiving significant attention in the growing field of behavioral finance (see Barberis and Thaler 2003, Hirshleifer 2001 for surveys of the field). On the other hand, the impact of higher order expectations (henceforth HOE) on the equilibrium asset price has received little attention and is not well understood. HOE refer to expectations that investors form of other investors' expectations of an asset's subsequent payoffs. In the words of Keynes,

We are grateful to John Campbell, Bernard Dumas, Hyun Shin, Narayana Kocherlakota, and two anonymous referees for comments on a previous draft. Financial support from the National Science Foundation (grant SES-0649442), the Bankard Fund for Political Economy, and the National Centre of Competence in Research "Financial Valuation and Risk Management" (NCCR FINRISK) is gratefully acknowledged.

Philippe Bacchetta *is Professor of Economics, University of Lausanne, Swiss Finance Institute and CEPR Research Fellow* (*E-mail:* philippe.bacchetta@unil.ch). Eric van Wincoop *is Professor of Economics, University of Virginia, and NBER Research Associate* (*E-mail:* vanwincoop@virginia.edu).

Received December 9, 2005; and accepted in revised form December 31, 2007.

Journal of Money, Credit and Banking, Vol. 40, No. 5 (August 2008)

investors "are concerned, not with what an investment is really worth to a man who buys it for keeps, but with what the market will value it at . . . three months or a year hence."[1]

HOE naturally play an important role in dynamic models with heterogeneous information. While this is well known, the role of HOE in asset pricing has not been formally analyzed until the recent paper by Allen, Morris, and Shin (2006). These authors show that in general the law of iterated expectations does not hold for average expectations, so that HOE differ from first order average expectations of the asset's payoff. Moreover, they explicitly solve an equilibrium asset price as a function of HOE of the asset's payoff in the context of a model with an asset that has a single terminal payoff. They show that the farther we are from the terminal date, the higher the order of expectations. The key implication of HOE emphasized in their paper is that more weight is given to public information as a result of HOE.

In this paper we further explore the role of HOE by analyzing the "higher order wedge," which captures the impact of HOE on the equilibrium price. It is equal to the difference between the equilibrium price and what it would be if HOE were replaced by first order expectations. The latter yields the standard asset pricing formula as the expected present value of future payoffs and risk premia. The higher order wedge therefore depends on the difference between higher order and first order expectations of future payoffs and risk premia. This wedge adds a third asset pricing component to standard models, where the price depends on expected payoffs and discount rates.

The goal of the paper is to analyze the determinants of the higher order wedge, and its impact on the equilibrium price, in the context of a dynamic noisy rational expectations (NRE) model. Information heterogeneity is a necessary condition for HOE to differ from first order expectations. NRE models have been widely used since the late 1970s to model information heterogeneity and therefore provide a natural framework for analyzing the role of HOE. Since unobserved asset supply or demand shocks introduce noise that prevent private information from being revealed through the asset price, they assure that agents will have heterogeneous expectations. Allen, Morris, and Shin (2006) also analyze the role of HOE in the context of an NRE model. However, like many NRE models, they consider the case of an asset with only one payoff at a terminal date. A distinctive feature of this paper is to consider a more standard dynamic asset pricing context with an infinitely lived asset.

The higher order wedge can be expressed as the sum of first order and higher order expectations of future market expectational errors about the present value of subsequent asset payoffs. However, we show that this expression can be reduced to first order average expectational errors about the mean set of private signals. We can also show that the higher order wedge depends linearly on expectational errors about

1. In a well-known paragraph he compares asset markets to a beauty contest, where contestants have to pick the faces that other competitors find the most beautiful. Keynes argues that third and higher order expectations matter as well: "We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practice the fourth, fifth, and higher degrees."

future asset payoffs based on errors in public signals. We find that the wedge is largest for intermediate levels of the quality of private signals.

Regarding the impact of the higher order wedge on the equilibrium price, we first show that it reduces asset price volatility. Second, it tends to reduce the impact of future asset payoff innovations on the equilibrium price and amplify the impact of unobserved supply or noise trading shocks on the equilibrium price. Third, it disconnects the equilibrium price from the present value of future asset payoffs. Fourth, we show that the impact of the higher order wedge on the equilibrium price can be quantitatively large.

The finding that the higher order wedge depends on first order expectational errors about the mean set of private signals is a key result from which many other results are derived. Intuitively it can be understood in two steps. First, portfolio holdings of investors will depend on expectational errors that investors expect the market to make next period about future payoffs. This is in line with Keynes' reasoning discussed above. If investors expect that the market will value the asset too high next period, they will buy the asset, pushing up its price. Second, investors expect the market to make expectational errors to the extent that they expect average private signals to differ from their own. When this is systematic there is an average expectational error about the mean set of private signals.

Public information plays a key role. Assume that public signals are overly favorable about future payoffs, so that there would be positive expectational errors about future payoffs based on public information alone. Since public information is more favorable than the average private signal, the majority of investors believe that their own private signal is relatively weak and others have more favorable private signals. In other words, there is an average expectational error about average private signals. When a majority of investors expect others to have more favorable private signals, and these signals are still relevant tomorrow, these investors expect the outlook of the market to be too favorable tomorrow. Investors buy the asset in anticipation of this, pushing up the price. Errors in public signals (in this case too favorable) therefore affect the equilibrium price by changing the expectations that agents have about private signals of others and therefore the expectations of others. This is captured by the higher order wedge.

The remainder of the paper proceeds as follows. In the next section we review the related literature. In Section 2, we develop a simple asset price equation that relates the price to first and HOE of future dividends and risk premia. We show that the equilibrium price is driven by three factors: expected payoffs, current and expected future risk premia, and the higher order wedge. In Section 3, we introduce a specific information structure in the context of a dynamic NRE model and use it to analyze the determinants of the higher order wedge. A specific example of the general information structure is discussed in Section 4, which also provides a numerical illustration of the findings. Section 5 concludes.

1. RELATED LITERATURE

While higher order beliefs have been studied in a wide range of contexts, two features make them of special interest in the context of financial markets. First, in financial markets the price today depends on the price tomorrow, so that investors naturally need to form expectations of future market expectations. This dynamic perspective differs from the analysis of "static" HOE, i.e., expectations of expectations within a period. This is the case when agents interact strategically, e.g., as in Morris and Shin (2002), Woodford (2003), or Amato and Shin (2003).[2] We abstract from strategic interactions by assuming atomistic investors. Second, in financial markets the price provides a mechanism through which idiosyncratic information is aggregated. In forming expectations of other investors' expectations, special attention is paid to the asset price as it is informative about the private information of others. This additional feature is often not present in the analysis of games with incomplete information, e.g., in global games.

HOE play a role in two types of asset pricing models. The first are models with short sales constraints. Harrison and Kreps (1978) first showed that the price of an asset is generally higher than its "fundamental value" when arbitrage is limited by short sales constraints (see also Scheinkman and Xiong 2003, Allen, Morris, and Postlewaite 1993, Biais and Bossaerts 1998). The difference is equal to an option value to resell the asset at a future date to investors with a higher valuation. HOE play a role in this context since the option value depends on the opinions of other investors' expectations at future dates. However, in this literature the price is equal to its fundamental value when the short sales constraints are removed.

The second type of models featuring HOE are dynamic NRE models without short sales constraints. However, these models are usually analyzed without any reference to HOE. This can be done because these models can be solved using a reduced form where HOE are not explicit. This was first shown by Townsend (1983) in the context of a dynamic business cycle model that features dynamic HOE.[3]

Most of this literature considers a special model where an asset has only one payoff at a terminal date (see Brunnermeier 2001 for a nice survey of the literature). Investors receive private information on the final payoff either at an initial date or every period. They trade every period and progressively learn about the final payoff by observing the price. Such a model is studied in particular by He and Wang (1995), Vives (1995), Foster and Viswanathan (1996), Brennan and Cao (1997), and Allen, Morris, and Shin (2006).[4] Among the issues analyzed are trading volume and intensity, market depth

2. Hellwig (2003) characterizes explicitly HOE in the model proposed by Woodford (2003). It is therefore related in spirit to our approach.

3. The general approach is the method of undetermined coefficients. In the context of asset pricing one first assumes some equilibrium asset price as a linear function of current and past innovations. Investors make decisions based on this conjectured price equation. The resulting equilibrium price equation is then equated to the conjectured one in order to solve for the coefficients.

4. Foster and Viswanathan (1996) consider a model with strategic trading, while the other papers consider competitive investors.

and liquidity, the informativeness of prices, and important aspects of the solution procedure. As mentioned in the introduction, only Allen, Morris, and Shin (2006) explicitly analyze the role of HOE in a terminal payoff model.

Although they do not explicitly study the asset pricing implications of HOE, He and Wang (1995) and Foster and Viswanathan (1996) do make some comments on static HOE within their model (the average expectation at time t of the average expectation at time t). While this does not correspond to the dynamic form of HOE that affect the equilibrium asset price, these authors do make the important point that HOE can be reduced to first order expectations. In this paper, we show that this remains true for the relevant dynamic HOE. The higher order wedge depends on the average first order expectational error about the mean set of private signals. It is important to stress that the ability to reduce higher order to first order expectations does not imply that they do not matter. The wedge created by HOE is an additional determinant of the asset price, separate from expected dividends and risk premia, and can be quantitatively very large.[5]

While the terminal payoff model is technically convenient, it is not very realistic and far removed from more standard dynamic asset pricing models. In this paper we will therefore consider a dynamic asset pricing model with an infinitely lived asset. Closely related, in Bacchetta and van Wincoop (2006) we solve an infinite horizon NRE model of exchange rate determination in which HOE arise. Using the results from the present paper, we show that HOE can help contribute to the puzzling disconnect between the exchange rate and observed macroeconomic aggregates. HOE in an infinite horizon framework were first analyzed in macroeconomics, in the business cycle model of Townsend (1983) in which firms have private information about demand from their own customers.

There are also private information models that exhibit some but not all features of NRE models. These models may or may not exhibit HOE. The two key characteristics of NRE models are that (i) agents have private information about future asset payoffs and (ii) this information is not fully revealed through the asset price due to unobserved net asset supply shocks. At least two ways of deviating from these assumptions have been considered in the recent literature.

One possibility that has been considered is where the asset supply is constant but agents have private information about different components of asset payoff innovations today, with each component providing different information about future payoffs. Kasa, Walker, and Whiteman (2007) show that even though there are no unobserved asset supply shocks in this case, the equilibrium price does not necessarily reveal all private information about the components of asset payoff innovations. If, dependent on various assumptions, the price does not reveal private information,

5. He and Wang (1995) and Foster and Viswanathan (1996) argue that the ability to reduce higher order to first order expectations helps solve the model since the infinite space of mean beliefs that Townsend (1983) alluded to is reduced to a space of only first order beliefs. However, the method of undetermined coefficients used by Townsend to solve the model does not make any reference to the space of mean beliefs and the solution methods in these two papers also make no use of the fact that HOE can be reduced to first order expectations.

agents will have heterogeneous expectations in the equilibrium of the model and the model may exhibit HOE.[6]

A second deviation from the NRE framework consists of models where agents do not have private information about future asset payoffs, but they do have private information about asset supply shocks. Walker (2007a) and Singleton (1987) develop such models.[7] Agents have private information about different components of net asset supply innovations today, which provide different information about net asset supply in future periods. In a model of this type developed by Walker (2007a) the price fully reveals private information and the model therefore does not exhibit HOE. Walker (2007a) shows that the same is the case for Singleton (1987).

The advantage of NRE models over these alternatives is that there is always information heterogeneity in equilibrium as the price does not fully reveal the private information. NRE models are therefore a natural framework for thinking about the role of HOE. It is important to emphasize though that information heterogeneity is a necessary but not a sufficient condition for HOE to differ from first order expectations. For example, in NRE models with a hierarchical information structure as in Wang (1993, 1994), there is information heterogeneity in equilibrium but HOE collapse to first order expectations. In the model presented below we will give precise conditions under which HOE differ from first order expectations.

2. A SIMPLE ASSET PRICING EQUATION

2.1 Assumptions and Equilibrium Price

In this section, we derive a simple asset price equation that relates the asset price to HOE of future payoffs. We adopt a share economy that is standard in the NRE literature and allows for an exact solution without using linearization methods. The basic assumptions are: (i) constant absolute risk aversion, (ii) investors invest for one period only (overlapping generations of two-period lived investors), (iii) an excess return that is normally distributed, (iv) a constant risk-free interest rate, (v) a share economy with a stochastic supply of shares, and (vi) a competitive market with a countable infinite set of agents $N = 1, 2, \ldots$ (the set of natural numbers).

These assumptions are commonly made in the NRE literature, but deserve some comments. First, assumption (i) leads to a simple optimal portfolio allocation without

6. If there are only two traders they find that the asset price fully reveals the private information and there are no HOE. If there are two investors with different private signals, the equilibrium price is only affected by these two private signals. Each type of investor can then derive exactly the signal of the other type. The two-type case is also considered in the first model of Townsend (1983) and analyzed by Pearlman and Sargent (2005). HOE only apply to section 8 of Townsend (1983) in which there is an infinity of firms (or markets).

7. Both Kasa, Walker, and Whiteman (2007) and Walker (2007a) use frequency domain methods to solve their models, following Futia (1981).

the need for any approximation. Assumption (ii) significantly simplifies the portfolio choice problem of investors. If agents have longer horizons the optimal portfolio includes a hedge against possible changes in expected returns, which unnecessarily complicates matters. Assumption (iii) is made exogenously for now, but in Section 3 it will be the endogenous outcome of the assumed information structure. The stochastic supply of shares in assumption (v) is important in that it prevents the equilibrium asset price from completely revealing the average of private information.[8] The per capita random supply of shares is X_t and is not observable. The countability of agents in assumption (vi) is frequently adopted as well (e.g., Hellwig 1980, Brown and Jennings 1989, He and Wang 1995). This allows us to assume that the average across agents of independent draws from a random variable is equal to the expected value of the random variable.[9]

Based on these preliminaries, we can examine the investors' decisions and the equilibrium asset price. Investors allocate optimally their wealth between a risky stock and a safe asset. Let P_t be the ex-dividend share price, D_t the dividend, and R the constant gross interest rate. The dollar excess return on one share is $Q_{t+1} = P_{t+1} + D_{t+1} - RP_t$. This leads to the standard asset demand equation

$$x_t^i = \frac{E_t^i(P_{t+1} + D_{t+1}) - RP_t}{\gamma \sigma_{it}^2}, \tag{1}$$

where $E_t^i(.)$ is the expectation of investor i, γ is the rate of absolute risk aversion and σ_{it}^2 is the conditional variance of next period's excess return.

The market equilibrium condition is

$$\int_i x_t^i = X_t. \tag{2}$$

We define the risk premium term as $\phi_t = \gamma \sigma_t^2 X_t / R$, where $\sigma_t^2 = \int_i \sigma_{it}^2$ (we will show that in equilibrium $\sigma_{it}^2 = \sigma_t^2\ \forall i$). Defining $\bar{E}_t(.) = \int_0^1 E_t^i(.)\, di$ as the average or *market* expectation, the market clearing condition gives:[10]

$$P_t = \frac{1}{R} \bar{E}_t(P_{t+1} + D_{t+1}) - \phi_t. \tag{3}$$

To compute the equilibrium price, we need to integrate (3) forward. In typical asset pricing formulas, this is done by applying the law of iterated expectations. While this

8. A typical justification for this assumption is that the *net* supply of shares is random. For example, He and Wang (1995) assume that the total number of shares is constant but changes in demand by exogenous liquidity traders makes the residual supply stochastic. Such exogenous traders are also commonly referred to as noise traders.

9. For a nice discussion of these issues see Vives (2008), who suggests the solution adopted in Feldman and Gilles (1985) and He and Wang (1995) with a countable infinite number of agents. See Judd (1985) and Sun (2006) for further details on the mathematical foundations of the law of large numbers.

10. Notice that, despite heterogeneity, we could express the price in terms of a stochastic discount factor. However, we do not follow this route.

law always holds for individual expectations, it may not hold for *market* expectations when investors have different information sets. For example, $\bar{E}_t \bar{E}_{t+1} D_{t+2} \neq \bar{E}_t D_{t+2}$. Thus, we define the average expectation of order k as

$$\bar{E}_t^k = \bar{E}_t \bar{E}_{t+1} \dots \bar{E}_{t+k-1} \tag{4}$$

for $k > 1$.[11] Moreover, $\bar{E}_t^0 x = x$, $\bar{E}_t^1 x = \bar{E}_t x$. The equilibrium price is then (ruling out bubbles):

$$P_t = \sum_{s=1}^{\infty} \frac{1}{R^s} \bar{E}_t^s D_{t+s} - \sum_{s=1}^{\infty} \frac{1}{R^s} \bar{E}_t^s \phi_{t+s} - \phi_t. \tag{5}$$

The stock price is equal to the present discounted value of expected dividends minus risk premia. The difference with a standard asset pricing equation is that first order expectations are replaced by HOE. A dividend accruing s periods ahead has an expectation of order s. For example, if $s = 2$, we need to compute the market expectation at time t of the market expectation at $t + 1$ of D_{t+2} rather than the first order expectation of D_{t+2}. This implies that investors have to predict the future market expectation of the dividend rather than the dividend itself. This is the “beauty contest” phenomenon described by Keynes. Moreover, with an infinite horizon, the order of expectation can obviously go to infinity.

2.2 The Higher Order Wedge

Standard representative agent asset pricing models imply that asset price fluctuations can be decomposed into two components: changes in expected payoffs and changes in expected discount rates. This decomposition is commonly adopted for example to determine the contribution of each component to overall asset price volatility (see Campbell, Lo, and MacKinlay 1997, ch. 7, for a nice discussion). Indeed, when all agents have the same information, we can simply write (5) as

$$P_t = \sum_{s=1}^{\infty} \frac{1}{R^s} E_t D_{t+s} - \sum_{s=1}^{\infty} \frac{1}{R^s} E_t \phi_{t+s} - \phi_t. \tag{6}$$

The first term on the right hand side captures expected dividends, while the last two terms capture expected future and current risk premia that determine current and future discount rates.

When agents have heterogeneous information there will in general be a third component of asset prices that we will call the higher order wedge. To see this, first define

11. We adopt the same notation as Allen, Morris, and Shin (2006). Notice that the time horizon changes with the order of expectation.

the sum of the two traditional asset pricing components as P_t^*, which is the same as (6) except that the expectations are the average across all agents:

$$P_t^* = \sum_{s=1}^{\infty} \frac{1}{R^s} \bar{E}_t D_{t+s} - \sum_{s=1}^{\infty} \frac{1}{R^s} \bar{E}_t \phi_{t+s} - \phi_t. \tag{7}$$

P_t^* is not an equilibrium asset price in a particular model. Rather, it is simply meant to capture the sum of the two traditional asset pricing components (expected payoffs and discount rates).

We define the *higher order wedge* as the difference between P_t and P_t^*:

$$\begin{aligned}\Delta_t = P_t - P_t^* = \sum_{s=1}^{\infty} \frac{1}{R^s} \left[\bar{E}_t^s D_{t+s} - \bar{E}_t D_{t+s}\right] \\ - \sum_{s=1}^{\infty} \frac{1}{R^s} \left[\bar{E}_t^s \phi_{t+s} - \bar{E}_t \phi_{t+s}\right].\end{aligned} \tag{8}$$

It depends on the present value of deviations between higher order and first order expectations of dividends minus risk premia. The higher order wedge Δ_t therefore adds a third element to the standard asset pricing equation:

$$P_t = \sum_{s=1}^{\infty} \frac{1}{R^s} \bar{E}_t D_{t+s} - \left(\sum_{s=1}^{\infty} \frac{1}{R^s} \bar{E}_t \phi_{t+s} + \phi_t \right) + \Delta_t. \tag{9}$$

The first term is associated with expected payoffs, the second term captures current and expected future risk premia (affecting discount rates), and the last term is the higher order wedge. For expositional purposes we will focus in this paper on the case where the second term in (8) is zero, so that the higher order wedge is only associated with the difference between higher and first order expectations of dividends. The information structure chosen in the next section will assure that this is the case because there will be only public information about future risk premia. While this simplifies by allowing us to focus on HOE associated with dividends only, Appendix B shows that all the results in the paper still go through under a more general information structure where the second term in (8) is not zero.

Before introducing more specific assumptions in the next section, we show that the difference between higher order and first order expectations in (8) can be written in terms of expectations of market expectational errors. This makes concrete the conjecture by Keynes (1936) that investors do not just make decisions based on their own perception of the "prospective yield" (expected future dividends) but worry about market expectations. It also allows us to adopt an iterative procedure in Section 3.4 to convert the wedge into an expression that depends on first order expectational errors about average private signals.

First consider $s = 2$. The difference between the second and first order expectation is equal to the average expectation at time t of the average expectational error at $t+1$ about D_{t+2}:

$$\bar{E}_t^2 D_{t+2} - \bar{E}_t D_{t+2} = \bar{E}_t(\bar{E}_{t+1} D_{t+2} - D_{t+2}).$$

The intuition behind this term is as follows. Investment decisions at time t are based on the expected price at $t+1$. This price will reflect the market expectation of subsequent dividends. An investor at time t therefore makes investment decisions not just based on what he believes the dividend at $t+2$ to be, but also on whether he believes the market to make an expectational error at $t+1$ about the dividend at $t+2$. When investors have common information, they expect no future market expectational errors. But as we show below, this is no longer the case when information is heterogeneous.

Next consider $s = 3$. The difference between the third and first order expectation is equal to the difference between the first and second order expectation plus the difference between the second and third order expectation. This can be written as the average expectation at t of the average expectational error at $t+1$ plus the second order expectation at t of the average expectational error at $t+2$:

$$\bar{E}_t^3 D_{t+3} - \bar{E}_t D_{t+3} = \bar{E}_t(\bar{E}_{t+1} D_{t+3} - D_{t+3}) + \bar{E}_t^2(\bar{E}_{t+2} D_{t+3} - D_{t+3}).$$

The last term can be understood as follows. Just as the price at time t depends on expected average expectational error at $t+1$, so does the price at $t+1$ depend on expected average expectational error at $t+2$. The expected return from t to $t+1$ then depends on the expectation at time t of the market's expectation at $t+1$ of the market's expectational error at $t+2$. In other words, investment decisions at time t depend on the second order expectation at t of the market's expectational error at $t+2$.

Proceeding along this line for expectations of even higher order, and defining the present value of future dividends as $PV_t = \sum_{s=1}^{\infty}(D_{t+s}/R^s)$, we can rewrite (8) as follows:[12]

$$\Delta_t = \sum_{s=1}^{\infty} \frac{1}{R^s} \bar{E}_t^s(\bar{E}_{t+s} PV_{t+s} - PV_{t+s}). \tag{10}$$

The higher order wedge therefore depends on first order and higher order expectations of future expectational errors of the subsequent present value of dividends: the market expectation at t of the market's expectational error at $t+1$ of PV_{t+1}, the second order expectation at t of the expectational error at $t+2$ of PV_{t+2}, and so on. Investors make decisions not just based on what they expect future dividends to be but also on what they expect the market's expectational error next period to be about those future dividends, and what they expect next period's market expectation of the expectational error in the subsequent period to be. In the rest of our analysis we will use (10) instead of (8) to interpret the wedge.

12. The detailed steps leading to (10) are found in a technical appendix available upon request.

3. A DYNAMIC NRE MODEL

3.1 Basic Setup

In order to describe what determines the expectations of future expectational errors, as expressed in (10), we need to be more precise about the information structure and the process of dividends. We develop an infinite horizon NRE framework in which there is a constant flow of information, specified below, leading to an equilibrium asset price that is a time-invariant function of shocks.

Dividends are observable and the process of dividends is known by all agents. We assume a general process:

$$D_t = \bar{D} + C(L)\varepsilon_t^d, \tag{11}$$

where $C(L) = c_1 + c_2 L + \cdots$ is an infinitely lagged polynomial with $c_1 \neq 0$, c_s approaching a constant as $s \to \infty$, and $\varepsilon_t^d \sim N(0, \sigma_d^2)$. Asset supply is not observable, but its process is known by all agents. We will simply assume that

$$X_t = \varepsilon_t^x, \tag{12}$$

where $\varepsilon_t^x \sim N(0, \sigma_x^2)$. We will show that this implies that HOE of future risk premia are equal to first order expectations, which allows us to focus on HOE associated with dividends only. In Appendix B, we show that all the results in the paper still go through in the more general case where $X_t = F(L)\varepsilon_t^x$, with $F(L) = f_1 + f_2 L + \cdots$ an infinitely lagged polynomial. In that case HOE of future risk premia will in general differ from first order expectations.

Each period investors obtain a vector of J independent (orthogonal) private signals about dividend innovations over the next T periods:

$$\mathbf{v}_t^i = \mathbf{\Omega}\boldsymbol{\epsilon}_t^d + \boldsymbol{\epsilon}_t^{vi}, \tag{13}$$

where $\boldsymbol{\epsilon}_t^{d\prime} = (\varepsilon_{t+1}^d, \varepsilon_{t+2}^d, \ldots, \varepsilon_{t+T}^d)$ and the J errors in $\boldsymbol{\epsilon}_t^{vi}$ are uncorrelated (independent signals) and each distributed $N(0, \sigma_v^2)$. $\mathbf{\Omega}$ is a $J \times T$ matrix. This general formulation allows for single signals as well as signals on linear combinations of future innovations. It is assumed that the last column of $\mathbf{\Omega}$ is non-zero, so that at least one private signal provides information about the dividend innovation T periods later.

Now consider the entire set of signals received from $t - T + 1$ to $t - 1$ that contain information about future dividends at time t. We can summarize these past private signals as

$$\mathbf{V}_{t-1}^i = \mathbf{\Gamma}\boldsymbol{\epsilon}_t^d + \boldsymbol{\epsilon}_{t-1}^{Vi}, \tag{14}$$

where $\mathbf{\Gamma}$ is a matrix composed of subsets of $\mathbf{\Omega}$ (the derivation of $\mathbf{\Gamma}$ can be found in the technical appendix available upon request) . The vectors $\mathbf{v}_t^i$ and $\mathbf{V}_{t-1}^i$ jointly contain all current and past private signals available to agent i at time t that are informative about future dividend innovations.

Given the assumption of uncorrelated errors in private signals, the average of current and past private signals are $\bar{\mathbf{v}}_t = \mathbf{\Omega}\boldsymbol{\epsilon}_t^d$ and $\bar{\mathbf{V}}_{t-1} = \mathbf{\Gamma}\boldsymbol{\epsilon}_t^d$. Note that the last column of $\mathbf{\Gamma}$ is zero because private signals only contain information about dividend innovations up to T periods later. Therefore, we will also write $\bar{\mathbf{V}}_{t-1} = \mathbf{G}\boldsymbol{\epsilon}_{t-1}^d$, where $\mathbf{G}$ has zeros in the first column and the remainder consists of the first $T-1$ columns of $\mathbf{\Gamma}$.

3.2 Solution

Appendix A provides technical details regarding the solution of the equilibrium price. Here we only provide a more descriptive summary of the steps involved. As standard, we consider solutions for the equilibrium price that are a linear function of model innovations. To be precise, we conjecture the following equilibrium:

$$P_t = \frac{\bar{D}}{R-1} + A(L)\varepsilon_{t+T}^d + B(L)\varepsilon_t^x \tag{15}$$

with $A(L) = a_1 + a_2 L + a_3 L^2 + \cdots$ an infinitely lagged polynomial and $B(L) = b_1 + b_2 L + b_3 L^2 + \cdots + b_T L^{T-1}$. Since investors use their private signals to form expectations, and the average of the private signals depends on future dividend innovations, it is reasonable to conjecture that the price at time t depends on dividend innovations over the next T periods.

Conditional on the conjectured equilibrium price we can compute the expectation and variance of $P_{t+1} + D_{t+1}$. This involves solving a signal extraction problem that is described below. The conditional variance σ^2 is the same for all agents and constant over time. Imposing market equilibrium then yields the equilibrium price (3), which can be written as

$$P_t = \frac{1}{R}\bar{E}_t(P_{t+1} + D_{t+1}) - \frac{\gamma}{R}\sigma^2\varepsilon_t^x. \tag{16}$$

After computing $\bar{E}_t(P_{t+1} + D_{t+1})$, the equilibrium price depends on the same model innovations as the conjecture in (15). The final step involves solving a fixed point problem, equating the coefficients of the conjectured price equation (coefficients of the polynomials $A(L)$ and $B(L)$) to those in the equilibrium price equation. The fixed point problem is highly non-linear in the set of parameters of the equilibrium price and therefore does not have an analytical solution. In the application in Section 4, we solve it with the non-linear equation routine in Gauss.

A couple of points about existence and multiplicity of equilibria are in order. It is well known that this type of NRE model can exhibit multiple equilibria. However, the source of the multiplicity is unrelated to information heterogeneity. The same multiplicity arises under common knowledge. It is associated with the endogeneity of the conditional variance σ^2. Bacchetta and van Wincoop (2006) and Walker (2007b) develop similar models and show that under common knowledge there are two equilibria, one associated with a high σ^2 and one with a low σ^2. Intuitively, if agents believe σ^2 to be high, then supply shocks have a big impact on the asset price through the regular risk-premium channel. The large impact of supply shocks on the price then indeed justifies the belief that σ^2 is high.

In the common knowledge version of the model one can distinguish between three cases (this is derived in the technical appendix available upon request). An equilibrium does not exist when

$$\frac{4}{R^2}\gamma^2\sigma_d^2\sigma_x^2 > K,$$

where K is a constant that depends on the parameters $C(L)$ of the dividend process. In the knife-edge case where this condition holds with equality there is exactly one equilibrium. Otherwise there will be two equilibria. Since we will focus on cases where an equilibrium does exist, in general there will be two equilibria.

We find similar results under information heterogeneity. Since there is no analytical solution under information heterogeneity, existence and multiplicity of equilibria can only be checked numerically. We find that the region of parameters for which a solution exists is similar to that for the common knowledge model. For example, a solution does not exist when the variance of either supply or dividend shocks is too large or the rate of risk aversion is too large. Conditional on the existence of an equilibrium, we again find two equilibria (again with the exception of a knife-edge case where there is one equilibrium). These equilibria can be found as follows. We first solve for the equilibrium price for an exogenously chosen σ^2 and then compare it to the theoretical σ^2, $\text{var}_t(P_{t+1} + D_{t+1})$. This leads to a mapping of σ^2 into itself that is found to have two fixed points when searching over a very wide space of σ^2. As in Bacchetta and van Wincoop (2003) only the low variance equilibrium is stable in the common knowledge model. In the numerical application in Section 4 we therefore focus on the low variance equilibrium.

3.3 Expectation of Future Dividend Innovations

Before deriving some specific results on the higher order wedge it is useful to compute the expectation of future dividend innovations. Investors have no private signals on dividends more than T periods from now, so that $E_t^i(\varepsilon_{t+j}^d) = 0$ for $j > T$. Investors form expectations of dividend innovations in the following T periods, $\boldsymbol{\epsilon}_t^d$, by using a combination of private and public signals. Public information takes two forms. First, the $N(0, \sigma_d^2)$ distribution of future dividend innovations is public information. One can summarize these public signals with the vector $\mathbf{o}^T$ of T zeros. The dividend innovations themselves are the errors in these zero-signals.

The other pieces of public information are current and past prices. The equilibrium price (15) depends on dividend innovations over the next T periods. The price therefore contains information about future dividend innovations. This information is imperfect since the price also depends on asset supply innovations at time t and earlier that cannot be observed. These are the errors in the public price signals. At time t only the supply innovations $\boldsymbol{\epsilon}_t^{x\prime} = (\varepsilon_{t-T+1}^x, \ldots, \varepsilon_t^x)$ are unknown since supply innovations at $t - T$

and earlier can be extracted from prices at time $t - T$ and earlier if the conjecture (15) is correct.[13]

It is useful to subtract from the price signal (15) the components of the right-hand side that are known at time t: $\bar{D}/(R-1)$, current and past dividend innovations. This leads to the following adjusted time t price signal:

$$\begin{aligned} P_t^a &= a_T \varepsilon_{t+1}^d + a_{T-1}\varepsilon_{t+2}^d + \cdots + a_1 \varepsilon_{t+T}^d \\ &\quad + b_T \varepsilon_{t-T+1}^x + b_{T-1}\varepsilon_{t-T+2}^x + \cdots + b_1 \varepsilon_t^x \\ &\equiv \mathbf{a}\boldsymbol{\epsilon}_t^d + \mathbf{b}\boldsymbol{\epsilon}_t^x. \end{aligned} \tag{17}$$

All prices between $t - T + 1$ and time t contain information about future dividend innovations. Subtracting the known components of the entire set of price signals that depend on current and past dividend innovations, as well as supply innovations more than T periods ago, the set of price signals $\mathbf{p}_t' = (P_t^a, P_{t-1}^a, \ldots, P_{t-T+1}^a)$ can be written as

$$\mathbf{p}_t = \mathbf{A}\boldsymbol{\epsilon}_t^d + \mathbf{B}\boldsymbol{\epsilon}_t^x, \tag{18}$$

where

$$\mathbf{A} = \begin{pmatrix} a_T & .. & .. & a_1 \\ a_{T-1} & .. & a_1 & 0 \\ .. & .. & .. & .. \\ a_1 & 0 & .. & 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} b_T & .. & .. & b_1 \\ b_{T-1} & .. & b_1 & 0 \\ .. & .. & .. & .. \\ b_1 & 0 & .. & 0 \end{pmatrix}.$$

The vector $\boldsymbol{\epsilon}_t^x$ of supply shocks prevents the vector $\mathbf{p}_t$ of price signals from revealing the vector $\boldsymbol{\epsilon}_t^d$ of future dividend innovations. This is immediately clear from the definition of the matrix $\mathbf{B}$. Since asset supply shocks have an immediate impact on the equilibrium price ($b_1 \neq 0$) it follows that the matrix $\mathbf{B}$ has full rank. Therefore, any linear combination of prices will still depend on asset supply shocks, which are unobservable. The price signals therefore do not reveal any linear combination of future dividends. This standard feature of NRE models is important as otherwise all agents would have the same expectations of future dividends and the higher order wedge would be zero.

Other than the zero prior signals, the signals of future dividend innovations can be written as

$$\begin{pmatrix} \mathbf{p}_t \\ \mathbf{v}_t^i \\ \mathbf{V}_{t-1}^i \end{pmatrix} = \mathbf{H}\boldsymbol{\epsilon}_t^d + \begin{pmatrix} \mathbf{B}\boldsymbol{\epsilon}_t^x \\ \boldsymbol{\epsilon}_t^{vi} \\ \boldsymbol{\epsilon}_t^{Vi} \end{pmatrix}, \tag{19}$$

where $\mathbf{H}$ stacks the matrices $\mathbf{A}$, $\boldsymbol{\Omega}$, and $\boldsymbol{\Gamma}$.

13. This assumes that the polynominal $B(L)$ is invertible (the roots of $B(L) = 0$ are outside the unit circle), as is usually the case in applications (e.g., see Bacchetta and van Wincoop 2006).

The model gets around the infinite regress problem mentioned by Townsend (1983) because the set of relevant unknown innovations $\boldsymbol{\epsilon}_t^d$ and $\boldsymbol{\epsilon}_t^x$ is finite. This is because (i) investors observe current and past dividends, (ii) signals are informative about future dividend innovations up to T periods into the future, and (iii) $B(L)$ is invertible. This implies that we can represent expectations in the Kalman form with finite matrices. Using that the prior signal of $\boldsymbol{\epsilon}_t^d$ has mean $\mathbf{o}^T$ and variance $\sigma_d^2\mathbf{I}$, with $\mathbf{I}$ an identity matrix of size T, and writing the variance of the errors of the signals in (19) as $\mathbf{R}$, the standard signal extraction formula implies

$$E_t^i\left(\boldsymbol{\epsilon}_t^d\right) = (\mathbf{I} - \mathbf{MH})\mathbf{o}^T + \mathbf{M}\begin{pmatrix} \mathbf{p}_t \\ \mathbf{v}_t^i \\ \mathbf{V}_{t-1}^i \end{pmatrix}, \tag{20}$$

where $\mathbf{M} = \sigma_d^2\mathbf{H}'[\sigma_d^2\mathbf{HH}' + \mathbf{R}]^{-1}$. This can also be written as

$$E_t^i\left(\boldsymbol{\epsilon}_t^d\right) = \mathbf{M}_1\mathbf{Z}_t + \mathbf{M}_2\mathbf{v}_t^i + \mathbf{M}_3\mathbf{V}_{t-1}^i, \tag{21}$$

where $\mathbf{Z}_t' = (\mathbf{o}^{T\prime}\ \mathbf{p}_t')$ contains all the public signals about future dividend innovations.

3.4 The Wedge as a First-Order Expectational Error

We are now in a position to derive the expectation of the present discounted value of dividends. From the definition of PV_{t+1} it follows that

$$E_{t+1}^i PV_{t+1} = \sum_{s=1}^{\infty} \frac{\hat{D}_{t+s+1}}{R^s} + \mathbf{d}' E_{t+1}^i \boldsymbol{\epsilon}_{t+1}^d, \tag{22}$$

where $\hat{D}_{t+s+1}$ is the known component of dividend D_{t+s+1} at $t+1$, e.g., $\hat{D}_{t+2} = \bar{D} + c_2\varepsilon_{t+1}^d + c_3\varepsilon_t^d + c_4\varepsilon_{t-1}^d + \cdots$, and $\mathbf{d}$ is a vector of length T with element i equal to[14]

$$\frac{1}{R^{i-1}} \sum_{s=1}^{\infty} \frac{c_s}{R^s}.$$

Substituting (21) at $t+1$ into (22) gives:

$$E_{t+1}^i PV_{t+1} = \sum_{s=1}^{\infty} \frac{\hat{D}_{t+s+1}}{R^s} + \boldsymbol{\theta}'\mathbf{V}_t^i + \boldsymbol{\beta}'\mathbf{v}_{t+1}^i + \boldsymbol{\gamma}'\mathbf{Z}_{t+1}, \tag{23}$$

wherever $\boldsymbol{\theta}' = \mathbf{d}'\mathbf{M}_3$, $\boldsymbol{\beta}' = \mathbf{d}'\mathbf{M}_2$, $\boldsymbol{\gamma}' = \mathbf{d}'\mathbf{M}_1$.

14. This sum is well defined because c_s is assumed to approach a finite number as $s \to \infty$.

It is also useful to know the expectation of average private signals $\bar{\mathbf{V}}_{t+1}$ at $t+1$. We know from Section 3.1 that $\bar{\mathbf{V}}_{t-1} = \mathbf{G}\boldsymbol{\epsilon}^d_{t-1}$. Therefore $E^i_{t+1}\bar{\mathbf{V}}_{t+1} = E^i_{t+1}\mathbf{G}\boldsymbol{\epsilon}^d_{t+1}$ and (21) at $t+1$ yields:

$$E^i_{t+1}\bar{\mathbf{V}}_{t+1} = \boldsymbol{\Psi}'\mathbf{V}^i_t + \boldsymbol{\mu}'\mathbf{v}^i_{t+1} + \boldsymbol{\lambda}'\mathbf{Z}_{t+1}, \tag{24}$$

where $\boldsymbol{\Psi}' = \mathbf{GM}_3$, $\boldsymbol{\mu}' = \mathbf{GM}_2$, $\boldsymbol{\lambda}' = \mathbf{GM}_1$.

We are now in a position to derive a key result regarding the higher order wedge. First note that the higher order wedge is only associated with dividends. Future risk premia depend on future values of X_t and therefore on future supply innovations. The only information that agents have about future supply innovations is the common knowledge that they have a $N(0, \sigma^2_x)$ distribution. Agents do not have private information about future risk premia. Therefore HOE and first order expectations of future risk premia are the same and both equal to zero. In Section 2.2, we showed that in this case the higher order wedge can be written as in (10). Using this result, together with (23) and (24), Appendix B derives the following proposition about the higher order wedge.

Proposition 1: *The deviation between higher and first order expectations that affects the equilibrium asset price is*

$$\Delta_t = \boldsymbol{\Pi}'(\bar{E}_t\bar{\mathbf{V}}_t - \bar{\mathbf{V}}_t), \tag{25}$$

where $\boldsymbol{\Pi} = \frac{1}{R}(\mathbf{I} - \frac{\boldsymbol{\Psi}}{R})^{-1}\boldsymbol{\theta}$.

The proposition tells us that the higher order wedge depends on the average expectational error at time t about the vector of average private signals that remain informative about future dividends at $t+1$. The proposition therefore reduces differences between higher and first order expectations to a first order expectational error.

In order to provide some intuition behind Proposition 1, it is useful to write

$$\boldsymbol{\Pi} = \frac{1}{R}\boldsymbol{\theta} + \sum_{s=1}^{\infty}\frac{1}{R^{s+1}}\boldsymbol{\Psi}^s\boldsymbol{\theta}.$$

Consider the first element of $\boldsymbol{\Pi}$, $\boldsymbol{\theta}/R$. It corresponds to the first element of (10), the average expectation at t of the market expectational error at $t+1$ about PV_{t+1}. An investor's expectation of this error can be written as $E^i_t(\bar{E}_{t+1}PV_{t+1} - PV_{t+1}) = E^i_t(\bar{E}_{t+1}PV_{t+1} - E^i_{t+1}PV_{t+1})$. From (23) it follows that:

$$\bar{E}_{t+1}PV_{t+1} - E^i_{t+1}PV_{t+1} = \boldsymbol{\theta}'\left(\bar{\mathbf{V}}_t - \mathbf{V}^i_t\right) + \boldsymbol{\beta}'\left(\bar{\mathbf{v}}_{t+1} - \mathbf{v}^i_{t+1}\right). \tag{26}$$

An investor expects the market to make expectational errors to the extent that the market is expected to have a different set of private signals. The second term in (26) is expected to be zero. Thus, an investor only expects the market to make expectational errors tomorrow if the average private signals $\bar{\mathbf{V}}_t$ are expected to be different from the investor's own private signals. Taking the expectation of (26) for investor i at time t yields $\boldsymbol{\theta}'(E^i_t\bar{\mathbf{V}}_t - \mathbf{V}^i_t)$. The average of this across investors is $\boldsymbol{\theta}'(\bar{E}_t\bar{\mathbf{V}}_t - \bar{\mathbf{V}}_t)$, which corresponds to the first element of $\boldsymbol{\Pi}$.

The second element in $\boldsymbol{\Pi}$ corresponds to the sum of HOE of future expectational errors. Consider, e.g., the second-order expectation of the market's expectational error at $t + 2$ about PV_{t+2}. Corresponding to the discussion above, the average expectation at $t + 1$ of the market's expectational error at $t + 2$ about PV_{t+2} is $\boldsymbol{\theta}'(\bar{E}_{t+1}\bar{\mathbf{V}}_{t+1} - \bar{\mathbf{V}}_{t+1})$. This depends itself on an average expectational error, this time not about future dividends but about average private signals. Using a similar argument as above, but using (24), the average expectation at time t of the market's expectational error at $t + 1$ about average private signals is equal to $\boldsymbol{\Psi}'(\bar{E}_t\bar{\mathbf{V}}_t - \bar{\mathbf{V}}_t)$. Following an iterative argument one can similarly derive third and HOE of future expectational errors. The key point is that these all depend on average expectational errors at time t about average private signals.

One can think of first and higher order expectations of future expectational errors as resulting from a chain effect. This explains why current expectational mistakes $\bar{E}_t\bar{\mathbf{V}}_t - \bar{\mathbf{V}}_t$ affect expectations of all orders. As an illustration consider the case where $\mathbf{V}^i_t$ has only one element, so that investors have only one private signal at time t that is still relevant at $t + 1$. Assume that a higher private signal $\mathbf{V}^i_t$ at time t makes the investor both more optimistic at $t+1$ about future payoffs ($\boldsymbol{\theta} > 0$) and more optimistic at $t + 1$ about average private signals ($\boldsymbol{\Psi} > 0$).

Now consider what happens when the average investor at time t expects others to have more favorable, and therefore too optimistic, private signals, i.e., $\bar{E}_t\bar{\mathbf{V}}_t > \bar{\mathbf{V}}_t$. The average investor then expects that (i) the market is too optimistic at $t + 1$ about future dividends and (ii) the market is too optimistic at $t + 1$ about average private signals. The first leads to first order expectations of positive expectational errors at $t + 1$ about PV_{t+1}. The second implies a first order expectation of positive expectational errors at $t + 1$ about private signals, i.e., a first order expectation at t that $\bar{E}_{t+1}\bar{\mathbf{V}}_{t+1} > \bar{\mathbf{V}}_{t+1}$. This leads to the next step in the chain. Following the same argument as above, it leads to second order expectations that the market is too optimistic at $t + 2$ about future dividends and average private signals. The latter leads to a third step in the chain, and so on.

Proposition 1 has several implications.

COROLLARY 1: *The higher order wedge is proportional to average expectational errors of future dividend innovations* $\bar{E}_t\epsilon^d_t - \epsilon^d_t$.

Corollary 1 follows almost immediately from Proposition 1. Since $\bar{\mathbf{V}}_t = \mathbf{G}\epsilon^d_t$, we have

$$\bar{E}_t\bar{\mathbf{V}}_t - \bar{\mathbf{V}}_t = \mathbf{G}\left(\bar{E}_t\epsilon^d_t - \epsilon^d_t\right). \tag{27}$$

Average expectational errors of future dividend innovations can only be associated with errors in public signals as the average errors of private signals are zero. There are two public signals in the model that provide information about future dividend innovations: the price signals $\mathbf{p}_t$ and the zero-signals $\mathbf{o}^T$. The errors in the price signals are summarized by $\mathbf{B}\epsilon_t^x$ in (19) and therefore depend on the supply innovations over the past T periods. The errors in the signals $\mathbf{o}^T$ are $\epsilon_t^o = \mathbf{o}^T - \epsilon_t^d$. Substituting (19) into (20), and taking the average across investors, we have

$$\bar{E}_t\epsilon_t^d - \epsilon_t^d = (\mathbf{I} - \mathbf{MH})\epsilon_t^o + \mathbf{M}_P\mathbf{B}\epsilon_t^x, \tag{28}$$

where $\mathbf{M}_P$ contains the first T columns of $\mathbf{M}$. Together with (27) and Proposition 1 it follows that the higher order wedge depends on the errors ϵ_t^o and $\mathbf{B}\epsilon_t^x$ in public signals. This result is summarized in Corollary 2:

COROLLARY 2: *The higher order wedge depends on expectational errors of future dividend innovations based on public signals.*

The significance of this result was already discussed in the introduction. Errors in public signals are the ultimate source of the higher order wedge. They provide a coordination mechanism through which the average investor believes that (on average) other investors will be either overly optimistic or pessimistic about the present value of future payoffs. For example, a supply shock that raises the asset price leads the average investor to believe that other investors have more favorable private signals than they do. The average investor therefore believes that the market will be overly optimistic. Even if investors themselves do not believe that future dividends will be higher, they will nonetheless buy more of the asset in the belief that the price tomorrow will be high due to overly optimistic beliefs by others. This component of the asset price is captured by the higher order wedge.

3.5 On the Existence and Magnitude of the Higher Order Wedge

Proposition 1 has two more implications that relate to the existence and magnitude of the wedge.

COROLLARY 3: *Within the context of the assumed information structure, a necessary and sufficient condition for the higher order wedge to be non-zero is that $T > 1$.*

First consider the *necessary* part of Corollary 3. When $T = 1$, the set $\bar{\mathbf{V}}_t$ is empty and therefore the higher order wedge is zero. In that case there are no private signals at time t that still contain information about future dividend innovations at $t + 1$. Intuitively, since an investor has no reason to expect that his private signals in future periods differ from the average, there is no reason to expect the market to make expectational errors in the future when predicting future dividends.

Next consider the *sufficient* part of Corollary 3. It is immediate from Proposition 1 that the higher order wedge is non-zero if both $\boldsymbol{\theta}$ is non-zero and all elements of $\bar{E}_t\bar{\mathbf{V}}_t - \bar{\mathbf{V}}_t$ are non-zero ($\mathbf{\Pi}$ is non-zero if and only if $\boldsymbol{\theta}$ is non-zero). First, consider $\boldsymbol{\theta}$. We know that PV_{t+1} depends on all future dividend innovations, starting with ε_{t+2}^d. The set $\mathbf{V}_t^i$ contains private signals received at time t and earlier that remain informative about future dividend innovations at time $t + 1$. This set is non-zero when $T > 1$. Since the errors in the signals $\mathbf{V}_t^i$ are uncorrelated with those of all other signals, and no combination of the other signals can fully reveal ϵ_{t+1}^d, the expectation of PV_{t+1} will depend on $\mathbf{V}_t^i$.[15] This implies that $\boldsymbol{\theta}$ is non-zero.

Next consider $\bar{E}_t \bar{\mathbf{V}}_t - \bar{\mathbf{V}}_t$. By assumption each element of $\bar{\mathbf{V}}_t$ depends on at least one dividend innovation between $t + 2$ and $t + T$. Since the errors in all signals have a non-zero variance, and there is no linear relation between the errors of the signals, no combination of them fully reveals any of the future dividend innovations. This implies that each of the elements of $\bar{E}_t \bar{\mathbf{V}}_t - \bar{\mathbf{V}}_t$ are non-zero.

A final implication of Proposition 1 relates to the size of the wedge.

COROLLARY 4: *The variance of Δ_t is largest for intermediate levels of the quality of private information as measured by $1/\sigma_v^2$. It vanishes when the variance σ_v^2 of the errors in private signals approach zero or infinity.*

The proof of this Corollary 4 is almost trivial. On the one hand, when $\sigma_v^2 \to 0$ the errors in private signals vanish, so that there are no expectational errors about future dividend innovations and the wedge disappears. On the other hand, when $\sigma_v^2 \to \infty$, private signals contain no information about future dividend innovations, so that $\boldsymbol{\theta} \to 0$ and again the wedge vanishes.

3.6 *Implication for Asset Price Volatility*

An important question is whether HOE increase the volatility of asset prices. We find, perhaps surprisingly, that the higher order wedge reduces the volatility of asset prices. This is made precise in Corollary 5.

COROLLARY 5: *Assume that dividends are stationary, such that the asset price P_t is stationary. Then* $\text{var}(P_t) < \text{var}(P_t^*)$.

We know from Proposition 1 that the higher order wedge depends on average expectational errors about $\bar{\mathbf{V}}_t$. Since the asset price P_t is in the information set of investors, expectational errors at time t should be orthogonal to the price, so that $\text{cov}(P_t, \Delta_t) = 0$. Using that $\Delta_t = P_t - P_t^*$, we have $\text{var}(P_t^*) = \text{var}(P_t - \Delta_t) = \text{var}(P_t) + \text{var}(\Delta_t)$, which implies Corollary 5.

In the early literature on excess volatility the emphasis was on expected dividends as a determinant of the stock price. However, time variation in expected returns was later recognized as an important second asset pricing determinant. These two traditional asset pricing determinants are reflected in P_t^*. To this we have now added a third asset pricing determinant, the higher order wedge. While the second asset pricing determinant can contribute to an increase in asset price volatility, the third one (the higher order wedge) reduces the volatility of P_t.

15. If some combination of other signals fully reveals ϵ_{t+1}^d, then $\text{var}_{t+1}(\epsilon_{t+1}^d) = \sigma_d^2(\mathbf{I} - \mathbf{MH}) = 0$, so that $\mathbf{I} = \mathbf{MH}$. Multiplying by $\mathbf{H}$ and adding $\mathbf{R}[\sigma_d^2\mathbf{HH}' + \mathbf{R}]^{-1}\mathbf{H}$ to both sides, using the definition of $\mathbf{M}$, gives $\mathbf{R}\,[\sigma_d^2\mathbf{HH}' + \mathbf{R}]^{-1}\mathbf{H} = 0$. Since $\mathbf{R}$ is an invertible matrix (no linear combination of the errors in the signals is zero), it would follow that $\mathbf{H} = 0$, which is clearly violated.

4. AN ILLUSTRATION

While the higher order wedge reduces asset price volatility, this does not imply that the response to all shocks is reduced by the higher order wedge. In order to determine the precise impact of the higher order wedge on the equilibrium price, we now turn to a specific example that is contained within the general information structure described so far and solve it numerically. The example illustrates that the higher order wedge tends to dampen the impact of future dividend innovations on the asset price while it tends to amplify the impact of unobserved supply shocks. Both of these disconnect the asset price from the present value of future dividends. The higher order wedge can therefore help explain the weak link between the price and the present value of future dividends that was first documented by Shiller (1981).

The specific example that we will consider in this section makes simplifying assumptions about the process of dividends and about private signals. We assume that dividends are i.i.d., so that:

$$D_t = \bar{D} + \varepsilon_t^d, \tag{29}$$

where $\varepsilon_t^d \sim N(0, \sigma_d^2)$. With regard to private signals we assume that each period investors obtain a single private signal about the dividend innovation T periods later:

$$v_t^i = \varepsilon_{t+T}^d + \varepsilon_t^{vi} \tag{30}$$

with $\varepsilon_t^{vi} \sim N(0, \sigma_v^2)$.

As T increases investors get information further in advance but also have a larger number of relevant private signals each period. Since $\mathbf{V}_t^i$ denotes the private information set at time t that is still valuable at time $t+1$, we have $\mathbf{V}_t^i = \{v_{t-T+2}^i, .., v_t^i\}$ and the average across agents is $\bar{\mathbf{V}}_t = \{\varepsilon_{t+2}^d, .., \varepsilon_{t+T}^d\}$ for $T \geq 2$.[16] We will write $\mathbf{\Pi}' = \{\pi_1, .., \pi_{T-1}\}$. Proposition 1 then implies

$$\Delta_t = \sum_{s=2}^{T} \pi_{s-1} \left(\bar{E}_t \varepsilon_{t+s}^d - \varepsilon_{t+s}^d \right). \tag{31}$$

Numerical results show that all the elements of $\mathbf{\Pi}$ are positive.[17] Therefore, the higher order wedge depends positively on expectational errors about future dividend innovations.

16. As discussed above in the context of Corollary 3, when $T = 1$ private signals today are no longer in the information set tomorrow since tomorrow's dividend is observed tomorrow. In that case HOE collapse to first order expectations.

17. To check this, we have varied the standard deviations σ_x, σ_d, and σ_v from 0 to 1, γ from 1 to 20, and T from 2 to 50 (setting $R = 1.02$).

The equilibrium price is given by:

$$P_t = \frac{\bar{D}}{R-1} + \sum_{s=1}^{T} a_s \varepsilon^d_{t+T+1-s} + \sum_{s=1}^{T} b_s \varepsilon^x_{t-s+1} \quad a_s > 0,\ b_s < 0. \tag{32}$$

Notice that even though supply shocks are not persistent, they have a persistent effect on the asset price since past equilibrium prices (which depend on past supply shocks) are informative about future dividends. The price at time t depends on the last T supply shocks since asset prices over the past T periods contain information about future dividends.

In order to obtain some insight regarding the role of the higher order wedge in the equilibrium price, consider the case where $T = 2$. In that case

$$\Delta_t = \pi\left(\bar{E}_t \varepsilon^d_{t+2} - \varepsilon^d_{t+2}\right). \tag{33}$$

We know from Corollary 2 that the expectation error of the time $t + 2$ dividend innovation depends on the errors in the public signals. In this case the adjusted price signals are

$$P^a_t = a_1 \varepsilon^d_{t+2} + a_2 \varepsilon^d_{t+1} + b_1 \varepsilon^x_t + b_2 \varepsilon^x_{t-1} \tag{34}$$

$$P^a_{t-1} = a_1 \varepsilon^d_{t+1} + b_1 \varepsilon^x_{t-1}. \tag{35}$$

These signals provide joint information about ε^d_{t+1} and ε^d_{t+2} with errors consisting of ε^x_{t-1} and ε^x_t. The other public signals in the model are the zero-signals, with errors $-\varepsilon^d_{t+1}$ and $-\varepsilon^d_{t+2}$. Thus, from Corollary 2 there are parameters $\delta_1, \ldots, \delta_4$ such that

$$\Delta_t = \delta_1 \varepsilon^x_t + \delta_2 \varepsilon^x_{t-1} + \delta_3 \varepsilon^d_{t+2} + \delta_4 \varepsilon^d_{t+1}. \tag{36}$$

The precise values of the coefficients δ_i, based on implementing (28) for the case where $T = 2$, are listed in Appendix C as a function of the parameters of the equilibrium price. Unfortunately it is impossible to obtain an analytical solution for the parameters of the equilibrium price even for $T = 2$ as the solution involves a highly non-linear set of equations in the four parameters of the equilibrium price function (coefficients on dividend innovations in the next two periods and on the current and last period's supply innovation).

The other component of the price is P^*_t, which from (7) can be written as

$$P^*_t = \bar{E}_t PV_t - \gamma\sigma^2 \frac{1}{R} \varepsilon^x_t. \tag{37}$$

Using (23), $\bar{E}_t PV_t$ is equal to $\bar{D}/(R-1)$ plus a linear combination of $\bar{\mathbf{V}}_{t-1} = \varepsilon^d_{t+1}$, $\bar{\mathbf{v}}_t = \varepsilon^d_{t+2}$ and the adjusted prices (34) and (35). It follows that, similar to the price P_t itself, P^*_t can be written as $\bar{D}/(R-1)$ plus a linear sum of the shocks ε^d_{t+1}, ε^d_{t+2}, ε^x_{t-1}, and ε^x_t. Not surprisingly, numerical analysis shows that P^*_t always depends positively on ε^d_{t+1} and ε^d_{t+2}: higher private signals at t and $t-1$, which themselves depend positively on ε^d_{t+1} and ε^d_{t+2}, raise the expectation of future dividends. Moreover, numerical analysis shows that P^*_t depends negatively on ε^x_{t-1} and ε^x_t: higher supply at time t lowers the price by raising the risk premium, while higher supply at $t-1$ lowers the price at $t-1$, which in turn lowers P^*_t by lowering the expectation of ε^d_{t+1} (see (35)).

Appendix C shows that the coefficients δ_1 and δ_3 in (36) are both negative. The negative sign of δ_3 is a result of the zero public signal, since based on the zero signal alone, the expectational error about ε^d_{t+2} is equal to $-\varepsilon^d_{t+2}$. The negative sign of δ_1 is a result of the error in the time t price signal. As can be expected from (34), an increase in P^a_t raises the expectation of ε^d_{t+2}. As P^a_t depends negatively on ε^x_t, this shock negatively affects the expectational error of ε^d_{t+2} and therefore the higher order wedge.

Since $P_t = P^*_t + \Delta_t$, the negative signs of δ_1 and δ_3 imply that the higher order wedge leads to a less positive coefficient on ε^d_{t+2} in the equilibrium price and a more negative coefficient on ε^x_t. The dampened impact of future dividend innovations and the amplified impact of unobserved supply shocks both weaken the link between the price P_t and the present value PV_t of future dividends.

It is not the case that the impact of each past dividend innovation is dampened and each past asset supply innovation is amplified: the weight of ε^d_{t+1} is amplified ($\delta_4 > 0$) and that of ε^x_{t-1} is dampened by the higher order wedge ($\delta_2 > 0$). Numerically, these effects are far outweighed though by the dampened impact of ε^d_{t+2} and the amplified impact of ε^x_t on the equilibrium price. It therefore remains the case that the higher order wedge weakens the link between the price P_t and the present value PV_t of future dividends.

It is nonetheless useful to develop some understanding for the dampened impact of ε^x_{t-1} on the price.[18] It can best be understood by considering the price signals (34) and (35). An increase in P^a_{t-1} raises the expectation of ε^d_{t+1}, which for a given P^a_t lowers the expectation of ε^d_{t+2}. As a result, the expectation of ε^d_{t+2} depends negatively on P^a_{t-1}. And since P^a_{t-1} itself depends negatively on ε^x_{t-1}, the time $t-1$ supply shock has a positive impact on the expectational error of ε^d_{t+2} and therefore on the higher order wedge ($\delta_2 > 0$). A similar argument can also account for the amplified impact of ε^d_{t+1}.

This result also illustrates that the finding by Allen, Morris, and Shin (2006) that HOE give more weight to public signals does not necessarily generalize to all public signals. In this case the higher order wedge depends negatively on the P^a_{t-1} public signal, while the same signal raises P^*_t as a higher $t-1$ price raises the expectation of the time $t+1$ dividend.

18. As shown in Appendix C, δ_2 depends positively on $a_1b_2 - a_2b_1$, which numerically is always positive. The higher order wedge therefore depends positively on ε^x_{t-1}, therefore dampening its impact on the price.

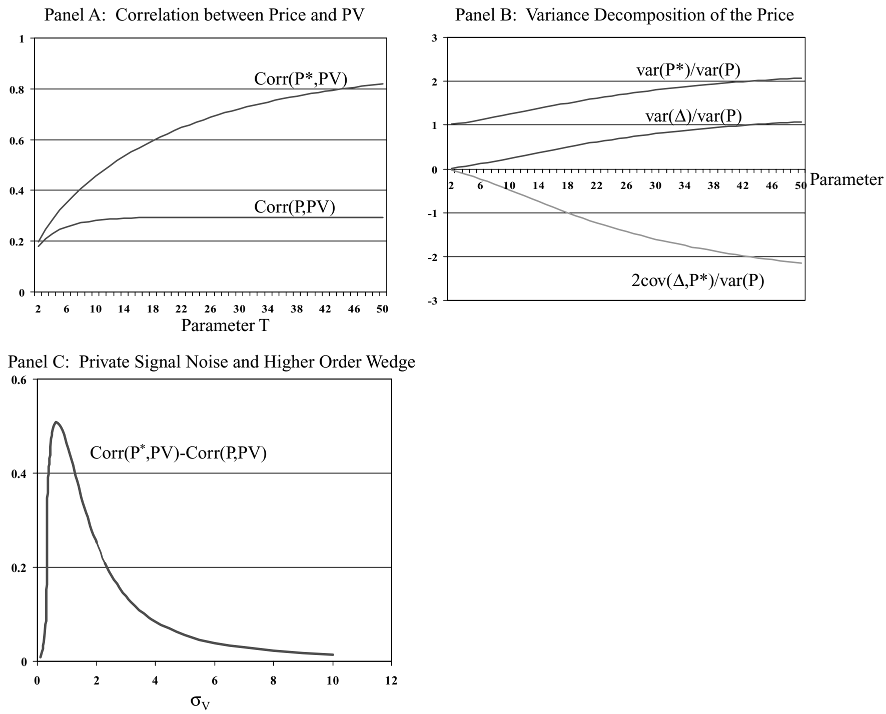

FIG. 1. Numerical Illustration.

NOTE: Benchmark parameters: $T = 30$, $\sigma_d = \sigma_v = \sigma_x = 0.4$; $R = 1.02$, $\gamma = 2$.

In order to provide a numerical illustration, Figure 1 shows some results for the parameterization $\sigma_v = \sigma_d = \sigma_x = 0.4$, $R = 1.02$, and $\gamma = 2$. In Panels A and B the parameter T is varied from 2 to 50. Panel A shows both corr(P_t^*, PV_t) and corr(P_t, PV_t) in order to illustrate the impact of the higher order wedge on the correlation between the price and the present value of future dividends. Consistent with the findings above, the higher order wedge weakens the correlation between the price and the present value of future dividends. The difference between corr(P_t^*, PV_t) and corr (P_t, PV_t) rises for larger values of T. When T is small, the information about most future dividend innovations is public in the form of the zero signals, so that HOE have little impact. The example shows that for $T = 50$ the impact of HOE is substantial, reducing the correlation between the present discounted value of dividends and the price from 0.82 to 0.29.

Consistent with Panel A, Panel B shows that the variance of the higher order wedge rises relative to the variance of price when T increases. The same is the case for P^*. When $T = 50$ the variance of the higher order wedge is larger than the variance of P, while the variance of P^* is more than twice the variance of P. The remaining factor contributing to the price variance is a large negative covariance between the higher order wedge and P^*. The higher order wedge therefore reduces asset price volatility, as reflected in values of $\text{var}(P^*)/\text{var}(P)$ above 1.

Panel C presents an implication of Corollary 4 for the case where $T = 30$. It shows that the impact of the higher order wedge on the correlation between the price and the present value of dividends is maximized for an intermediate level of the quality of private information. The reduction in the correlation is largest (0.51) for $\sigma_v = 0.7$. The impact on the correlation vanishes to zero when either $\sigma_v \to \infty$ or $\sigma_v \to 0$.

5. CONCLUSION

This paper has analyzed the role of HOE for asset pricing in the context of an NRE model. We have shown that an asset price can be written as the sum of the two traditional asset pricing components, expected asset payoffs and discount rates, plus a new component that we have called the "higher order wedge." This third asset pricing component captures the difference between higher order and first order expectations. The paper sheds light both on what determines this new asset pricing component and how it affects the equilibrium price. A key result is that it depends on expectational errors that weaken the relationship between the price and future dividends.

While our analysis assumes full rationality of investors, the recent literature in behavioral finance implies that expectational errors could be caused by deviations from rationality, such as overconfidence or changing market mood. We conjecture that the insights from our general analysis also apply when expectational errors are caused by factors different from noisy public signals. In particular, the impact of these errors would be amplified by HOE. Combining the dimension of market psychology with our analysis of HOE would bring us close to Keynes' reasoning on asset prices and closer to understanding asset price movements.

Another natural direction for future research is to quantify the importance of the higher order wedge as an asset pricing determinant. While we have shown that it can be quantitatively very large, its magnitude needs to be evaluated in the context of a somewhat more realistic setup that is calibrated to actual data. In particular, we have maintained the standard assumption in NRE models of constant absolute risk aversion. While this simplifies the solution significantly, a more realistic constant rate of relative risk aversion needs to be adopted when confronting the model to the data. More realistic assumptions about the process of dividends and the information structure would need to be considered as well.

APPENDIX A: EQUILIBRIUM PRICE

We describe the solution method in the general case where $X_t = F(L)\varepsilon_t^x$. The assumption $X_t = \varepsilon_t^x$ adopted in the text is clearly nested within the general case.

The overall approach to the solution of the equilibrium price is as follows. First conjecture that the price takes the form (15) where in the general case $B(L)$ is an infinitely lagged polynomial. Conditional on this compute $\bar{E}_t(P_{t+1} + D_{t+1})$ and $\mathrm{var}_t(P_{t+1} + D_{t+1})$ and then impose the market equilibrium condition (3):

$$P_t = \frac{1}{R}\bar{E}_t(P_{t+1} + D_{t+1}) - \frac{\gamma}{R}\mathrm{var}_t(P_{t+1} + D_{t+1})F(L)\varepsilon_t^x. \tag{A1}$$

This gives a new price equation of the same form as (15). This therefore provides a mapping from a set of parameters of the conjectured price equation to a new set of parameters of the equilibrium price equation. We then need to solve a fixed point problem that equates the conjectured to the equilibrium parameters.

We will now show that the equilibrium price (A1) indeed takes the conjectured form (15) conditional on the conjecture (15) being correct. First consider the expectation of $P_{t+1} + D_{t+1}$. It is useful to express $P_{t+1} + D_{t+1}$ in function of observable and unobservable variables:

$$\begin{aligned} P_{t+1} + D_{t+1} = {} & \frac{R}{R-1}\bar{D} + a_1\varepsilon_{t+T+1}^d + b_1\varepsilon_{t+1}^x + \mathbf{a}^*\boldsymbol{\epsilon}_t^d + \mathbf{b}^*\boldsymbol{\epsilon}_t^x \\ & + A^*(L)\varepsilon_t^d + B^*(L)\varepsilon_{t-T}^x, \end{aligned} \tag{A2}$$

where $\mathbf{a}^* = (a_{T+1} + c_1, a_T, \ldots, a_2)$, $\mathbf{b}^* = (b_{T+1}, \ldots, b_2)$, $A^*(L) = (a_{T+2} + c_2) + (a_{T+3} + c_3)L + \ldots$, and $B^*(L) = b_{T+2} + b_{T+3}L + \ldots$. It follows that

$$\bar{E}_t(P_{t+1} + D_{t+1}) = \frac{R}{R-1}\bar{D} + \mathbf{a}^*\bar{E}_t\boldsymbol{\epsilon}_t^d + \mathbf{b}^*\bar{E}_t\boldsymbol{\epsilon}_t^x + A^*(L)\varepsilon_t^d + B^*(L)\varepsilon_{t-T}^x. \tag{A3}$$

From (18) we have

$$\bar{E}_t\boldsymbol{\epsilon}_t^x = \mathbf{B}^{-1}\mathbf{p}_t - \mathbf{B}^{-1}\mathbf{A}\bar{E}_t\boldsymbol{\epsilon}_t^d = \mathbf{B}^{-1}\mathbf{A}\left(\boldsymbol{\epsilon}_t^d - \bar{E}_t\boldsymbol{\epsilon}_t^d\right) + \boldsymbol{\epsilon}_t^x. \tag{A4}$$

Substituting this into (A3), we get

$$\begin{aligned} \bar{E}_t(P_{t+1} + D_{t+1}) = {} & \frac{R}{R-1}\bar{D} + \boldsymbol{\psi}\bar{E}_t\boldsymbol{\epsilon}_t^d + \mathbf{b}^*\boldsymbol{\epsilon}_t^x + \mathbf{b}^*\mathbf{B}^{-1}\mathbf{A}\boldsymbol{\epsilon}_t^d \\ & + A^*(L)\varepsilon_t^d + B^*(L)\varepsilon_{t-T}^x, \end{aligned} \tag{A5}$$

where $\boldsymbol{\psi} = \mathbf{a}^* - \mathbf{b}^*\mathbf{B}^{-1}\mathbf{A}$. Averaging (20) across agents and substituting (19) gives $\bar{E}_t\boldsymbol{\epsilon}_t^d$ as a function of $\boldsymbol{\epsilon}_t^d$ and $\boldsymbol{\epsilon}_t^x$. Substituting this into (A5) it follows that $\bar{E}_t(P_{t+1} + D_{t+1})$ takes the form

$$\bar{E}_t(P_{t+1} + D_{t+1}) = \frac{R}{R-1}\bar{D} + \tilde{A}(L)\varepsilon_{t+T}^d + \tilde{B}(L)\varepsilon_t^x \tag{A6}$$

with $\tilde{A}(L)$ and $\tilde{B}(L)$ polynomials defined analogously to $A(L)$ and $B(L)$.

For the variance we have:

$$\text{var}_t(P_{t+1} + D_{t+1}) = a_1^2\sigma_d^2 + b_1^2\sigma_x^2 + \psi'\text{var}_t\left(\boldsymbol{\epsilon}_t^d\right)\psi. \tag{A7}$$

The standard signal extraction formulas give $\text{var}_t(\boldsymbol{\epsilon}_t^d) = \sigma_d^2(\mathbf{I} - \mathbf{MH})$, so that

$$\text{var}_t(P_{t+1} + D_{t+1}) = a_1^2\sigma_d^2 + b_1^2\sigma_x^2 + \sigma_d^2\psi'(\mathbf{I} - \mathbf{MH})\psi. \tag{A8}$$

Substituting (A6) and (A8) into (A1) gives an equilibrium price that indeed takes the conjectured form in (15) on which our computation of $\bar{E}_t(P_{t+1} + D_{t+1})$ and $\text{var}_t(P_{t+1} + D_{t+1})$ are based. The equilibrium price equation depends on the parameters of the polynomials $A(L)$ and $B(L)$ in the conjectured price equation. Equating the conjectured price equation to the equilibrium price equation then involves computing a fixed point problem in the parameters $A(L)$ and $B(L)$ of the price equation. Finally, it is easy to see that in the case where $X_t = \varepsilon_t^x$ only the first T parameters of the polynomial $B(L)$ are non-zero as conjectured below equation (15). This conjecture implies that $B^*(L) = 0$, so that $\bar{E}_t(P_{t+1} + D_{t+1})$ only depends on the supply shocks in the vector $\boldsymbol{\epsilon}_t^x$. Therefore the equilibrium price (A1) also only depends on supply innovations in $\boldsymbol{\epsilon}_t^x$, which implies that the parameters b_s in the polynominal $B(L)$ are zero for $s > T$.

APPENDIX B

Proof of Proposition 1: Given (13), the investor's expectation of (26) is:

$$E_t^i\left(\bar{E}_{t+1}PV_{t+1} - E_{t+1}^i PV_{t+1}\right) = \boldsymbol{\theta}'\left(E_t^i\bar{\mathbf{V}}_t - \mathbf{V}_t^i\right). \tag{B1}$$

Taking the average across investors, we have

$$\bar{E}_t(\bar{E}_{t+1}PV_{t+1} - PV_{t+1}) = \boldsymbol{\theta}'(\bar{E}_t\bar{\mathbf{V}}_t - \bar{\mathbf{V}}_t). \tag{B2}$$

The other terms in (10) involve HOE of future expectational errors. Consider the term at $t+s$: $\bar{E}_t^s(\bar{E}_{t+s}PV_{t+s} - PV_{t+s})$. It can be rewritten as $\bar{E}_t^{s-1}\bar{E}_{t+s-1}(\bar{E}_{t+s}PV_{t+s} - PV_{t+s})$. Using (B2) at time $t+s-1$ we can write:

$$\bar{E}_t^s(\bar{E}_{t+s}PV_{t+s} - PV_{t+s}) = \boldsymbol{\theta}'\bar{E}_t^{s-1}(\bar{E}_{t+s-1}\bar{\mathbf{V}}_{t+s-1} - \bar{\mathbf{V}}_{t+s-1}). \tag{B3}$$

Using (24) and following the same reasoning as to get (B2), it then follows that

$$\bar{E}_t(\bar{E}_{t+1}\bar{\mathbf{V}}_{t+1} - \bar{\mathbf{V}}_{t+1}) = \boldsymbol{\Psi}'(\bar{E}_t\bar{\mathbf{V}}_t - \bar{\mathbf{V}}_t).$$

Similarly $\bar{E}_{t+s-2}(\bar{E}_{t+s-1}\bar{\mathbf{V}}_{t+s-1} - \bar{\mathbf{V}}_{t+s-1}) = \boldsymbol{\Psi}'(\bar{E}_{t+s-2}\bar{\mathbf{V}}_{t+s-2} - \bar{\mathbf{V}}_{t+s-2})$. This can be substituted into (B3) and we can work backward using (24) to get

$$\bar{E}_t^s(\bar{E}_{t+s}PV_{t+s} - PV_{t+s}) = \boldsymbol{\theta}'(\boldsymbol{\Psi}')^{s-1}(\bar{E}_t\bar{\mathbf{V}}_t - \bar{\mathbf{V}}_t). \tag{B4}$$

Substituting this into (10) gives the expression for Δ_t in Proposition 1.

While in the text we have focused on the case where $X_t = \varepsilon_t^x$, so that HOE are only associated with dividends, it is easy to see that Proposition 1 still holds for the more general case where $X_t = F(L)\varepsilon_t^x$. In this more general case both HOE of future dividends and future risk premia will in general differ from first order expectations. If we allow the second term in (8) to be non-zero, (10) still holds if we replace the present value of future dividends by the present value of future dividends minus risk premia. In other words, $PV_t = \sum_{s=1}^{\infty} \frac{1}{R^s}(D_{t+s} - \phi_{t+s})$. All that is left to show then is that (23) still holds with this new definition of PV_t since otherwise the proof of Proposition 1 remains unchanged. Equation (22) now becomes

$$E_{t+1}^i PV_{t+1} = \sum_{s=1}^{\infty} \frac{\hat{D}_{t+s+1} - \frac{1}{R}\gamma\sigma^2 \hat{X}_{t+s+1}}{R^s} + \mathbf{d}' E_{t+1}^i \boldsymbol{\epsilon}_{t+1}^d - \frac{1}{R}\gamma\sigma^2 \mathbf{e}' E_{t+1}^i \boldsymbol{\epsilon}_{t+1}^x, \tag{B5}$$

where $\hat{X}_{t+s+1}$ now is equal to the known component of X_{t+s+1} at $t+1$, e.g., $\hat{X}_{t+2} = f_{T+2}\varepsilon_{t-T+1}^x + f_{T+3}\varepsilon_{t-T}^x + \ldots$, and $\mathbf{e}$ is a vector of length T with element $T+1-i$ equal to $\sum_{s=1}^{\infty} \frac{1}{R^s} f_{s+i}$.

In Appendix A, we showed that in the general case where $X_t = F(L)\varepsilon_t^x$ the equilibrium price still takes the conjectured form (15), with the polynomial $B(L)$ now being infinitely lagged. Given this equilibrium price all results of Section 3.3 go through unchanged. From (18) we then have

$$E_{t+1}^i \boldsymbol{\epsilon}_{t+1}^x = \mathbf{B}^{-1}\mathbf{p}_{t+1} - \mathbf{B}^{-1}\mathbf{A} E_{t+1}^i \boldsymbol{\epsilon}_{t+1}^d. \tag{B6}$$

Substituting this into (B5), we have

$$E_{t+1}^i PV_{t+1} = \sum_{s=1}^{\infty} \frac{\hat{D}_{t+s+1} - \frac{1}{R}\gamma\sigma^2 \hat{X}_{t+s+1}}{R^s} + \tilde{\mathbf{d}}' E_{t+1}^i \boldsymbol{\epsilon}_{t+1}^d - \tilde{\mathbf{e}}'\mathbf{p}_{t+1}, \tag{B7}$$

where $\tilde{\mathbf{d}}' = \mathbf{d}' + \frac{1}{R}\gamma\sigma^2\mathbf{e}'\mathbf{B}^{-1}\mathbf{A}$ and $\tilde{\mathbf{e}}' = \frac{1}{R}\gamma\sigma^2\mathbf{e}'\mathbf{B}^{-1}$. Substituting (21) at $t+1$ then gives an equation analogous to (23):

$$E_{t+1}^i PV_{t+1} = \sum_{s=1}^{\infty} \frac{\hat{D}_{t+s+1} - \frac{1}{R}\gamma\sigma^2 \hat{X}_{t+s+1}}{R^s} + \boldsymbol{\theta}'\mathbf{V}_t^i + \boldsymbol{\beta}'\mathbf{v}_{t+1}^i + \boldsymbol{\gamma}'\mathbf{Z}_{t+1}, \tag{B8}$$

where $\boldsymbol{\theta}' = \tilde{\mathbf{d}}'\mathbf{M}_3$, $\boldsymbol{\beta}' = \tilde{\mathbf{d}}'\mathbf{M}_2$, $\tilde{\boldsymbol{\gamma}}' = \tilde{\mathbf{d}}'\mathbf{M}_1 + (\mathbf{0}_T, -\tilde{\mathbf{e}}')$. Here $\mathbf{0}_T$ is a $1 \times$ T vector of zeros. Equation (B8) has the same form as (23) with the exception for the extra term involving $\hat{X}$. This term is the same for all investors and plays no role in the proof of Proposition 1, so that the proposition continues to go through in this more general case. The five corollaries in the paper follow directly from Proposition 1 and therefore continue to go through as well. □

APPENDIX C: THE HIGHER ORDER WEDGE FOR $T = 2$

Without going over the extensive algebra, implementation of (28) for the case where $T = 2$ delivers (36) where

$$\delta_1 = \pi \frac{\sigma_d^2 \sigma_v^2 g_1}{G} \left(\sigma_v^2 \sigma_d^2 + \left(\sigma_v^2 + \sigma_d^2\right) g_1^2 \sigma_x^2\right) \tag{C1}$$

$$\delta_2 = \pi \frac{\sigma_d^4 \sigma_v^4 g_2}{G} \tag{C2}$$

$$\delta_3 = -\pi \frac{1}{G} \left(g_1^4 \sigma_v^2 \sigma_x^4 \left(\sigma_v^2 + \sigma_d^2\right) + \sigma_d^2 \sigma_v^4 \sigma_x^2 \left(g_1^2 + g_2^2\right)\right) \tag{C3}$$

$$\delta_4 = -\pi \frac{\sigma_d^2 \sigma_v^4 g_1 g_2 \sigma_x^2}{G} \tag{C4}$$

and where

$$G = \sigma_d^4 \sigma_v^4 + \left(2g_1^2 + g_2^2\right) \left(\sigma_d^2 + \sigma_v^2\right) \sigma_d^2 \sigma_v^2 \sigma_x^2 + \left(\sigma_d^2 + \sigma_v^2\right)^2 g_1^4 \sigma_x^4 > 0 \tag{C5}$$

$$g_1 = \frac{b_1}{a_1} < 0 \tag{C6}$$

$$g_2 = \frac{a_1 b_2 - a_2 b_1}{a_1^2}. \tag{C7}$$

Numerically g_2 is always positive since b_2 is substantially smaller in absolute size than b_1. Therefore $\delta_1 < 0$, $\delta_2 > 0$, $\delta_3 < 0$, and $\delta_4 > 0$.

LITERATURE CITED

Allen, Franklin, Stephen Morris, and Andrew Postlewaite. (1993) "Finite Bubbles with Short Sale Constraints and Asymmetric Information." *Journal of Economic Theory,* 61, 206–29.

Allen, Franklin, Stephen Morris, and Hyun Song Shin. (2006) "Beauty Contests and Iterated Expectations in Asset Markets." *Review of Financial Studies,* 19, 719–52.

Amato, Jeffery D., and Hyun Song Shin. (2003) "Public and Private Information in Monetary Policy Models." BIS Working Paper No. 138.

Bacchetta, Philippe, and Eric van Wincoop. (2003) "Can Information Heterogeneity Explain the Exchange Rate Determination Puzzle?" NBER Working Paper 9498.

Bacchetta, Philippe, and Eric van Wincoop. (2006) "Can Information Heterogeneity Explain the Exchange Rate Determination Puzzle?" *American Economic Review,* 96, 552–76.

Barberis, Nicholas, and Richard Thaler. (2003) "A Survey of Behavioral Finance." In *Handbook of the Economics of Finance*, edited by George Constantinides, Milton Harris, and Rene Stulz, pp. 1053–128. Amsterdam: Elsevier.

Biais, Bruno, and Peter Bossaerts. (1998) "Asset Prices and Trading Volume in a Beauty Contest." *Review of Economic Studies,* 65, 307–40.

Brennan, Michael J., and H. Henry Cao. (1997) "International Portfolio Investment Flows." *Journal of Finance,* 52, 1851–80.

Brown, David P., and Robert H. Jennings. (1989) "On Technical Analysis." *Review of Financial Studies,* 2, 527–52.

Brunnermeier, Markus K. (2001) *Asset Pricing under Asymmetric Information*. Oxford, UK: Oxford University Press.

Campbell, John Y., Andrew W. Lo, and A. Craig Mackinlay. (1997) *The Econometrics of Financial Markets*. Princeton, NJ: Princeton University Press.

Feldman, Mark, and Christian Gilles. (1985) "An Expository Note on Individual Risk without Aggregate Uncertainty." *Journal of Economic Theory,* 35, 26–32.

Foster, F. Douglas, and S. Viswanathan. (1996) "Strategic Trading When Agents Forecast the Forecasts of Others." *Journal of Finance,* 51, 1437–78.

Futia, Carl A. (1981) "Rational Expectations in Stationary Linear Models." *Econometrica,* 49, 171–92.

Harrison, J. Michael, and David M. Kreps. (1978) "Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations." *Quarterly Journal of Economics,* 92, 323–36.

He, Hua, and Jiang Wang. (1995) "Differential Information and Dynamic Behavior of Stock Trading Volume." *Review of Financial Studies,* 8, 919–72.

Hellwig, Christian. (2003) "Public Announcements, Adjustment Delays and the Business Cycle." Mimeo, UCLA.

Hellwig, Martin. (1980) "On the Aggregation of Information in Competitive Markets." *Journal of Economic Theory,* 22, 477–98.

Hirshleifer, David. (2001) "Investor Psychology and Asset Pricing." *Journal of Finance,* 56, 1533–97.

Judd, Kenneth L. (1985) "The Law of Large Numbers for a Continuum of IID Random Variables." *Journal of Economic Theory,* 35, 19–25.

Kasa, Kenneth, Todd B. Walker, and Charles H. Whiteman. (2007) "Asset Prices in a Time Series Model with Perpetually Disparately Informed, Competitive Traders." Mimeo.

Keynes, John Maynard. (1936) *The General Theory of Employment, Interest and Money*. London: MacMillan.

Morris, Stephen, and Hyun Song Shin. (2002) "Social Values of Public Information." *American Economic Review,* 92, 1521–34.

Pearlman, Joseph G., and Thomas J. Sargent. (2005) "Knowing the Forecasts of Others." *Review of Economic Dynamics,* 8, 480–97.

Scheinkman, Jose A., and Wei Xiong. (2003) "Overconfidence and Speculative Bubbles." *Journal of Political Economy,* 6, 1183–219.

Shiller, Robert. (1981) "Do Stock Prices Move Too Much to Be Justified by Subsequent Changes in Dividends?" *American Economic Review,* 71, 421–36.

Singleton, Kenneth J. (1987) "Asset Prices in a Time-Series Model with Disparately Informed, Competitive Traders." In *New Approaches to Monetary Economics*, edited by W. A. Barnett and Kenneth J. Singleton, pp. 249–72. Cambridge, UK: Cambridge University Press.

Sun, Yeneng. (2006) "The Exact Law of Large Numbers via Fubini Extension and Characterization of Uninsurable Risks." *Journal of Economic Theory,* 126, 31–69.

Townsend, Robert M. (1983) "Forecasting the Forecasts of Others." *Journal of Political Economy,* 91, 546–88.

Vives, Xavier. (1995) "Short-Term Investment and the Informational Efficiency of the Market." *Review of Financial Studies,* 8, 125–60.

Vives, Xavier. (2008) *Information and Learning in Markets*. Princeton, NJ: Princeton University Press.

Walker, Todd B. (2007a) "How Equilibrium Prices Reveal Information in Time Series Models with Disparately Informed, Competitive Traders." *Journal of Economic Theory*, 137, 512–37.

Walker, Todd B. (2007b) "Multiple Equilibria in a Simple Asset Pricing Model." *Economic Letters*, 97, 191–96.

Wang, Jiang. (1993) "A Model of Asset Prices under Asymmetric Information." *Review of Economic Studies,* 60, 249–82.

Wang, Jiang. (1994) "A Model of Competitive Stock Trading Volume." *Journal of Political Economy,* 102, 127–68.

Woodford, Michael. (2003) "Imperfect Common Knowledge and the Effects of Monetary Policy." In *Knowledge, Information and Expectations in Modern Macroeconomics: In Honor of Edmund S. Phelps*, edited by Philippe Aghion, R. Frydman, J. Stiglitz, and M. Woodford, pp. 25–58. Princeton, NJ: Princeton University Press.