CBDC as Imperfect Substitute to Bank Deposits: a Macroeconomic Perspective. Online Appendix

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A Model with Cash

We now present a model including three types of money: bank deposits, cash and CBDC. In our modeling, central-bank-issued money d^c is a composite of cash and CBDC

$$d_t^c = (\alpha_{cash}(cash_t)^{\frac{\varepsilon-1}{\varepsilon}} + \alpha_{cbdc}(cbdc_t)^{\frac{\varepsilon-1}{\varepsilon}})^{\frac{\varepsilon}{\varepsilon-1}} \tag{A.1}$$

with

$$\alpha_{cash}^{\varepsilon} + \alpha_{cbdc}^{\varepsilon} = 1 \tag{A.2}$$

The transaction cost of consumption $s(x_t)$ is still a function of money velocity $x = pc/d$ and the composite money instrument d_t is still given by

$$d_t = \left(\alpha_c(d_t^c)^{\frac{\epsilon_{cb}-1}{\epsilon_{cb}}} + \alpha_b(d_t^b)^{\frac{\epsilon_{cb}-1}{\epsilon_{cb}}}\right)^{\frac{\epsilon_{cb}}{\epsilon_{cb}-1}} \tag{A.3}$$

as in (1). However d_t^c is now reinterpreted as the composite (A.1), while d_t^b is still the composite of bank deposits (2). Cash pays zero interest and CBDC pays interest r_t^{cbdc}.

The household budget constraint is now

$$\begin{aligned}(1-\tau_h)w_t h_t + (1+r_t^*)a_{t-1} + \int(1+r_{t-1}^b(j))d_{t-1}^b(j)dj + (1+r_{t-1}^{cbdc})cbdc_{t-1} + cash_{t-1}\\ + \zeta(1-\tau_b)\Pi_t^b = p_t c_t(1+s_t) + \int d_t^b(j)dj + cbdc_t + a_t + cash_t + p_t t_t \end{aligned} \tag{A.4}$$

First-order conditions (42),(43),(44),(46) are unchanged, however the FOC with respect to the central-bank-issued money (45) needs to be replaced with two conditions, with respect to cash and CBDC, respectively

$$\lambda_t\left(1-(Ax_t^2-B)\alpha_c\alpha_{cash}\left(\frac{d}{d_c}\right)^{\frac{1}{\epsilon_{cb}}}\left(\frac{d_c}{cash}\right)^{\frac{1}{\varepsilon}}\right) = \frac{\lambda_{t+1}}{c_{t+1}(1+2Ax_{t+1}-2\sqrt{AB})} \tag{A.5}$$

$$\lambda_t\left(1-(Ax_t^2-B)\alpha_c\alpha_{cbdc}\left(\frac{d}{d_c}\right)^{\frac{1}{\epsilon_{cb}}}\left(\frac{d_c}{cbdc}\right)^{\frac{1}{\varepsilon}}\right) = \frac{\lambda_{t+1}(1+r^{cbdc})}{c_{t+1}(1+2Ax_{t+1}-2\sqrt{AB})} \tag{A.6}$$

(A.5)and (A.6), together with (46), imply that the optimal cash and CBDC holdings satisfy

$$\frac{cash_t}{cbdc_t} = \left(\frac{\alpha_{cash}}{\alpha_{cbdc}}\frac{r_t^* - r_t^{cbdc}}{r_t^*}\right)^{\varepsilon} \tag{A.7}$$

and imply the equilibrium relationship

$$cash_t r_t^* + cbdc_t(r_t^* - r_t^{cbdc}) = d_t^c(r^* - r_t^c) \tag{A.8}$$

where r_t^c is now defined via the relationship

$$(r^* - r_t^c) = \left((r^*)^{1-\varepsilon}\alpha_{cash}^{\varepsilon} + (r_t^* - r_t^{cbdc})^{1-\varepsilon}\alpha_{cbdc}^{\varepsilon}\right)^{\frac{1}{1-\varepsilon}} \tag{A.9}$$

Furthermore, from the Euler equations (44) and (A.6) we have

$$\frac{d_t^c}{d_t^b} = \left(\frac{\alpha_c}{\alpha_b}\ \alpha_{cbdc}\left(\frac{d_t^c}{cbdc_t}\right)^{\frac{1}{\varepsilon}}\frac{r_t^* - r_t^b}{r_t^* - r_t^{cbdc}}\right)^{\varepsilon} \tag{A.10}$$

By combining (A.7) and (A.1)

$$d_t^c = \frac{cbdc_t}{\alpha_b^{\varepsilon}}\left(\frac{r_t^* - r_t^{cbdc}}{r - r^c}\right)^{\varepsilon} \tag{A.11}$$

Inserting (A.11) in (A.10) we re-obtain the relationship

$$d_t^b = \left(\frac{\alpha_b}{\alpha_c} \times \frac{r_t^* - r_t^c}{r_t^* - r_t^b}\right)^{\epsilon_{cb}} d_t^c \tag{A.12}$$

showing that in equilibrium resources are split between bank deposits and the "basket" of cash and CBDC the same way that they were split between bank deposits and CBDC in the model with two instruments. (A.7) also implies that he total opportunity cost of holding money is

$$d_t^b(r^* - r_t^b) + cash_t r_t^* + cbdc_t(r_t^* - r_t^{cbdc}) = d_t^b(r^* - r_t^b) + d_t^c(r^* - r_t^c) \tag{A.13}$$

This analysis shows that with cash as a third instrument, economic outcomes may be unchanged. Cash pays zero interest by construction. If the composite interest defined by (A.9), that can be interpreted as the interest paid by the "basket" of cash and CBDC, equals the interest paid by CBDC in the model with only two instruments, all outcomes are identical: households allocate the same resources in money instruments – implying that they incur the same transaction cost of consumption – and pay the same opportunity cost (A.13) of holding money.

B Model with Deposits Affecting Banks' Marginal Funding Cost

In this extended model, as in the baseline model, each bank j can borrow in the form of deposits d_j^b or bonds. However, in the spirit of the model of Wang, Whited, Wu and

Xiao (2022), we now posit that the cost of "non-reservable borrowing" is increasing in the amount borrowed in this form, as a fraction of total borrowing.[23] In particular, we posit that, if bank j needs to finance loans $l(j)$ and deposits are $d^b(j) < l(j)$, purchasers of the additional debt $l(j) - d^b(j)$ demand an interest equal to $r^* + \frac{\chi}{2}\left(\frac{l(j)-d^b(j)}{l(j)}\right)$. As in the baseline model, at time $t-1$ banks choose the rates $r^b_{t-1}(j)$ and $r^l_{t-1}(j)$ to maximize time-t profits (B.1), subject to the deposit demand (9). Profits are now

$$\begin{aligned}
\Pi^b_t(j) &= [(1-\phi)r^*_{t-1} + \phi r^m_{t-1} - (r^b_{t-1}(j) + c^b)]d^b_{t-1}(j) + [r^l_{t-1}(j) - c^l - r^*_{t-1}]l_{t-1}(j) \\
&- \frac{\chi}{2}\left(\frac{l_{t-1}(j) - d^b_{t-1}(j)}{l_{t-1}(j)}\right)(l_{t-1}(j) - d^b_{t-1}(j)) \\
&= [r^*_{t-1} - r^b_{t-1}(j) - \tilde{c}^b_{t-1}]d^b_{t-1}(j) + [r^l_{t-1}(j) - c^l - r^*_{t-1}]l_{t-1}(j) \\
&- \frac{\chi}{2}\left(\frac{l_{t-1}(j) - d^b_{t-1}(j)}{l_{t-1}(j)}\right)(l_{t-1}(j) - d^b_{t-1}(j)) \qquad \text{(B.1)}
\end{aligned}$$

where in the last equality we defined $\tilde{c}^b \equiv c^b + \phi(r^* - r^m)$, assuming for simplicity the spread $(r^* - r^m)$ to be constant. As in the baseline model, banks maximize profits subject to the deposit demand (9) and the loan demand (24). The difference now is that the choice of the two rates are not independent. The FOCs of the bank's problem are

$$\begin{aligned}
(1-\epsilon_b)d^b_{t-1}\left(\frac{r^*_{t-1} - r^b_{t-1}(j)}{r^*_{t-1} - r^b_{t-1}}\right)^{-\epsilon_b} &+ \epsilon_b d^b_{t-1}(\tilde{c}^b_{t-1} - \chi)\frac{(r^*_{t-1} - r^b_{t-1}(j))^{-\epsilon_b - 1}}{(r^*_{t-1} - r^b_{t-1})^{-\epsilon_b}} \\
&+ \epsilon_b \chi \frac{(d^b)^2}{l}\frac{(r^*_{t-1} - r^b_{t-1}(j))^{-2\epsilon_b - 1}}{(r^*_{t-1} - r^b_{t-1})^{-2\epsilon_b}}\left(\frac{r^l_{t-1}(j)}{r^l_{t-1})}\right)^{\epsilon_l} = 0 \qquad \text{(B.2)} \\
(1-\epsilon_l)l\left(\frac{r^l_{t-1}(j)}{r^l_{t-1}}\right)^{-\epsilon_l} &+ \epsilon_l d^b_{t-1}\left(r^*_{t-1} + c^l + \frac{\chi}{2}\right)\frac{(r^l_{t-1}(j))^{-\epsilon_l - 1}}{(r^l_{t-1})^{-\epsilon_l}} \\
&- \epsilon_l \frac{\chi}{2}\frac{(d^b)^2}{l}\left(\frac{r^*_{t-1} - r^b_{t-1}(j)}{r^*_{t-1} - r^b_{t-1}}\right)^{-2\epsilon_b}\left(\frac{r^l_{t-1}(j)}{r^l_{t-1})}\right)^{\epsilon_l} = 0 \qquad \text{(B.3)}
\end{aligned}$$

(remember that rates, loans and deposits carrying the argument j refer to an individual bank, whereas those not carrying the argument j refer to the aggregate quantities).

[23] Wang, Whited, Wu and Xiao (2022) motivate the additional cost with the fact that non-reservable borrowing does not benefot from FDIC insurance, hence purchasers of this debt carry default risk. It is worth noticing that, if the introduction of CBDC causes banks' disintermediation, it is possible that the government would extend some form of insurance to some forms of bank debt other than deposits. In this case our baseline model, where the marginal funding cost is the risk-free rate, would be more appropriate.

In a symmetric equilibrium in which all banks choose the same deposit rate $r^b(j) = r^b$ and loan rate $r^l(j) = r^l$, the solution to (B.2)-(B.3) is

$$r^b_{t-1}(j) = r^*_{t-1} - \frac{\epsilon_b}{\epsilon_b - 1}\left(\tilde{c}^b_{t-1} - \chi\left(1 - \frac{d^b_{t-1}}{l_{t-1}}\right)\right) \tag{B.4}$$

$$r^l_{t-1}(j) = \frac{\epsilon_l}{\epsilon_l - 1}\left(r^*_{t-1} + c^l + \frac{\chi}{2}\left(1 - \left(\frac{d^b_{t-1}}{l_{t-1}}\right)^2\right)\right) \tag{B.5}$$

As the introduction of CBDC decreases the demand for bank deposits, banks react in two ways. On the one hand banks increase non-reservable borrowing, which pushes up the interest on this part of the debt and which in turn induces banks to increase the loan rate (see (B.5)), with a negative effect on loan demand (61). On the other hand banks also increase the deposit rate (B.4), in order to attract a higher deposit demand and thus mitigate the increase in their marginal funding cost.

To evaluate the effect of introducing CBDC in this model we set $\chi = 0.5\%$, in line with the estimation of Wang, Whited, Wu and Xiao (2022).

Table B: CBDC-induced changes in the economy

τ_h=**25%**	"case a"	"case b"	τ_h=**45%**	"case a"	"case b"
HH Cons.	+24 bps (+27)	+46 bps (+54)	HH Cons.	+30 bps (+41)	+49 bps (+62)
Bankers' Cons.	/	-130 bps (−119)	Bankers' Cons.	/	-140 bps (−117)
Deposit rate	3.08% (2%)	3.45% (2%)	Dep. rate	3.29% (2%)	3.70% (2%)
Loan rate	5.14% (5%)	5.17% (5%)	Loan rate	5.16% (5%)	5.19% (5%)
Welfare	+13 bps (+9)	+37 bps (+40)	Welfare	+23 bps (+20)	+44 bps (+47)

Table B shows the effects of the introduction of CBDC on the economy in the new model. For comparison, the table reports also the impact in the baseline model (in parenthesis). Notice that, when evaluating the new model, we adjust the model parameters ϵ_b and ϵ_l to keep the pre-CBDC deposit rate at 2% and the pre-CBDC loan rate at 5%. After CBDC is introduced, in the new model the deposit rate increases by more than 1 percentage point, whereas the loan rate increases by less than 20 bps. This contrasts with the baseline model, in which these two rates are unchanged after the introduction of CBDC (see Lemma 1).

The modest increase in the loan rate after the introduction of CBDC translates into an even smaller increase of the cost of capital $r^K = (1-\varphi)r^* + \varphi r^l$: since in our calibration the working capital requirement is $\varphi = 0.2$ the cost of capital increases by at most 5 bps. On the other hand, the increase in the deposit rate has a beneficial effect on consumption and welfare.

Overall, the effect of the inroduction of CBDC on consumption and welfare is very similar in the two models. In fact, we see that in some cases welfare increases even slightly more in the new model than in the baseline one: the slightly higher cost of capital has a negative impact on consumption but also on labor, with the result that in some cases consumption may increase less than in the baseline model but welfare may increase more.

C Proof of Proposition 1

If $\alpha_b^{\epsilon_{cb}}\epsilon_{cb} > 1$ and the marginal cost of managing deposits is negligible:

- a) *The interest rate r^c that maximizes seigniorage is larger than the interest rate on deposits r^b,*

  Neglecting c^c, the component of seigniorage due to CBDC is $\mathcal{S}^{cbdc} = (r^* - r^c)d^c$. Given equations (11), (17) and (13), demand for CBDC can be written as

  $$\frac{d^c}{pc} = \alpha_c^{\epsilon_{cb}}(r^* - r^c)^{-\epsilon_{cb}}(r^* - r^{comp})^{\epsilon_{cb}}\sqrt{\frac{A(1+r^*)}{r^* - r^{comp} + B(1+r^*)}} \tag{C.1}$$

  so that

  $$\mathcal{S}^{cbdc} = \alpha_c^{\epsilon_{cb}} pc(r^* - r^c)^{1-\epsilon_{cb}}(r^* - r^{comp})^{\epsilon_{cb}}\sqrt{\frac{A(1+r^*)}{r^* - r^{comp} + B(1+r^*)}} \tag{C.2}$$

  with $(r^* - r^{comp})$ given by (14). $\mathcal{S}^{cbdc}$ takes a more convenient form in terms of the variable x, defined as

  $$x \equiv \left(\alpha_b^{\epsilon_{cb}} + \alpha_c^{\epsilon_{cb}}\frac{(r^* - r^c)^{1-\epsilon_{cb}}}{(r^* - r^b)^{1-\epsilon_{cb}}}\right)^{\frac{1}{1-\epsilon_{cb}}} \tag{C.3}$$

  In terms of x, we can write $\mathcal{S}^{cbdc}$ as

  $$\mathcal{S}^{cbdc} = k\ (x^{1-\epsilon_{cb}} - \alpha_b^{\epsilon_{cb}})x^{\epsilon_{cb}}(a + bx)^{-1/2} \tag{C.4}$$

  with $a \equiv B(1+r^*)$, $b \equiv (r^* - r^b)$ and $k \equiv pc(r^* - r^b)\sqrt{A(1+r^*)}$. Remember that $r^* - r^b$ is given per Lemma 1, so that x is basically a function of r^c.

Hence

$$\mathcal{S}^{cbdc} = k\ (x - \alpha_b^{\epsilon_{cb}} x^{\epsilon_{cb}})(a + bx)^{-1/2} \tag{C.5}$$

The FOC wrt x is

$$\left(1 - \alpha_b^{\epsilon_{cb}} \epsilon_{cb} x^{\epsilon_{cb}-1} - \frac{1}{2}(x - \alpha_b^{\epsilon_{cb}} x^{\epsilon_{cb}}) \frac{b}{(a + bx)}\right) = 0 \tag{C.6}$$

It is easy to see that the term $\frac{1}{2}(x - \alpha_b^{\epsilon_{cb}} x^{\epsilon_{cb}}) \frac{b}{(a+bx)}$ is positive, since it is equal to $\frac{b}{2k}(a + bx)^{-\frac{1}{2}} \mathcal{S}^{cbdc}$ and seigniorage is positive for any value of $r^c < r^*$ (a, b, k and x are also all positive). Hence (C.6) shows that, for x to be an interior maximum of seigniorage,[24] it must be

$$\alpha_b^{\epsilon_{cb}} \epsilon_{cb} x^{\epsilon_{cb}-1} < 1 \tag{C.7}$$

So if $\alpha_b^{\epsilon_{cb}} \epsilon_{cb} > 1$, then it must be $x^{\epsilon_{cb}-1} < 1$. Hence, since $\epsilon_{cb} > 1$, it must be $x < 1$. From the definition of x (C.3) and the relationship $\alpha_b^{\epsilon_{cb}} + \alpha_b^{\epsilon_{cb}} = 1$, it follows that, in order for $x < 1$, we need $(r^* - r^c) < (r^* - r^b)$, or $r^c > r^b$.

- b) *The optimal value of r^c is decreasing in α_c, if $\epsilon_{cb} > 1.5$*

Define $\tilde{\alpha}_b \equiv \alpha_b^{\epsilon_{cb}}$, and

$$\mu(x, \tilde{\alpha}_b) = \left(1 - \tilde{\alpha}_b \epsilon_{cb} x^{\epsilon_{cb}-1} - \frac{1}{2}(x - \tilde{\alpha}_b x^{\epsilon_{cb}}) \frac{b}{(a + bx)}\right) \tag{C.8}$$

where x is defined in (C.3). For each value of $\tilde{\alpha}_b$ the value of x maximizing seigniorage we have $\mu(x(\tilde{\alpha}_b), \tilde{\alpha}_b) = 0$.

The implicit value theorem tells us how the seigniorage-maximizing value of x varies with $\tilde{\alpha}_b$:

$$\frac{\partial x}{\partial \tilde{\alpha}_b} = -\frac{\frac{\partial \mu}{\partial \tilde{\alpha}_b}}{\frac{\partial \mu}{\partial x}} \tag{C.9}$$

and we have

$$\frac{\partial \mu}{\partial \tilde{\alpha}_b} = -x^{\epsilon_{cb}-1}\left(\epsilon_{cb} - \frac{1}{2}\frac{bx}{(a + bx)}\right) < 0 \tag{C.10}$$

The above inequality is obvious since $\epsilon_{cb} > 1$ and $0 < \frac{bx}{a+bx} < 1$ (the latter holds since a, b and x are positive).

$$\frac{\partial \mu}{\partial x} = \tilde{\alpha}_b x^{\epsilon_{cb}-2}\left(-\epsilon_{cb}\left[(\epsilon_{cb} - 1) - \frac{1}{2}\frac{bx}{a + bx}\right] - \frac{1}{2}\frac{(bx)^2}{(a + bx)^2}\right) - \frac{1}{2}\frac{ab}{(a + bx)^2} \tag{C.11}$$

[24] Seigniorage is a continuous function of r^c for $0 \leq r^c \leq r^*$ so it must have a maximum in this range. It can be verified that seigniorage is increasing at $r^c = 0$ and decreasing at $r^c = r^*$, so the maximum must be an interior one.

Since $0 < \frac{bx}{a+bx} < 1$, the term in square parenthesis on the RHS of (C.11) is positive for $\epsilon_{cb} > 1.5$. If so, then it is easy to see $\frac{\partial \mu}{\partial x} < 0$. Hence we have

$$\frac{\partial x}{\partial \tilde{\alpha}_b} = -\frac{\frac{\partial \mu}{\partial \tilde{\alpha}_b}}{\frac{\partial \mu}{\partial x}} < 0 \tag{C.12}$$

Since $\tilde{\alpha}_c \equiv \alpha_c^{\epsilon_{cb}} = 1 - \alpha_b^{\epsilon_{cb}}$, we deduce that the seigniorage-maximizing value of x is increasing in α_c (and $\tilde{\alpha}_c$), hence, from the definition of x (C.3), the seigniorage-maximizing CBDC spread $r^* - r^c$ is increasing in α_c,[25] or equivalently, the seigniorage-maximizing r^c is decreasing in α_c.

- c) *The peak value of seigniorage in the r^c dimension is increasing in the CBDC liquidity parameter $\tilde{\alpha}_c = \alpha_c^{\epsilon_{cb}}$, and is increasing in the substitutability parameter ϵ_{cb} while $\tilde{\alpha}_c$ remains fixed.*

  (C.5) implies that the component of seigniorage due to CBDC can be written as

$$\mathcal{S}^{cbdc} = k\ (x - (1 - \tilde{\alpha}_c)x^{\epsilon_{cb}})(a + bx)^{-1/2} \tag{C.13}$$

  The seigniorage-maximizing value of x is a function of $\tilde{\alpha}_c$ and ϵ_{cb} (as specified in the statement of the proposition, we think of $\tilde{\alpha}_c$ and ϵ_{cb} as two independent dimensions), hence $\mathcal{S}^* = \mathcal{S}^*(x(\tilde{\alpha}_c, \epsilon_{cb}), \tilde{\alpha}_c, \epsilon_{cb})$ where $\mathcal{S}^* = max_{r^c}\mathcal{S}$. By the envelope theorem peak seigniorage satisfies

$$\frac{d\mathcal{S}^*}{d\tilde{\alpha}_c} = \frac{\partial \mathcal{S}^*}{\partial \tilde{\alpha}_c} \tag{C.14}$$

$$\frac{d\mathcal{S}^*}{d\epsilon_{cb}} = \frac{\partial \mathcal{S}^*}{\partial \epsilon_{cb}} \tag{C.15}$$

  We have $\frac{\partial \mathcal{S}^*}{\partial \tilde{\alpha}_c} = x^{\epsilon_{cb}}(a+bx)^{-1/2} > 0$ and $\frac{\partial \mathcal{S}^*}{\partial \epsilon_{cb}} = -log(x)(1-\tilde{\alpha}_c)x^{\epsilon_{cb}}(a+bx)^{-1/2}$ which is positive when $x < 1$. Since, as proved in point a), the seigniorage-maximizing value of x is smaller than 1, we have indeed that $\frac{\partial \mathcal{S}^*}{\partial \tilde{\alpha}_c} > 0$ and $\frac{\partial \mathcal{S}^*}{\partial \epsilon_{cb}} > 0$, i.e. the peak value of seigniorage in the r^c dimension is increasing in $\tilde{\alpha}_c$ and ϵ_{cb}.

D Proof of Proposition 2

Suppose that the government can choose the elasticity of substitution ϵ_{cb}, but the relative

[25]Using $\alpha_b^{\epsilon_{cb}} + \alpha_b^{\epsilon_{cb}} = 1$ we can write $x = \left(1 + \alpha_c^{\epsilon_{cb}}\left(\frac{(r^*-r^c)^{1-\epsilon_{cb}}}{(r^*-r^b)^{1-\epsilon_{cb}}} - 1\right)\right)^{\frac{1}{1-\epsilon_{cb}}}$. Given that the optimal r^c is such that $r^* - r^c < r^* - r^b$ (proved in point a) of this Proposition), and $\epsilon_{cb} > 1$, the only way that x can be increasing in α_c is if $r^* - r^c$ is increasing in α_c.

liquidity between CBDC and bank deposits ($\alpha_c^{\epsilon_{cb}}/\alpha_b^{\epsilon_{cb}}$) is fixed. Under the conditions of Proposition 1 the maximum of seigniorage is achieved in the limit $\epsilon_{cb} \to \infty$ (so that the two monies are perfect substitutes) and r^c infinitesimally higher than r^b.

First, notice that the assumption that $\alpha_c^{\epsilon_{cb}}/\alpha_b^{\epsilon_{cb}}$ is fixed, together with the conditions $\alpha_b^{\epsilon_{cb}} + \alpha_c^{\epsilon_{cb}} = 1$, $0 < \alpha_b^{\epsilon_{cb}} < 1$ and $0 < \alpha_c^{\epsilon_{cb}} < 1$, which we hold throughout the paper (see (3)), implies that both $\alpha_b^{\epsilon_{cb}}$ and $\alpha_c^{\epsilon_{cb}}$ are fixed, and that $lim_{\epsilon_{cb}\to\infty}\alpha_b = lim_{\epsilon_{cb}\to\infty}\alpha_c = 1$.

Next, consider that, as per point a) of Proposition 1, for a given value of ϵ_{cb} the optimal r^c is larger than r^b. Moreover, as per point c) of Proposition 1), the peak value of seigniorage in the r^c dimension is increasing in the ϵ_{cb} dimension. So the maximum value of seigniorage (across the ϵ_{cb} and r^c dimensions) is in the region/limit $r^c > r^b$ and $\epsilon_{cb} \to \infty$. We first show that in the above region/limit $(r^* - r^{comp}) \to (r^* - r^c)$. We have

$$(r^* - r^{comp}) = \alpha_c^{\frac{\epsilon_{cb}}{1-\epsilon_{cb}}}(r^* - r^c)\left(1 + \frac{\alpha_b^{\epsilon_{cb}}}{\alpha_c^{\epsilon_{cb}}}\frac{(r^* - r^b))^{1-\epsilon_{cb}}}{(r^* - r^c))^{1-\epsilon_{cb}}}\right)^{\frac{1}{1-\epsilon_{cb}}} \tag{D.1}$$

Since $lim_{\epsilon_{cb}\to\infty}\alpha_c = 1$, $lim_{\epsilon_{cb}\to\infty}\alpha_c^{\frac{\epsilon_{cb}}{1-\epsilon_{cb}}} = 1$.

Moreover, with $r^c > r^b$, $\left(1 + \frac{\alpha_b^{\epsilon_{cb}}}{\alpha_c^{\epsilon_{cb}}}\frac{(r^* - r^b))^{1-\epsilon_{cb}}}{(r^* - r^c))^{1-\epsilon_{cb}}}\right)^{\frac{1}{1-\epsilon_{cb}}} \to 1$ for $\epsilon_{cb} \to \infty$. We thus obtain $(r^* - r^{comp}) \to (r^* - r^c)$. Intuitively, for $\epsilon_{cb} \to \infty$, bank deposits and CBDC are perfect substitutes and, as long as $(r^* - r^c) < (r^* - r^b)$, bank deposits are driven out of the market, the composite liquid asset becomes effectively CBDC and the composite rate becomes the CBDC rate.

Hence, in our region of interest ($r^c > r^b$ and $\epsilon_{cb} \to \infty$) the component of seigniorage due to CBDC (C.2) becomes

$$\mathcal{S}^{cbdc} = pc\ \alpha_c^{\epsilon_{cb}}(r^* - r^c)\sqrt{\frac{A(1+r^*)}{r^* - r^c + B(1+r^*)}} \tag{D.2}$$

(D.2) shows that in this region $\mathcal{S}^{cbdc}$ is decreasing in r^c, so the supremum is reached as r^c approaches r^b from above.

E Properties of the optimal interest on CBDC

In section 3 we presented some qualitative arguments suggesting that the optimal interest on CBDC is decreasing in government spending and the share of banks held by households. The numerical results in Section 5 are consistent with this hypothesis.

In this section we present an analytical proof of this statement under some conditions. In particular, one of the conditions we need to impose is related to the effect of a change in the CBDC rate on labor supply. A change in r^c affects the money velocity x, which in turn affects the transaction cost, and affects seigniorage, which in turn affects the tax rate. Condition C1 in Proposition E1 posits that the effect of r^c on labor supply through the transaction cost is negligible relative to the effect through the tax rate.

To state condition C1 in precise terms we need to introduce two quantities. First, define $\tilde{S}$ as the sum of all government revenues other than those coming from labor taxation, i.e. seigniorage plus revenues from taxation on profits:
$\tilde{S} \equiv \tau_b \hat{\Pi}^b + (r^* - r^m)\hat{m} + (r^* - r^c - c^c)\hat{d}^c = \tau_b \hat{\Pi}^b + \mathcal{S}$.
Second, define $\bar{h}$ as labor supply when the labor tax rate is zero: from (58) we have $\bar{h} = \frac{\hat{w}^{\frac{1}{\gamma}}}{c^{\frac{1}{\gamma}}(1+s(x)+xs'(x))^{\frac{1}{\gamma}}}$.

Condition C2 is used in the second part of the proof – related to the welfare-maximizing r^c – to simplify the algebra.

Proposition E1: The interest rate on CBDC that maximizes consumption and welfare is decreasing in government consumption and is decreasing in the share of banks held by households, under the following conditions:

- **C1:** $\left|\frac{\partial \tilde{S}}{\partial r^c}\right| >> \left|\hat{w}\bar{h}(2s'(x) + s''(x))\frac{\partial x}{\partial r^c}\right|$ **at the optimum.**
- **C2: Banker's consumption is big relative to household consumption, and/or bankers represent a small share of the population, so that** $\lambda \frac{c}{c^{bo}} \rightarrow 0$.
- **C3: The cost of managing CBDC** c^c **is negligible:** $c^c \rightarrow 0$.

1. The consumption-maximizing r^c

We first focus on the interest on CBDC that maximizes consumption. Let us first outline the general strategy to prove this proposition, and subsequently get into the details. The steady state equations, (51)-(64) allow us to find all the endogenous variables of the model in terms of the model parameters (including the parameter ζ, share of banks held by households), and the government policy variables g (government consumption) and r^c. Let us focus on on the budget constraint equation (55), that we rewrite here for convenience

$$c(1 + s(x)) = (1 - \tau_h)\hat{w}h + \upsilon^{hh}\hat{r} - \hat{d}^b(r^* - r^b) - \hat{d}^c(r^* - r^c) + \zeta(1 - \tau_b)\hat{\Pi}^b - \hat{t}$$

Using (62) and (64) we rewrite this equation as [26]

$$\nu(c, r^c, g, \zeta) \equiv c(1+s(x)) - \hat{w}h + g + c^c\hat{d}^c + \alpha\hat{d}^b - \zeta(1-\tau^b)(\hat{d}^b(r - \tilde{r}^b) + \hat{l}(r^* - r^l - c^l)) + k = 0 \tag{E.1}$$

with

$$\begin{aligned} \alpha &= (r^* - r^b) - \tau^b((r^* - \tilde{r}^b) \\ k &= \hat{t} - \upsilon^{hh}\,\hat{r} - \tau^b\,\hat{l}(r^* - r^l - c^l) \\ \tilde{r}^b &= r^b + c^b + \phi(r^* - r^m) \end{aligned}$$

ν can be seen as a function of the arguments $\{c,\ r^c\ g,\ \zeta\}$ since all the quantities appearing on the RHS of (E.1), given the other steady-state equations – (51)-(54) and (56)-(64) – are functions of these arguments (and of the other model parameters). The table below summarizes all the dependencies, as well as their directions *when fully unambiguous*, and references the corresponding steady-state equations (all the quantities appearing in (E.1) but not listed on this table depend only on model parameters).

Table 4		
variable	depends on	reference
x	$r^c\downarrow$	(54)
h	$c,\ r^c,\ g,$	(58)
$\hat{d}^b$	$c\uparrow,\ r^c\downarrow$	(56)
$\hat{d}^c$	$c\uparrow,\ r^c\uparrow$	(57)

(E.1) implicitly defines consumption as $c(r^c, g, \zeta)$. For any given value of g and ζ, the value of r^c that maximizes consumption needs to satisfy $\frac{\partial c}{\partial r^c} = 0$. By the implicit function theorem

$$\frac{\partial c}{\partial r^c} = -\frac{\frac{\partial \nu}{\partial r^c}}{\frac{\partial \nu}{\partial c}} \tag{E.2}$$

[26] use (64) to make the replacement $\tau_h\hat{w}h = g - \tau_b\hat{\Pi}^b - (r^* - r^m)\hat{m} - (r^* - r^c - c^c)\hat{d}^c$ and replace bank profits Π^b with its expression (62) in terms of profits from deposits and loans

Provided that $\frac{\partial \nu}{\partial c} \neq 0$,[27] (E.2) is satisfied iff

$$\mu(c, r^c, g, \zeta) \equiv \frac{\partial \nu}{\partial r^c} = 0 \tag{E.3}$$

(E.1) and (E.3) are two equations that implicitly define the pair $\{r^c, c\}$ at the optimal (consumption-maximizing) point, as functions of ζ and g. We can then use the implicit function theorem again to find how the value of $\{r^c, c\}$ solving the system (E.1) and (E.3) changes with ζ and g:

$$\begin{aligned} \begin{pmatrix} \frac{\partial r^c}{\partial g} & \frac{\partial r^c}{\partial \zeta} \\ \frac{\partial c}{\partial g} & \frac{\partial c}{\partial \zeta} \end{pmatrix} &= -\begin{pmatrix} \frac{\partial \nu}{\partial r^c} & \frac{\partial \nu}{\partial c} \\ \frac{\partial \mu}{\partial r^c} & \frac{\partial \mu}{\partial c} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \nu}{\partial g} & \frac{\partial \nu}{\partial \zeta} \\ \frac{\partial \mu}{\partial g} & \frac{\partial \mu}{\partial \zeta} \end{pmatrix} \\ &= -\begin{pmatrix} 0 & \frac{\partial \nu}{\partial c} \\ \frac{\partial \mu}{\partial r^c} & \frac{\partial \mu}{\partial c} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \nu}{\partial g} & \frac{\partial \nu}{\partial \zeta} \\ \frac{\partial \mu}{\partial g} & \frac{\partial \mu}{\partial \zeta} \end{pmatrix} \end{aligned} \tag{E.4}$$

where the last equality follows from the fact that $\frac{\partial \nu}{\partial r^c} = 0$ by (E.3). The two elements on the first row in the LHS matrix of (E.4), $\frac{\partial r^c}{\partial g}$ and $\frac{\partial r^c}{\partial \zeta}$, are the quantities we are interested in:

$$\frac{\partial r^c}{\partial g} = \frac{\frac{\partial \mu}{\partial c}\frac{\partial \nu}{\partial g}}{\frac{\partial \nu}{\partial c}\frac{\partial \mu}{\partial r^c}} - \frac{\frac{\partial \mu}{\partial g}}{\frac{\partial \mu}{\partial r^c}} \tag{E.5}$$

$$\frac{\partial r^c}{\partial \zeta} = \frac{\frac{\partial \mu}{\partial c}\frac{\partial \nu}{\partial \zeta}}{\frac{\partial \nu}{\partial c}\frac{\partial \mu}{\partial r^c}} - \frac{\frac{\partial \mu}{\partial \zeta}}{\frac{\partial \mu}{\partial r^c}} \tag{E.6}$$

The strategy to prove this proposition will be to show that the expressions in (E.5) and (E.6) are both negative. We further note that at the consumption-maximizing value of r^c it is necessary that $\frac{d^2 c}{d(r^c)^2} \leq 0$, and this implies $\frac{\partial \mu}{\partial (r^c)} \geq 0$.[28]

Hence

[27] $\frac{\partial \nu}{\partial c}$ is unambiguously positive (see (E.18)). Intuitively, increasing consumption violates the budget constraint, everything else equal.

[28] This is because, since $\frac{\partial c}{\partial r^c} = -\frac{\frac{\partial \nu}{\partial r^c}}{\frac{\partial \nu}{\partial c}}$,

$$\frac{\partial^2 c}{\partial (r^c)^2} = -\frac{\partial}{\partial r^c}\left(\frac{\frac{\partial \nu}{\partial r^c}}{\frac{\partial \nu}{\partial c}}\right) = \frac{\frac{\partial^2 \nu}{\partial (r^c)^2}}{\frac{\partial \nu}{\partial c}} - \frac{\frac{\partial \nu}{\partial r^c}}{\left(\frac{\partial \nu}{\partial c}\right)^2}\frac{\partial^2 \nu}{\partial c \partial r^c} \tag{E.7}$$

Given that $\frac{\partial \nu}{\partial r^c} = 0$ at the optimum, the second term in (E.7) is zero, and given that $\frac{\partial \nu}{\partial c} > 0$ (see (E.18)), for $\frac{\partial^2 c}{\partial (r^c)^2}$ to be non-positive, we need $\frac{\partial^2 \nu}{\partial (r^c)^2} = \frac{\partial \mu}{\partial r^c}$ to be non-negative.

- Given (E.6) and $\frac{\partial\mu}{\partial r^c} \geq 0$, to show that $\frac{\partial r^c}{\partial\zeta} < 0$ we need to show that

$$\frac{\partial\mu}{\partial\zeta}\frac{\partial\nu}{\partial c} - \frac{\partial\mu}{\partial c}\frac{\partial\nu}{\partial\zeta} > 0 \tag{E.8}$$

- Given (E.5) and $\frac{\partial\mu}{\partial r^c} \geq 0$, to show that $\frac{\partial r^c}{\partial g} < 0$ we need to show that

$$\frac{\partial\mu}{\partial g}\frac{\partial\nu}{\partial c} - \frac{\partial\mu}{\partial c}\frac{\partial\nu}{\partial g} > 0 \tag{E.9}$$

We can now follow our strategy and fill in the details. $\mu(c, r^c, g, \zeta)$, is obtained by differentiating the function $\nu(c, r^c, g, \zeta)$ with respect to r^c.

$$\begin{aligned}\mu(c, r^c, g, \zeta) &= cs'(x)\frac{\partial x}{\partial r^c} - \hat{w}\frac{\partial h}{\partial r^c} + c^c\frac{\partial d^c}{\partial r^c} \\ &+ \frac{\partial d^b}{\partial r^c}(\alpha - \zeta(1-\tau^b)(r - \tilde{r}^b)) = 0\end{aligned} \tag{E.10}$$

Given (58) and (64), labor h is itself defined through an implicit function

$$h = \frac{\hat{w}^{\frac{1}{\gamma}}(1-\tau_h)^{\frac{1}{\gamma}}}{c^{\frac{1}{\gamma}}(1+s(x)+xs'(x))^{\frac{1}{\gamma}}} = \frac{\hat{w}^{\frac{1}{\gamma}}\left(1-\frac{g-\tilde{S}}{\hat{w}h}\right)^{\frac{1}{\gamma}}}{c^{\frac{1}{\gamma}}(1+s(x)+xs'(x))^{\frac{1}{\gamma}}} \tag{E.11}$$

If the tax rate is zero h reaches its highest value $\bar{h} = \frac{\hat{w}^{\frac{1}{\gamma}}}{c^{\frac{1}{\gamma}}(1+s(x)+xs'(x))^{\frac{1}{\gamma}}}$. To avoid working with a third implicit function (in addition to the implicit functions μ and ν) which would make the algebra even more complex, we approximate h by performing a Tayor explansion in $\frac{g-\tilde{S}}{\hat{w}\bar{h}}$. Notice that, given that the labor tax rate is $\tau_h = \frac{g-\tilde{S}}{wh}$ and $\bar{h} = h(1-\tau_h)^{-\frac{1}{\gamma}}$, we have

$$\frac{g-\tilde{S}}{\hat{w}\bar{h}} = \tau_h(1-\tau_h)^{\frac{1}{\gamma}} \tag{E.12}$$

The smaller is this quantity, the more accurate is our Taylor expansion.[29]

We want to collect all terms up to first order in $\frac{g-\tilde{S}}{\hat{w}\bar{h}}$ contributing to expressions (E.8) and (E.9). For this purpose, since (E.8) and (E.9) contain second derivatives of h with respect to combinations of r^c, g and c (all of which affect $\frac{g-\tilde{S}}{\hat{w}\bar{h}}$), we need to expand h up to third order in $\frac{g-\tilde{S}}{\hat{w}\bar{h}}$. Later, when computing the first (second) derivatives of h we will keep all terms up to second (first) order. We obtain:

$$h = \bar{h}\left(1 - \frac{1}{\gamma}\frac{g-\tilde{S}}{\hat{w}\bar{h}} - \frac{1}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\left(\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)^2 - \frac{1}{3}\left(\frac{1}{\gamma}+\frac{3}{\gamma^2}-\frac{1}{\gamma^3}\right)\left(\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)^3 + \mathcal{O}\left(\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)^4\right) \tag{E.13}$$

[29] In the baseline scenario in our paper, τ_h ranges between 0.25 and 0.45, and $\gamma = 1$. This corresponds to the quantity $\frac{g-\tilde{S}}{\hat{w}\bar{h}}$ ranging between 0.19 and 0.25.

From this we can obtain an expression for $\hat{w}\frac{\partial h}{\partial r^c}$

$$\hat{w}\frac{\partial h}{\partial r^c} = -\frac{\hat{w}\bar{h}}{\gamma}\frac{(2s'(x)+s''(x))}{(1+s(x)+xs'(x))}\left(1+\frac{1}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\left(\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)^2\right)\frac{\partial x}{\partial r^c}$$
$$+ \left(\frac{1}{\gamma}+\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\frac{g-\tilde{S}}{\hat{w}\bar{h}}+\left(\frac{1}{\gamma}+\frac{3}{\gamma^2}-\frac{1}{\gamma^3}\right)\left(\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)^2\right)\frac{\partial \tilde{S}}{\partial r^c} \tag{E.14}$$

Inserting (E.14) in (E.10) we obtain

$$\mu(c,r^c,g,\zeta) = cs'(x)\frac{\partial x}{\partial r^c}+c^c\frac{\partial d^c}{\partial r^c}+\frac{\partial d^b}{\partial r^c}(\alpha-\zeta(1-\tau^b)(r^*-\tilde{r}^b))$$
$$+ \frac{\hat{w}\bar{h}}{\gamma}\frac{(2s'(x)+s''(x))}{(1+s(x)+xs'(x))}\left(1+\frac{1}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\left(\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)^2\right)\frac{\partial x}{\partial r^c}$$
$$- \left(\frac{1}{\gamma}+\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\frac{g-\tilde{S}}{\hat{w}\bar{h}}+\left(\frac{1}{\gamma}+\frac{3}{\gamma^2}-\frac{1}{\gamma^3}\right)\left(\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)^2\right)\frac{\partial \tilde{S}}{\partial r^c} = 0 \tag{E.15}$$

Since the quantities x and d^b are decreasing in r^c (see Table 4), the quantity $(\alpha-\zeta(1-\tau^b)(r^*-\tilde{r}^b))$ is positive,[30] the function $s(x)$ is increasing and convex, all the terms in (E.15) other than $c^c\frac{\partial d^c}{\partial r^c}$ and the term proportional to $\frac{\partial \tilde{S}}{\partial r^c}$ are negative. The term $c^c\frac{\partial d^c}{\partial (r^c)}$ is positive but negligible since c^c is negligible by assumption, so for the function μ to be 0 at the optimal point it must be $\frac{\partial \tilde{S}}{\partial r^c}<0$, and it must also be

$$\left|\left(\frac{1}{\gamma}+\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\frac{g-\tilde{S}}{\hat{w}\bar{h}}+\left(\frac{1}{\gamma}+\frac{3}{\gamma^2}-\frac{1}{\gamma^3}\right)\left(\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)^2\right)\frac{\partial \tilde{S}}{\partial r^c}\right|$$
$$> \left|\frac{\hat{w}\bar{h}}{\gamma}\frac{(2s'(x)+s''(x))}{(1+s(x)+xs'(x))}\left(1+\frac{1}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\left(\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)^2\right)\frac{\partial x}{\partial r^c}\right| \tag{E.16}$$

Later on we will assume something somewhat stronger: that $\left|\frac{\partial \tilde{S}}{\partial r^c}\right| >> \left|\hat{w}\bar{h}(2s'(x)+s''(x))\frac{\partial x}{\partial r^c}\right|$ (condition $C1$), so that the term on the RHS of (E.16) can be considered negligible relative to that on the LHS. The economic meaning is the following: $\left|\frac{\partial \tilde{S}}{\partial r^c}\right|$ represents the extra revenues that the government can collect by lowering r^c, and avoid extracting from workers' wages; $\left|\hat{w}\bar{h}(2s'(x)+s''(x))\frac{\partial x}{\partial r^c}\right|$ represents the part of the worker's wages that is "lost" to transaction cost as a result of lowering r^c of the same amount. Our assumption

[30] Using (E.2), the quantity $(\alpha-\zeta(1-\tau^b)(r^*-\tilde{r}^b))$ can be rewritten as $(r^*-r^b)-(\tau^b+\zeta(1-\tau^b))(r^*-\tilde{r}^b)$, which is positive because $(r^*-\tilde{r}^b)<(r^*-r^b)$ (see (E.2)) and $(\tau^b+\zeta(1-\tau^b))\leq 1$.

is therefore that the amount "gained" by workers as a result of tax lowering is much more significant that the amount "lost" to transaction cost, as we decrease r^c. In other words, the assumption is that a change in r^c affects labor (and labor income) essentially only through its effect on the tax rate, whereas its effect through the transaction cost is negligible.

After inserting (E.13) in (E.1) and (E.14) in (E.10) we obtain the following (to first order in $\frac{g-\tilde{S}}{\hat{w}\bar{h}}$)[31]

$$\frac{\partial \nu}{\partial c} = 1 + s(x) + \frac{1}{\gamma}\frac{\hat{w}\bar{h}}{c} + c^c \frac{d^c}{c} + \frac{d^b}{c}(\alpha - \zeta(1-\tau^b)(r-\tilde{r}^b)) > 0 \tag{E.18}$$

$$\begin{aligned} \frac{\partial \mu}{\partial c} &= -\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\frac{\hat{w}\bar{h}}{c}\frac{(2s'(x)+s''(x))}{(1+s(x)+xs'(x))}\frac{\partial x}{\partial r^c} \\ &- \frac{1}{\gamma c}\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\frac{g-\tilde{S}}{\hat{w}\bar{h}}\frac{\partial \tilde{S}}{\partial r^c} > 0 \end{aligned} \tag{E.19}$$

$$\frac{\partial \nu}{\partial g} = 1 + \frac{1}{\gamma} + \left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\frac{g-\tilde{S}}{\hat{w}\bar{h}} > 0 \tag{E.20}$$

$$\begin{aligned} \frac{\partial \mu}{\partial g} &= \frac{1}{\gamma}\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\frac{(2s'(x)+s''(x))}{(1+s(x)+xs'(x))}\frac{g-\tilde{S}}{\hat{w}\bar{h}}\frac{\partial x}{\partial r^c} \\ &- \left(\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)+2\left(\frac{1}{\gamma}+\frac{3}{\gamma^2}-\frac{1}{\gamma^3}\right)\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)\frac{1}{\hat{w}\bar{h}}\frac{\partial \tilde{S}}{\partial r^c} > 0 \end{aligned} \tag{E.21}$$

$$\frac{\partial \nu}{\partial \zeta} = -(1-\tau^b)(d^b(r-\tilde{r}^b) + l(r^* - r^l - c^l)) < 0 \tag{E.22}$$

$$\frac{\partial \mu}{\partial \zeta} = -(1-\tau^b)(r-\tilde{r}^b)\frac{\partial d^b}{\partial r^c} > 0 \tag{E.23}$$

The signs of the derivatives (E.18)-(E.23) are obtained the following way:

[31] Notice that (E.19) is the derivative $\frac{\partial \mu}{\partial c}$ at the $\{c, r^c\}$ point where $\mu = 0$. Indeed, given that d^b, d^c and hence $\tilde{S}$ are proportional to c (see (56) and (57)), and $\bar{h}$ is proportional to $c^{-\frac{1}{\gamma}}$, by differentiating (E.15) we obtain (ignoring terms in $\left(\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)^2$ or higher-order)

$$\begin{aligned} \frac{\partial \mu}{\partial c} &= s'(x)\frac{\partial x}{\partial r^c} + \frac{c^c}{c}\frac{\partial d^c}{\partial r^c} + \frac{1}{c}\frac{\partial d^b}{\partial r^c}(\alpha - \zeta(1-\tau^b)(r-\tilde{r}^b)) \\ &- \frac{1}{\gamma^2}\frac{\hat{w}\bar{h}}{c}\frac{(2s'(x)+s''(x))}{(1+s(x)+xs'(x))}\frac{\partial x}{\partial r^c} - \frac{1}{c}\left(\frac{1}{\gamma}+\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)\frac{\partial \tilde{S}}{\partial r^c} \\ &- \frac{1}{\gamma c}\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\frac{g-\tilde{S}}{\hat{w}\bar{h}}\frac{\partial \tilde{S}}{\partial r^c} \\ &= \frac{\mu}{c} - \left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\frac{\hat{w}\bar{h}}{c}\frac{(2s'(x)+s''(x))}{(1+s(x)+xs'(x))}\frac{\partial x}{\partial r^c} - \frac{1}{\gamma c}\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)\frac{g-\tilde{S}}{\hat{w}\bar{h}}\frac{\partial \tilde{S}}{\partial r^c} \end{aligned} \tag{E.17}$$

(E.17) is then equal to (E.19) given $\mu = 0$ at the optimum.

- In (E.18), (E.19) and (E.20) and (E.23) all the terms on the RHS are positive. Regarding (E.23), remember that $\frac{\partial d^b}{\partial r^c} < 0$ (see Table 4).

  Regarding (E.19), the first term is positive since $s(x)$ is an increasing and convex function and x is a decreasing function of r^c (see Table 4). The second term is positive because $\frac{\partial \tilde{S}}{\partial r^c} < 0$ at the $\{c, r^c\}$ point where $\mu = 0$ (see (E.15), (E.16) and the related discussion). Using the assumption $\left|\frac{\partial \tilde{S}}{\partial r^c}\right| >> \left|\hat{w}\bar{h}(2s'(x) + s''(x))\frac{\partial x}{\partial r^c}\right|$ we can say that $\frac{\partial \mu}{\partial c} > 0$.

- In (E.22) all the terms on the RHS are negative.

- In (E.21) the first term in the RHS is negative and the second term is positive for the reasons discussed in the previous point.

(E.8), together with (E.18)-(E.23) then shows that $\frac{\partial r^c}{\partial \zeta} < 0$. This shows that the consumption-maximizing r^c is decreasing in ζ.

Finally, to show that the consumption-maximizing r^c is decreasing in g, we need to show that $\frac{\partial \mu}{\partial g}\frac{\partial \nu}{\partial c} - \frac{\partial \mu}{\partial c}\frac{\partial \nu}{\partial g} > 0$ (see (E.9)) Given (E.18)-(E.23), the sign of the first term in the latter expression is positive, however the second term is negative.

Neglecting again the terms proportional to $\left|\hat{w}\bar{h}(2s'(x) + s''(x))\frac{\partial x}{\partial r^c}\right|$ relative to those proportional to $\left|\frac{\partial \tilde{S}}{\partial r^c}\right|$ in (E.19) and (E.21), and using $\frac{\partial \nu}{\partial c} > 1 + \frac{1}{\gamma}\frac{\hat{w}\bar{h}}{c}$ (since all the terms in (E.18) are positive), we can write

$$\begin{aligned}
\frac{\partial \mu}{\partial g}\frac{\partial \nu}{\partial c} - \frac{\partial \mu}{\partial c}\frac{\partial \nu}{\partial g} &> -\left(1 + \frac{1}{\gamma}\frac{\hat{w}\bar{h}}{c}\right)\left(\left(\frac{1}{\gamma} + \frac{1}{\gamma^2}\right) + 2\left(\frac{1}{\gamma} + \frac{3}{\gamma^2} - \frac{1}{\gamma^3}\right)\frac{g - \tilde{S}}{\hat{w}\bar{h}}\right)\frac{1}{\hat{w}\bar{h}}\frac{\partial \tilde{S}}{\partial r^c} \\
&+ \frac{1}{\gamma c}\left(\frac{1}{\gamma} + \frac{1}{\gamma^2}\right)\left(1 + \frac{1}{\gamma}\right)\frac{g - \tilde{S}}{\hat{w}\bar{h}}\frac{\partial \tilde{S}}{\partial r^c} \\
&= -\left[\frac{1}{\gamma} + \frac{1}{\gamma^2} + 2\left(\frac{1}{\gamma} + \frac{3}{\gamma^2} - \frac{1}{\gamma^3}\right)\frac{g - \tilde{S}}{\hat{w}\bar{h}}\right]\frac{1}{\hat{w}\bar{h}}\frac{\partial \tilde{S}}{\partial r^c} \\
&- \frac{1}{\gamma c}\left[\frac{1}{\gamma} + \frac{1}{\gamma^2} + \left(\frac{1}{\gamma} + \frac{4}{\gamma^2} - \frac{3}{\gamma^3}\right)\frac{g - \tilde{S}}{\hat{w}\bar{h}}\right]\frac{\partial \tilde{S}}{\partial r^c} > 0 \qquad \text{(E.24)}
\end{aligned}$$

The two terms after the equality sign are both positive for any $\gamma > 0$.[32]

[32] This is easy to see when $\gamma \geq 1$ (given $\frac{\partial \tilde{S}}{\partial r^c} < 0$). When $0 < \gamma < 1$, this is less immediate to see because the terms $\left(\frac{1}{\gamma} + \frac{3}{\gamma^2} - \frac{1}{\gamma^3}\right)$ and $\left(\frac{1}{\gamma} + \frac{4}{\gamma^2} - \frac{3}{\gamma^3}\right)$ can become negative for low γ. However, notice that $\frac{g - \tilde{S}}{\hat{w}\bar{h}} = \tau_h(1 - \tau_h)^{\frac{1}{\gamma}}$ takes a maximum possible value of $\frac{\gamma}{(1+\gamma)^{\frac{1}{\gamma}+1}}$ (when $\tau_h = \frac{\gamma}{1+\gamma}$). It is easy to verify that even for $\frac{g - \tilde{S}}{\hat{w}\bar{h}} = \frac{\gamma}{(1+\gamma)^{\frac{1}{\gamma}+1}}$ (and *a fortiori* for lower values of $\frac{g - \tilde{S}}{\hat{w}\bar{h}}$) the two expressions in square parenthesis are positive for any positive γ.

(E.24) shows indeed that the optimal value of r^c is decreasing in g.

2. The welfare-maximizing r^c

To prove this part of the proposition we adopt a very similar strategy. As before, the optimal r^c is defined by two functions: the function ν, which is the same as before, given that consumption still needs to satisfy the budget constraint, and a new function that we call μ^W, which states that the optimal r^c is the one maximizing welfare, derived as follows. Remember that welfare is $W = log(c) - \frac{h^{1+\gamma}}{1+\gamma} + \lambda log(c^{bo})$ (see (39)). (58), together with (54), (60) and (64) can be used to define h as a function of c and r^c, while (55) (i.e. the ν function) defines $c(r^c)$, with $\frac{\partial c}{\partial r^c} = -\frac{\frac{\partial \nu}{\partial r^c}}{\frac{\partial \nu}{\partial c}}$. All these equations therefore define $h(r^c, c(r^c))$. Similarly, we can define $c^{bo}(r^c, c(r^c))$.

The value of r^c that maximizes welfare must therefore satisfy

$$\frac{1}{c}\frac{\partial c}{\partial r^c} - h^\gamma\left(\frac{\partial h}{\partial c}\frac{\partial c}{\partial r^c} + \frac{\partial h}{\partial r^c}\right) + \frac{\lambda}{c^{bo}}\left(\frac{\partial c^{bo}}{\partial r^c} + \frac{\partial c^{bo}}{\partial c}\frac{\partial c}{\partial r^c}\right) = 0$$

Given (E.2) and $\frac{\partial \nu}{\partial c} > 0$, this condition can be written as

$$\frac{\partial \nu}{\partial r^c} - c\ h^\gamma \frac{\partial h}{\partial c}\frac{\partial \nu}{\partial r^c} + c\frac{\partial \nu}{\partial c}\ h^\gamma \frac{\partial h}{\partial r^c} + \lambda\frac{c}{c^{bo}}\left(\frac{\partial c^{bo}}{\partial r^c}\frac{\partial \nu}{\partial r^c} - \frac{\partial \nu}{\partial c}\frac{\partial c^{bo}}{\partial r^c}\right) = 0 \qquad \text{(E.25)}$$

Since by assumption we work in the limit $\lambda\frac{c}{c^{bo}} \to 0$ we define our second function $\mu^W(c, r^c, g, \zeta)$ as

$$\mu^W(c, r^c, g, \zeta) = \frac{\partial \nu}{\partial r^c} - c\ h^\gamma \frac{\partial h}{\partial c}\frac{\partial \nu}{\partial r^c} + c\frac{\partial \nu}{\partial c}\ h^\gamma \frac{\partial h}{\partial r^c} = 0 \qquad \text{(E.26)}$$

The functions ν and μ^W define c and the welfare-maximizing r^c as functions of g and ζ. Following the same reasoning as in the previous subsection, to show that the optimal r^c is decreasing in ζ and g we need to show that

$$\frac{\partial \mu^W}{\partial \zeta}\frac{\partial \nu}{\partial c} - \frac{\partial \mu^W}{\partial c}\frac{\partial \nu}{\partial \zeta} > 0$$

and

$$\frac{\partial \mu^W}{\partial g}\frac{\partial \nu}{\partial c} - \frac{\partial \mu^W}{\partial c}\frac{\partial \nu}{\partial g} > 0$$

respectively.

We have (using (E.10) and (E.18))

$$\begin{aligned}
\frac{\partial \nu}{\partial r^c} &= cs'(x)\frac{\partial x}{\partial r^c} + c^c\frac{\partial d^c}{\partial r^c} + \frac{\partial d^b}{\partial r^c}(\alpha - \zeta(1-\tau^b)(r-\tilde{r}^b)) - \hat{w}\frac{\partial h}{\partial r^c} \\
&= f_1 + \frac{\partial d^b}{\partial r^c}(\alpha - \zeta(1-\tau^b)(r-\tilde{r}^b)) - \hat{w}\frac{\partial h}{\partial r^c} \qquad \text{(E.27)} \\
\textit{with } f_1 &= cs'(x)\frac{\partial x}{\partial r^c} + c^c\frac{\partial d^c}{\partial r^c} + \frac{\partial d^b}{\partial r^c}(\alpha - \zeta(1-\tau^b)(r-\tilde{r}^b)) < 0
\end{aligned}$$

(remember that s is an increasing function of x, $\frac{\partial x}{\partial r^c} < 0$, $\frac{\partial d^b}{\partial r^c} < 0$ and c^c is negligible).

$$\begin{aligned}
c\frac{\partial \nu}{\partial c} &= c - \hat{w}c\frac{\partial h}{\partial c} + f_2 \qquad \text{(E.28)} \\
\textit{with } f_2 &= cs(x) + c^c d^c + d^b(\alpha - \zeta(1-\tau^b)(r-\tilde{r}^b)) > 0
\end{aligned}$$

Inserting (E.27)-(E.28) into (E.25) we obtain

$$\begin{aligned}
\mu^W &= \left(f_1 - \hat{w}\frac{\partial h}{\partial r^c}\right)\left(1 - c\, h^\gamma\frac{\partial h}{\partial c}\right) + \left(c - \hat{w}c\frac{\partial h}{\partial c} + f_2\right) h^\gamma \frac{\partial h}{\partial r^c} \\
&= f_1\left(1 - c\, h^\gamma\frac{\partial h}{\partial c}\right) - \hat{w}\frac{\partial h}{\partial r^c} + (c+f_2)h^\gamma\frac{\partial h}{\partial r^c} \qquad \text{(E.29)}
\end{aligned}$$

We use (E.14) to express $\frac{\partial h}{\partial r^c}$ up to second order in $\frac{g-\tilde{S}}{\hat{w}\bar{h}}$, ignoring terms proportional to $\left|\hat{w}\bar{h}(2s'(x)+s''(x))\frac{\partial x}{\partial r^c}\right|$ relative to those proportional to $\left|\frac{\partial \tilde{S}}{\partial r^c}\right|$. We also compute $\frac{\partial h}{\partial c}$ (again to second order in $\frac{g-\tilde{S}}{\hat{w}\bar{h}}$):

$$\frac{\partial h}{\partial c} = -\frac{1}{\gamma}\frac{\bar{h}}{c}\left(1 + \frac{1}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma^2}\right)\frac{g-\tilde{S}}{\hat{w}\bar{h}}^2\right) \qquad \text{(E.30)}$$

so that

$$\begin{aligned}
\mu^W &= f_1\left(1 + \frac{\bar{h}^{\gamma+1}}{\gamma}\left(1 - \frac{g-\tilde{S}}{\hat{w}\bar{h}} - \left(\frac{1}{2\gamma} - \frac{1}{2\gamma^2}\right)\left(\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)^2\right)\right) \\
&+ f_2\frac{\bar{h}^\gamma}{\gamma\hat{w}}\frac{\partial \tilde{S}}{\partial r^c}\left(1 + \frac{1}{\gamma}\frac{g-\tilde{S}}{\hat{w}\bar{h}} + \left(\frac{1}{\gamma} - \frac{1}{\gamma^2}\right)\left(\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)^2\right) \\
&- \frac{\partial \tilde{S}}{\partial r^c}\left(\frac{1}{\gamma}\frac{g-\tilde{S}}{\hat{w}\bar{h}} + \left(\frac{1}{\gamma} + \frac{2}{\gamma^2}\right)\left(\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)^2\right) = 0 \qquad \text{(E.31)}
\end{aligned}$$

The term in the first line of (E.31) is negative since f_1 is negative (see (E.27)), the term in the second line is also negative since f_2 is positive (see (E.28)) and $\frac{\partial \tilde{S}}{\partial r^c}$, as in the previous section, is negative;[33] the term on the third line is positive. For $\mu^W = 0$ to be satisfied, the term in the third line must be bigger in absolute value that the one on the second line. It is possible to show that this implies $f_2 \frac{\bar{h}^\gamma}{\gamma \hat{w}} < 1$ if $0 < \gamma \lesssim 5$, that is, for every realistic value of the inverse Frisch elasticity, especially in a macroeconomic context.[34] We have

$$\begin{aligned}
\frac{\partial \mu^W}{\partial g} &= \frac{1}{\hat{w}\bar{h}} \Bigg[-f_1 \frac{\bar{h}^{\gamma+1}}{\gamma} \left(1 + \left(\frac{1}{\gamma} - \frac{1}{\gamma^2} \right) \frac{g - \tilde{S}}{\hat{w}\bar{h}} \right) \\
&\quad + f_2 \frac{\bar{h}^\gamma}{\gamma \hat{w}} \left(\frac{1}{\gamma} + 2 \left(\frac{1}{\gamma} - \frac{1}{\gamma^2} \right) \frac{g - \tilde{S}}{\hat{w}\bar{h}} \right) \frac{\partial \tilde{S}}{\partial r^c} \\
&\quad - \left(\frac{1}{\gamma} + 2 \left(\frac{1}{\gamma} + \frac{2}{\gamma^2} \right) \frac{g - \tilde{S}}{\hat{w}\bar{h}} \right) \frac{\partial \tilde{S}}{\partial r^c} \Bigg] > 0 \qquad \text{(E.32)} \\
\frac{\partial \mu^W}{\partial c} &= \frac{f_1}{c} \left(1 - \frac{\bar{h}^{\gamma+1}}{\gamma^2} \right) + \frac{f_2}{c\hat{w}} \frac{\partial \tilde{S}}{\partial r^c} \bar{h}^\gamma \left(\frac{1}{\gamma} + \frac{1}{\gamma^2} \left(1 + \frac{1}{\gamma} \right) \frac{g - \tilde{S}}{\hat{w}\bar{h}} \right) \\
&- \frac{1}{c} \frac{\partial \tilde{S}}{\partial r^c} \frac{1}{\gamma} \left(1 + \frac{1}{\gamma} \right) \frac{g - \tilde{S}}{\hat{w}\bar{h}} > 0 \qquad \text{(E.33)} \\
\frac{\partial \mu^W}{\partial \zeta} &= -(1 - \tau_b)(r - \tilde{r}^b) \left[\frac{\partial d^b}{\partial r^c} \left(1 + \frac{\bar{h}^\gamma}{\gamma} \left(1 - \frac{g - \tilde{S}}{\hat{w}\bar{h}} \right) \right) \right. \\
&+ \left. d^b \frac{\partial \tilde{S}}{\partial r^c} \bar{h}^\gamma \left(\frac{1}{\gamma} + \frac{1}{\gamma^2} \frac{g - \tilde{S}}{\hat{w}\bar{h}} \right) \right] > 0 \qquad \text{(E.34)}
\end{aligned}$$

Notice that to obtain (E.34) we need to keep in mind that $f_1 \propto c$ (see (E.28)), $\bar{h} \propto c^{-\frac{1}{\gamma}}$, $\frac{\partial \tilde{S}}{\partial r^c} \propto c$.

- In (E.32) the terms in the first and third line are positive, the one in the second line is negative. But since, as discussed before, $f_2 \frac{\bar{h}^\gamma}{\gamma \hat{w}} < 1$, the term in the third

[33] Another way to see that $\frac{\partial \tilde{S}}{\partial r^c}$ is always negative at the optimum is the following: seigniorage increases in r^c for low values of r^c and increases for high values of r^c. The optimal value is the one the reaches the best tradeoff between maximizing seigniorage (which would require $\frac{\partial \tilde{S}}{\partial r^c} = 0$), and maximizing the other channels (which would require $r^c = r^*$). Therefore, the optimal value must be above the seigniorage-maximizing one, i.e. must be in the region where seigniorage is decreasing.

[34] As previously noted, since $\frac{g - \tilde{S}}{\hat{w}\bar{h}} = \tau_h (1 - \tau_h)^{\frac{1}{\gamma}}$, $\frac{g - \tilde{S}}{\hat{w}\bar{h}}$ takes a maximum possible value of $\frac{\gamma}{(1+\gamma)^{1+\frac{1}{\gamma}}}$ (when $\tau_h = \frac{\gamma}{1+\gamma}$). It is easy to verify that the factor in parenthesis in the third line of (E.31) is smaller than the factor in parenthesis in the second line for any feasible value of $\frac{g - \tilde{S}}{\hat{w}\bar{h}}$ and every γ s.t. $\gamma \lesssim 5$. Therefore, for $\gamma \lesssim 5$, for the term in the third line to be bigger in absolute value than the term in the second line, it must be $f_2 \frac{\bar{h}^\gamma}{\gamma \hat{w}} < 1$.

line is bigger in absolute value than the one in the second line, which is sufficient to conclude that $\frac{\partial \mu^W}{\partial g} > 0$.

- Concerning (E.34), to show that $\frac{\partial \mu^W}{\partial c} > 0$, similarly to what we did in (E.17), we can write (E.34) as

$$\begin{aligned} \frac{\partial \mu^W}{\partial c} &= \frac{\mu^W}{c} - \frac{f_1}{c}\frac{\bar{h}^{\gamma+1}}{\gamma}\left(\left(1+\frac{1}{\gamma}\right) - \frac{g-\tilde{S}}{\hat{w}\bar{h}}\right) + \frac{f_2}{\hat{w}c}\frac{\partial \tilde{S}}{\partial r^c}\frac{\bar{h}^\gamma}{\gamma^3}\frac{g-\tilde{S}}{\hat{w}\bar{h}} - \frac{1}{c}\frac{\partial \tilde{S}}{\partial r^c}\frac{1}{\gamma^2}\frac{g-\tilde{S}}{\hat{w}\bar{h}} \\ &= -\frac{f_1}{c}\frac{\bar{h}^{\gamma+1}}{\gamma}\left(\left(1+\frac{1}{\gamma}\right) - \frac{g-\tilde{S}}{\hat{w}\bar{h}}\right) + \frac{f_2}{\hat{w}c}\frac{\partial \tilde{S}}{\partial r^c}\frac{\bar{h}^\gamma}{\gamma^3}\frac{g-\tilde{S}}{\hat{w}\bar{h}} - \frac{1}{c}\frac{\partial \tilde{S}}{\partial r^c}\frac{1}{\gamma^2}\frac{g-\tilde{S}}{\hat{w}\bar{h}} > 0 \quad \text{(E.35)} \end{aligned}$$

  The second equality in (E.35) is obtained using $\mu^W = 0$ at the optimum (see (E.31); on the last line of (E.35), the first and the third term are positive, the second term is negative but smaller in absolute value than the third term since, as discussed before, $f_2\frac{\bar{h}^\gamma}{\gamma\hat{w}} < 1$.

- Concerning (E.34), $\frac{\partial \mu^W}{\partial \zeta}$ is unambiguously positive since both $\frac{\partial d^b}{\partial r^c}$ and $\frac{\partial \tilde{S}}{\partial r^c}$ are negative.

Given (E.34), (E.34), (E.19) and (E.23), we have unambiguously

$$\frac{\partial \mu^W}{\partial \zeta}\frac{\partial \nu}{\partial c} - \frac{\partial \mu^W}{\partial c}\frac{\partial \nu}{\partial \zeta} > 0 \quad \text{(E.36)}$$

which shows that the optimal r^c is decreasing in ζ.

Finally, since $\frac{\partial \nu}{\partial c} > 1 + \frac{1}{\gamma}\frac{\hat{w}\bar{h}}{c}$ (see (E.19)) and $\frac{\partial \mu^W}{\partial c} < -\frac{f_1}{c}\frac{\bar{h}^{\gamma+1}}{\gamma}\left(\left(1+\frac{1}{\gamma}\right) - \frac{g-\tilde{S}}{\hat{w}\bar{h}}\right) - \frac{1}{c}\frac{\partial \tilde{S}}{\partial r^c}\frac{1}{\gamma^2}\frac{g-\tilde{S}}{\hat{w}\bar{h}}$ (see (E.35)) we have (also using (E.21) for $\frac{\partial \nu}{\partial g}$ and (E.32) for $\frac{\partial \mu^W}{\partial g}$)

$$\begin{aligned} \frac{\partial \mu^W}{\partial g}\frac{\partial \nu}{\partial c} - \frac{\partial \mu^W}{\partial c}\frac{\partial \nu}{\partial g} > \frac{1}{c}\Bigg[&-f_1\frac{\bar{h}^{\gamma+1}}{\gamma^2}\left(1+\frac{2}{\gamma}\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right) \\ &+f_2\frac{\bar{h}^\gamma}{\gamma\hat{w}}\left(\frac{1}{\gamma^2} + 2\left(\frac{1}{\gamma^2}-\frac{1}{\gamma^3}\right)\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)\frac{\partial \tilde{S}}{\partial r^c} \\ &-\left(\frac{1}{\gamma^2} + \left(\frac{1}{\gamma^2}+\frac{3}{\gamma^3}\right)\frac{g-\tilde{S}}{\hat{w}\bar{h}}\right)\frac{\partial \tilde{S}}{\partial r^c}\Bigg] > 0 \end{aligned} \quad \text{(E.37)}$$

The only negative term in (E.37) is the second term on the RHS, however given $f_2\frac{\bar{h}^\gamma}{\gamma\hat{w}} < 1$ (for $0 < \gamma \lesssim 5$) this term is smaller in absolute value than the last term in the same range. (E.37) shows that the optimal r^c is decreasing in g.

References

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