Online Appendix

Exchange Rate Determination under Limits to CIP Arbitrage

Philippe Bacchetta, J. Scott Davis and Eric van Wincoop

August 2024

This Online Appendix has 3 sections. Section A derives the optimal portfolio of European households. Section B derives the spot market equilibrium. Section C discusses the pre-shock equilibrium.

A Portfolio Foreign Households

In period 1 Foreign households hold dollar and euro money balances, as well as euro bonds, and borrow dollars both synthetically and from the ECB. Their period 2 consumption is

$$
\begin{aligned}
P_2^* C_{F,2} &= Y_{F,2} + \Pi_{FCB,2} + \frac{1}{S_2} M_{F,1}^{\$} + M_{F,1}^{€} + (1 + i_1^{€}) B_{F,1}^{€} \\
&\quad - \frac{1}{S_2}(1 + i_1^{\$,syn}) D_{F,1}^{\$} - \frac{1}{S_2}(1 + i_1^{\$,ECB}) D_{swap,1}^{\$} \qquad \text{(A.1)}
\end{aligned}
$$

Using that period 1 financial wealth is equal to

$$
W_{F,1} = B_{F,1}^{€} - \frac{1}{S_1}\left(D_{F,1}^{\$} + D_{swap,1}^{\$}\right) \qquad \text{(A.2)}
$$

we can also write this as

$$
\begin{aligned}
P_2^* C_{F,2} &= Y_{F,2} + \Pi_{FCB,2} + \frac{1}{S_2} M_{F,1}^{\$} + M_{F,1}^{€} + (1 + i_1^{€}) W_{F,1} \qquad \text{(A.3)} \\
&\quad - \left(\frac{1 + i_1^{\$,syn}}{S_2} - \frac{1 + i_1^{€}}{S_1}\right) D_{F,1}^{\$} - \left(\frac{1 + i_1^{\$,ECB}}{S_2} - \frac{1 + i_1^{€}}{S_1}\right) D_{swap,1}^{\$}
\end{aligned}
$$

Period 2 income is

$$
Y_{F,2} = e^{-\kappa_F \theta s_2} \qquad \text{(A.4)}
$$

Using that $p_2^* = -0.5 s_2$, linearizing (A.3) around $C_{F,2} = \bar{C}_{F,2}$, $s_2 = s_1 = 0$ and zero interest rates, we have

$$
\begin{aligned}
C_{F,2} &= 1 - \rho s_2 + \Pi_{FCB,2} + (1 - s_2) M_{F,1}^{\$} + M_{F,1}^{€} + (1 + i_1^{€}) W_{F,1} \qquad \text{(A.5)} \\
&\quad - (i_1^{\$,syn} - i_1^{€} - s_2 + s_1) D_{F,1}^{\$} - (i_1^{\$,ECB} - i_1^{€} - s_2 + s_1) D_{swap,1}^{\$}
\end{aligned}
$$

where $\rho = \kappa_F \theta - 0.5\bar{C}_{F,2}$.

Agents maximize the following mean-variance objective:

$$EC_{F,2} - 0.5\gamma var(C_{F,2}) \tag{A.6}$$

This is

$$\begin{aligned} &1 + \Pi_{FCB,2} + M^{\$}_{F,1} + M^{€}_{F,1} + (1 + i^{€}_1)W_{F,1} - (i^{\$,syn}_1 - i^{€}_1 + s_1)D^{\$}_{F,1} \\ &-(i^{\$,ECB}_1 - i^{€}_1 + s_1)D^{\$}_{swap,1} - 0.5\gamma\left(-\rho - M^{\$}_{F,1} + D^{\$}_{F,1} + D^{\$}_{swap,1}\right)^2 var(s_2) \end{aligned} \tag{A.7}$$

The first-order condition with respect to $D^{\$}_{F,1}$ gives

$$D^{\$}_{F,1} = \rho + M^{\$}_{F,1} - D^{\$}_{swap,1} - \frac{i^{\$,syn}_1 - i^{€}_1 + s_1}{\gamma var(s_2)} \tag{A.8}$$

B Spot Market Equilibrium

In order to derive the spot market equilibrium, we need to track changes in foreign currency positions of all agents.

B.1 European Households

European households enter period 1 with $M^{\$}_{F,0}$ dollar balances, while at the end of period 1 they have $M^{\$}_{F,1}$ dollar balances. During period 1 these dollar balances change due to dollar invoiced income, dollar invoiced consumption, new synthetic dollar borrowing, dollar borrowing from the ECB, interest and principal payments on period 0 synthetic dollar borrowing and spot market purchases of dollars in period 1.

With regards to income, European households receive dollar income from exports to the US that are invoiced in dollars. This is equal to

$$Y^{\$}_{F,1} = C_{HF,1} + C^{o}_{HF,1} \tag{B.9}$$

Their spending in dollars on goods imported from the US that are invoiced in dollars is $C_{FH,1}$.

We then have

$$M^{\$}_{F,1} = M^{\$}_{F,0} + Y^{\$}_{F,1} - C_{FH,1} + D^{\$}_{F,1} - (1 + i^{\$,syn}_0)D^{\$}_{F,0} + D^{\$}_{swap,1} + Q^{\$,spot}_{F,1} \tag{B.10}$$

This shows how their dollar money balances at the end of period 1 are determined. Foreign households start with $M^{\$}_{F,0}$ dollar money balances from period 0. Then they receive $Y^{\$}_{F,1}$ dollars from exports. They need to make a dollar payment of $C_{FH,1}$ for buying US goods invoiced in dollars. Their dollar balances increase in period 1 due to borrowing $D^{\$}_{F,1}$ dollars synthetically. They drop due to principal and interest payments on their synthetic dollar borrowing from period 0, $(1+i_0^{\$,syn})D^{\$}_{F,0}$. They increase due to borrowing $D^{\$}_{swap,1}$ from the ECB. Finally, $Q^{\$,spot}_{F,1}$ are dollars purchased in the spot market in period 1 in exchange for euros.

It follows that the spot market purchases by European households are ($dX = X_1 - X_0$)

$$Q^{\$,spot}_{F,1} = dM^{\$}_{F,1} - Y^{\$}_{F,1} + C_{FH,1} - D^{\$}_{F,1} + (1+i_0^{\$,syn})D^{\$}_{F,0} - D^{\$}_{swap,1} \tag{B.11}$$

we can also write this as

$$Q^{\$,spot}_{F,1} = dM^{\$}_{F,1} - Y^{\$}_{F,1} + C_{FH,1} - dD^{\$}_{F,1} + i_0^{\$,syn}D^{\$}_{F,0} - D^{\$}_{swap,1} \tag{B.12}$$

We do not need to consider US households as they only hold dollar bonds, receive dollars from exports and make payments in dollars for imports. They do not hold euro money balances.

B.2 Other Agents

The remaining agents are the CIP arbitrageurs, UIP arbitrageurs and noise traders. They are all US agents. CIP arbitrageurs do not enter the spot market. Their profits in period 1 from their period 0 positions are denominated in dollars, while all the goods they consume are invoiced in dollars.

The US UIP arbitrageurs will enter the spot market. They receive $(1+i_0^{€})B^{€}_{UIP,0}$ euros from their time 0 euro position. They will also buy $B^{€}_{UIP,1}$ new euro bonds. They therefore sell $(1+i_0^{€})B^{€}_{UIP,0} - B^{€}_{UIP,1}$ euros, which is equivalent to buying

$$Q^{\$,spot}_{UIP,1} = -S_1 dB^{€}_{UIP,1} + S_1 i_0^{€} B^{€}_{UIP,0}$$

dollars.

In analogy to the UIP arbitrageurs, but setting positions at time zero equal to zero, net demand for dollars by noise traders is

$$Q^{\$,spot}_{noise,1} = -S_1 B^{€}_{noise,1} \tag{B.13}$$

B.3 Equilibrium

Spot market equilibrium is

$$Q^{\$,spot}_{F,1} + Q^{\$,spot}_{UIP,1} + Q^{\$,spot}_{noise,1} = 0 \tag{B.14}$$

Substituting the expressions above gives

$$dM^{\$}_{F,1} - Y^{\$}_{F,1} + C_{FH,1} - dD^{\$}_{F,1} + i^{\$,syn}_0 D^{\$}_{F,0} - D^{\$}_{swap,1} - S_1 dB^{€}_{UIP,1} + S_1 i^{€}_0 B^{€}_{UIP,0} - S_1 B^{€}_{noise,1} = 0 \tag{B.15}$$

The US trade account is equal to

$$TA^{\$}_{H,1} = Y_{H,1} - P_1 C_{H,1} - P_1 C^{o}_{H,1} \tag{B.16}$$

We also have

$$Y_{H,1} = C_{HH,1} + C_{FH,1} + C^{o}_{HH,1} \tag{B.17}$$
$$P_1 C_{H,1} = C_{HH,1} + C_{HF,1} \tag{B.18}$$
$$P_1 C^{o}_{H,1} = C^{o}_{HH,1} + C^{o}_{HF,1} \tag{B.19}$$

Therefore

$$TA^{\$}_{H,1} = C_{FH,1} - C_{HF,1} - C^{o}_{HF,1} \tag{B.20}$$

It follows that

$$-Y^{\$}_{F,1} + C_{FH,1} = -C_{HF,1} - C^{o}_{HF,1} + C_{FH,1} = TA^{\$}_{H,1} \tag{B.21}$$

We can then write the spot market equilibrium as

$$dM^{\$}_{F,1} + TA^{\$}_{H,1} - dD^{\$}_{F,1} + i^{\$,syn}_0 D^{\$}_{F,0} - D^{\$}_{swap,1} - S_1 dB^{€}_{UIP,1} + S_1 i^{€}_0 B^{€}_{UIP,0} - S_1 B^{€}_{noise,1} = 0 \tag{B.22}$$

We can write net factor income as

$$i^{\$,syn}_0 B^{\$}_{CIP,0} + S_1 i^{€}_0 B^{€}_{UIP,0} \tag{B.23}$$

The first term captures interest income earned by US CIP arbitrageurs for lending synthetic dollars to Europe. The last term is interest paid to US UIP arbitrageurs on euro bonds. Using that $B^{\$}_{CIP,0} = D^{\$}_{F,0}$, we then have

$$CA^{\$}_{H,1} = TA^{\$}_{H,1} + i^{\$,syn}_0 D^{\$}_{F,0} + S_1 i^{€}_0 B^{€}_{UIP,0} \tag{B.24}$$

We can then write the spot market equilibrium as

$$dM^{\$}_{F,1} + CA^{\$}_{H,1} - dD^{\$}_{F,1} - D^{\$}_{swap,1} - S_1 dB^{€}_{UIP,1} - S_1 dB^{€}_{noise,1} = 0 \tag{B.25}$$

Since $D^{\$}_{F,1} = B^{\$}_{CIP,1}$, we can also write this as

$$dM^{\$}_{F,1} + CA^{\$}_{H,1} - dB^{\$}_{CIP,1} - D^{\$}_{swap,1} - S_1 dB^{€}_{UIP,1} - S_1 dB^{€}_{noise,1} = 0 \tag{B.26}$$

or

$$CA^{\$}_{H,1} = dB^{\$}_{CIP,1} + D^{\$}_{swap,1} + S_1 dB^{€}_{UIP,1} + S_1 dB^{€}_{noise,1} - dM^{\$}_{F,1} \tag{B.27}$$

This says that the current account is equal to net capital outflows. The first four terms on the right are capital outflows. The first term involves US CIP arbitrageurs lending synthetic dollars to Europe. The second term involves the ECB lending dollars to European households, which is in turn associated with the swap line from the Fed to the ECB. This results in the Fed holding more euro money balances, which is a capital outflow. We could also think about the dollar loan by the ECB to European households as a loan from existing dollar foreign exchange reserves. In that case the drop in their foreign exchange reserves reduces US external liabilities and is therefore a net capital outflow as well. The third and fourth terms terms involve US UIP arbitrageurs and noise traders purchasing euro bonds. The last term is a US capital inflow in the form of European households buying liquid dollar assets.

C Pre-Shock Equilibrium

In the pre-shock equilibrium we assume that $D^{\$}_{swap,0} = D^{\$}_{swap,1} = 0$ and $n_1 = n_0 = 0$. Period 1 variables are equal to period 0 variables. For the exchange rate this implies $s_1 = s_0 = 0$. This also implies that $P_1 = P^*_1 = 1$. Consumption is smoothed in that period 1 consumption by households is equal to period 2 consumption when $s_2 = 0$. We denote pre-shock period 1 variables with a bar. They are equal to corresponding period 0 variables.

In the pre-shock equilibrium saving of Home and Foreign households is zero, so that wealth is the same in period 1 as in period 0. This implies

$$\bar{C}_{H,1} = \bar{Y}_{H,1} + \bar{\Pi}_{HCB,1} + i^{\$}_0 W_{H,0} \tag{C.28}$$

$$\bar{C}_{F,1} = \bar{Y}_{F,1} + \bar{\Pi}_{FCB,1} - i^{\$,syn}_0 D^{\$}_{F,0} + i^{€}_0 B^{€}_{F,0} \tag{C.29}$$

This sets period 1 consumption equal to income, which is the sum of income from production and interest income and transfers of central bank profits back to the households. Here $\bar{\Pi}_{HCB,1} = i_0^{\$} M_0^{\$}$ and $\bar{\Pi}_{FCB,1} = i_0^{€} M_0^{€}$. One of these equations is redundant as aggregate world saving is zero. So we remove the last equation.

We have consumption smoothing in that period 1 consumption is equal to period 2 consumption when $s_2 = 0$. Then

$$\bar{C}_{H,1} = 1 + \bar{\Pi}_{HCB,2} + \bar{M}_{H,1}^{\$} + (1 + \bar{i}_1^{\$})\bar{W}_{H,1} \tag{C.30}$$

$$\bar{C}_{F,1} = 1 + \bar{\Pi}_{FCB,2} + \bar{M}_{F,1}^{\$} + \bar{M}_{F,1}^{€} + (1 + \bar{i}_1^{€})\bar{W}_{F,1} - (\bar{i}_1^{\$,syn} - \bar{i}_1^{€})\bar{D}_{F,1}^{\$} \tag{C.31}$$

The last two equations needed to derive the pre-shock equilibrium are

$$\bar{C}_{H,1} + \bar{C}_{F,1} + \bar{C}_{H,1}^{o} = 2 \tag{C.32}$$

$$\bar{B}_{CIP,1}^{\$} = \bar{D}_{F,1}^{\$} \tag{C.33}$$

These correspond to the period 2 world goods market equilibrium, replacing $\bar{C}_{h,2} = \bar{C}_{h,1}$, and the period 1 swap market equilibrium. We then have a total of 5 equations: (C.28) and (C.30)-(C.33). This system can be solved by substituting expressions for money balances, portfolio holdings, central bank profits and period 1 production, setting $\bar{i}_1^{\$,syn} = i_0^{\$,syn}$, $s_1 = s_0 = 0$ and $\bar{W}_{h,1} = W_{h,0}$. We then have 5 equations in 5 variables: the 2 period 1 consumption levels, the 2 initial wealth levels $W_{h,0}$ and $\overline{cip}$. The interest differential $\bar{i}_1^d = i_0^d$ is exogenous.

Start with expressions for consumption of the "other agents":

$$\bar{C}_{H,1}^{o} = \frac{\overline{cip}^2}{\phi\Gamma} + \frac{\left(\bar{i}^d\right)^2}{\Gamma} \tag{C.34}$$

We have

$$\bar{Y}_{H,1} = \bar{C}_{HH,1} + \bar{C}_{FH,1} + \bar{C}_{HH,1}^{o} = (1-\omega)\left(\bar{C}_{H,1} + \bar{C}_{H,1}^{0}\right) + \omega\bar{C}_{F,1} \tag{C.35}$$

Regarding central bank profits in the pre-shock equilibrium, we have

$$\bar{\Pi}_{HCB,2} = \bar{\Pi}_{HCB,1} = \psi i_0^{\$}\left(\bar{C}_{H,1} + \omega\bar{C}_{F,1}\right) \tag{C.36}$$

$$\bar{\Pi}_{FCB,2} = \bar{\Pi}_{FCB,1} = \psi i_0^{€}(1-\omega)\bar{C}_{F,1} \tag{C.37}$$

This uses that central bank bond holdings are equal to the money supply, which is equal to money demand.

Money demand expressions are

$$\bar{M}^{\$}_{H,1} = \psi \bar{C}_{H,1} \tag{C.38}$$

$$\bar{M}^{\$}_{F,1} = \psi \omega \bar{C}_{F,1} \tag{C.39}$$

$$\bar{M}^{€}_{F,1} = \psi (1-\omega) \bar{C}_{F,1} \tag{C.40}$$

Substituting these expressions into the 5 equations, we have

$$\omega \bar{C}_{H,1} = \omega \bar{C}_{F,1} + (1-\omega)\left[\frac{\overline{cip}^2}{\phi\Gamma} + \frac{\left(\bar{i}^d\right)^2}{\Gamma}\right] + \psi i_0^{\$}\left(\bar{C}_{H,1} + \omega \bar{C}_{F,1}\right) + i_0^{\$} W_{H,0} \tag{C.41}$$

$$(1-\psi)\bar{C}_{H,1} = 1 + \psi i_0^{\$}\left(\bar{C}_{H,1} + \omega \bar{C}_{F,1}\right) + (1+i_0^{\$}) W_{H,0} \tag{C.42}$$

$$(1-\psi)\bar{C}_{F,1} = 1 + \psi i_0^{€}(1-\omega)\bar{C}_{F,1} + (1+i_0^{€}) W_{F,0} - \frac{\overline{cip}(\overline{cip} + \bar{i}^d)}{\phi\Gamma} \tag{C.43}$$

$$\bar{C}_{H,1} + \bar{C}_{F,1} + \frac{\overline{cip}^2}{\phi\Gamma} + \frac{\left(\bar{i}^d\right)^2}{\Gamma} = 2 \tag{C.44}$$

$$\rho + \psi \omega \bar{C}_{F,1} - \frac{\overline{cip} + \bar{i}^d}{\Gamma} = \frac{\overline{cip}}{\phi\Gamma} \tag{C.45}$$